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Taylor Series If $f:[a,b]\to\mathbb{R}$ and $f^{(N+1)}:[a,b]\to\mathbb{R}$ is continuous then

$$f(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_N(x-a)^N + \text{Error}(N),$$

where
$$a_\ell = \frac{1}{\ell!} f^{(\ell)}(a)$$ and there exists $c \in (a,x)$
such that $$\text{Error}(N) = \frac{1}{(N+1)!} f^{(N+1)}(c)(x-a)^{N+1}.$$

Stone-Weierstrass If $f:[a,b]\to\mathbb{C}$ is a continuous function then there exists a sequence of polynomials $(p_1, p_2, \ldots)$ such that $(p_1, p_2, \ldots)$ converges uniformly to f.
So, essentially, any continuous function is close to a power series
$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$$

Trigonometric series

$$f(x) = c_0 + c_1 e^{ix} + c_{-1} e^{-ix} + c_2 e^{2ix} + c_{-2} e^{-ix} + \cdots$$

If $k \in \mathbb{Z}$ then

$$\sin kx = \frac{e^{kix} - e^{-kix}}{2i} \quad \text{and} \quad \cos kx = \frac{e^{kix} + e^{ikx}}{2}$$

and $e^{ikx} = \cos kx + i \sin kx$ and $e^{-ikx} = \cos kx - i \sin kx$.

So

$$f(x) = c_0 + c_1(\cos x + i\sin x) + c_2(\cos 2x + i \sin 2x) + \dots$$
$$+ c_{-1}(\cos x - i \sin x) + c_{-2}(\cos 2x - i \sin 2x)$$

$$= c_0 + (c_1 + c_{-1})\cos x + i(c_1 - c_{-1})\sin x$$
$$+ (c_2 + c_{-2})\cos 2x + i(c_2 - c_{-2})\sin 2x + \dots$$

$$= a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \dots$$

where
$a_0 = c_0$, $a_k = c_k + c_{-k}$ and $b_k = i(c_k - c_{-k})$.

<u>Proposition</u> Let $k \in \mathbb{Z}$.

(a) $\displaystyle \frac{1}{2\pi}\int_0^{2\pi} e^{ikx}\,dx = \begin{cases} 0, & \text{if } k \neq 0 \\ 1, & \text{if } k = 0 \end{cases}$

(b) $\displaystyle \frac{1}{2\pi}\int_0^{2\pi} e^{ikx} e^{-ilx}\,dx = \delta_{kl}$, where $\delta_{kl} = \begin{cases} 0, & \text{if } k \neq l \\ 1, & \text{if } k = l \end{cases}$

(c) If $f(x) = c_0 + c_1 e^{ix} + c_{-1} e^{-ix} + c_2 e^{i2x} + c_{-2} e^{i(-2x)} + \dots$

then $$c_k = \frac{1}{2\pi}\int_0^{2\pi} f(x)\, e^{i(-kx)}\, dx$$

<u>Proof</u> (a)

If $k \neq 0$ then $\frac{1}{2\pi}\int_0^{2\pi} e^{ikx}\,dx = \frac{1}{2\pi}\, e^{ikx}\Big|_{x=0}^{x=2\pi}$

$$= \frac{1}{2\pi}\left(e^{2\pi i} - e^{0}\right) = \frac{1}{2\pi}(1-1) = \frac{1}{2\pi}\cdot 0 = 0,$$

If $k = 0$ then $\frac{1}{2\pi}\int_0^{2\pi} e^{ikx}\,dx = \frac{1}{2\pi}\int_0^{2\pi} e^{0}\,dx$

$$= \frac{1}{2\pi}\int_0^{2\pi} dx = \frac{1}{2\pi}\left(x\Big|_{x=0}^{x=2\pi}\right) = \frac{1}{2\pi}(2\pi - 0) = 1.$$

(b)
$$\frac{1}{2\pi}\int_0^{2\pi} e^{ikx} e^{i(-\ell x)}\,dx = \frac{1}{2\pi}\int_0^{2\pi} e^{i(k-\ell)x}\,dx$$

$$= \begin{cases} 0, & \text{if } k \neq \ell, \\ 1, & \text{if } k = \ell, \end{cases} \quad \text{by part (a).}$$

(c)
$$\frac{1}{2\pi}\int_0^{2\pi} f(x)\, e^{i(-kx)}\,dx = \frac{1}{2\pi}\int_0^{2\pi}\Big(\sum_{\ell\in\mathbb{Z}} c_\ell e^{i\ell x}\Big) e^{i(-kx)}\,dx$$

$$= \frac{1}{2\pi}\sum_{\ell\in\mathbb{Z}} c_\ell \int_0^{2\pi} e^{i(\ell-k)x}\,dx = c_k, \quad \cancel{\text{if } k \neq 0}$$

Example Consider $f:[0,2\pi]\to\mathbb{R}$ given by $f(x)=x^2$. Write

$$f(x)=c_0+c_1e^{ix}+c_{-1}e^{-ix}+c_2e^{2ix}+c_{-2}e^{-2ix}+\dots$$

Well

$$c_0=\frac{1}{2\pi}\int_0^{2\pi}f(x)\,dx=\frac{1}{2\pi}\int_0^{2\pi}x^2\,dx=\frac{1}{2\pi}\left(\frac{x^3}{3}\Big|_{x=0}^{x=2\pi}\right)$$

$$=\frac{1}{2\pi}\frac{(2\pi)^3}{3}=\frac{4\pi^2}{3}$$

and

$$c_k=\frac{1}{2\pi}\int_0^{2\pi}x^2e^{-ikx}\,dx.$$

$$\int x^2e^{-ikx}\,dx=\frac{1}{-ik}x^2e^{-ikx}-\int\frac{2xe^{-ikx}}{-ik}\,dx$$

$$=\frac{ix^2}{k}e^{-ikx}-\left(\frac{2xe^{-ikx}}{(-ik)^2}-\int\frac{2e^{-ikx}}{(-ik)^2}\,dx\right)$$

$$=\frac{ix^2}{k}e^{-ikx}+\frac{2xe^{-ikx}}{k^2}+\frac{2e^{-ikx}}{(-ik)^3}.$$

So

$$c_k=\frac{1}{2\pi}\int_0^{2\pi}x^2e^{-ikx}\,dx=\frac{i(2\pi)^2}{2\pi k}e^{-ik2\pi}+\frac{2(2\pi)e^{-ik2\pi}}{2\pi k^2}$$

$$=\frac{2\pi i}{k}+\frac{2}{k^2}.$$

So

$$a_k=c_k+c_{-k}=\frac{2\pi i}{k}+\frac{2}{k^2}+\frac{2\pi i}{-k}+\frac{2}{k^2}=\frac{4}{k^2}$$

and $b_k = i(c_k - c_{-k}) = i\left(\left(\frac{2\pi i}{k} + \frac{2}{k^2}\right) - \left(\frac{2\pi i}{-k} + \frac{2}{k^2}\right)\right) = i\,2\cdot\frac{2\pi i}{k} = -\frac{4\pi}{k}.$

So

$$x^2 = \frac{4\pi^2}{3} + 4\cos x - 4\pi \sin x + \frac{4}{4}\cos 2x - \frac{4\pi}{2}\sin 2x + \cdots$$

$$= \frac{4\pi^2}{3} + \sum_{k=1}^{\infty}\left(\frac{4}{k^2}\cos kx - \frac{4\pi}{k}\sin kx\right)$$

If $x=\pi$ then

$$\pi^2 = \frac{4\pi^2}{3} + \sum_{k=1}^{\infty} \frac{4}{k^2}\cos k\pi - \frac{4\pi}{k}\sin k\pi$$

$$= \frac{4\pi^2}{3} + \sum_{k=1}^{\infty} (-1)^k \frac{4}{k^2}$$

So

$$-\frac{\pi^2}{3} = \sum_{k=1}^{\infty}(-1)^k\frac{4}{k^2} \quad \text{and} \quad \frac{\pi^2}{12} = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{k^2}$$