620-295 Real Analysis with applications, lect 26, Improper Integrals

6 Oct. 2009.

First kind:

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx.$$

Second kind:

$$\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx \qquad \text{or}$$

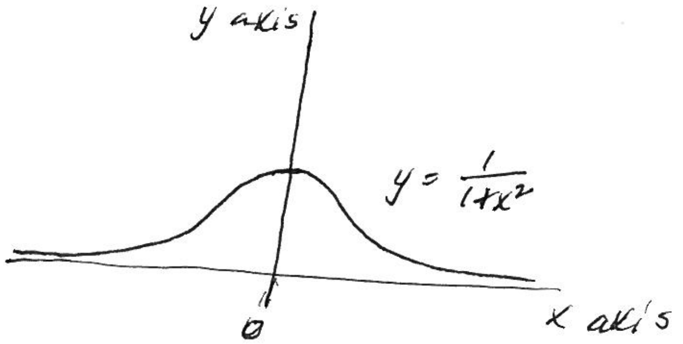
$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx.$$

Example

$$\int_0^\infty \frac{dx}{1+x^2} = \lim_{t\to\infty} \int_0^t \frac{dx}{1+x^2}$$

$$= \lim_{t\to\infty} \left(\tan^{-1} x \Big|_{x=0}^{x=t} \right) = \lim_{t\to\infty} \left(\tan^{-1} t - \tan^{-1} 0\right)$$

$$= \lim_{t\to\infty} \tan^{-1} t = \frac{\pi}{2}.$$



<u>Example</u> Let $p \in \mathbb{R}$, $p > 1$. Then

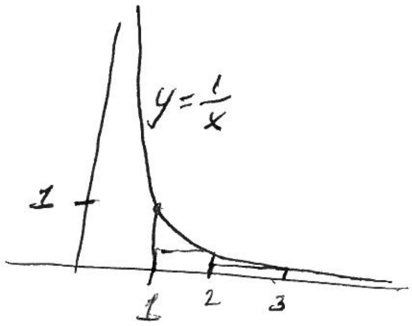
$$\int_1^\infty \frac{dx}{x^p} = \lim_{t\to\infty} \int_1^t \frac{dx}{x^p} = \lim_{t\to\infty} \int_1^t x^{-p}\,dx$$

$$= \lim_{t\to\infty} \left(+ \frac{x^{-p+1}}{-p+1} \Bigg|_{x=1}^{x=t} \right) = \lim_{t\to\infty} \left(\frac{t^{-p+1}}{-p+1} - \frac{1}{-p+1} \right)$$

$$= \lim_{t\to\infty} \left(\frac{t^{-(p-1)}}{-(p-1)} - \frac{1}{-(p-1)} \right) = 0 + \frac{1}{p-1} = \frac{1}{p-1}.$$

If $p = 1$ then

$$\int_1^\infty \frac{dx}{x}$$



So

$$\int_1^\infty \frac{dx}{x} > \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

So $\int_1^\infty \frac{dx}{x}$ diverges

If $0 < p < 1$ then

$$\int_1^\infty \frac{dx}{x^p} > \int_1^\infty \frac{dx}{x}.$$ So $\int_1^\infty \frac{dx}{x^p}$ diverges.

<u>Example</u>

$$\int_0^1 \frac{1}{x^{1/2}}\,dx = \lim_{t\to 0} \int_t^1 x^{-\frac{1}{2}}\,dx$$

$$= \lim_{t\to 0} \left(2x^{\frac{1}{2}} \Big|_{x=t}^{x=1} \right) = \lim_{t\to 0} \left(2\cdot 1^{\frac{1}{2}} - 2t^{\frac{1}{2}} \right) = 2.$$

<u>Example</u>

$$\int_{-1}^1 \frac{dx}{\sqrt{1-x^2}} = \int_{-1}^1 \frac{dx}{\sqrt{1-x^2}} = \cancel{\lim_{t\to -1} \int_t^1 \frac{dx}{\sqrt{1-x^2}}}$$

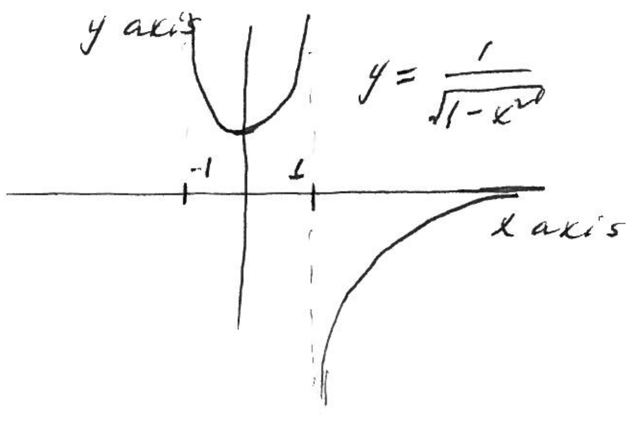
$$= \int_{-1}^0 \frac{dx}{\sqrt{1-x^2}} + \int_0^1 \frac{dx}{\sqrt{1-x^2}}$$

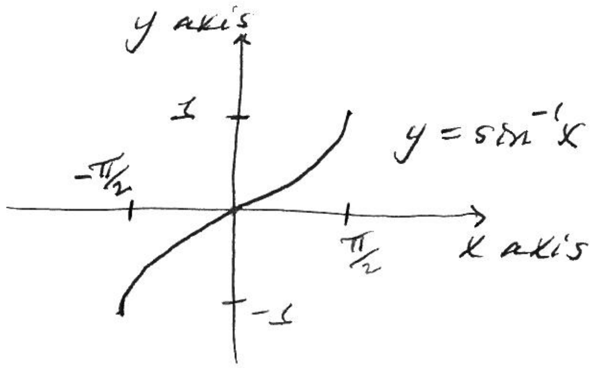
$$= \lim_{t\to -1} \int_t^0 \frac{dx}{\sqrt{1-x^2}} + \lim_{t\to 1} \int_0^t \frac{dx}{\sqrt{1-x^2}}$$

$$= \lim_{t\to -1} \left(\sin^{-1}x \Big|_{x=t}^{x=0} \right) + \lim_{t\to 1} \left(\sin^{-1}x \Big|_{x=0}^{x=t} \right)$$

$$= \lim_{t\to -1} \left(\sin^{-1}0 - \sin^{-1}t \right) + \lim_{t\to 1} \left(\sin^{-1}t - \sin^{-1}0 \right)$$

$$= 0 - \sin^{-1}(-1) + \sin^{-1}1 - \sin^{-1}0 = -\left(-\frac{\pi}{2}\right) + \frac{\pi}{2} = \pi.$$





Example $\int_0^1 \frac{dx}{x^p}$

Let $p \in [0,1)$. Then

$$\int_0^1 \frac{dx}{x^p} = \lim_{t\to 0} \int_t^1 \frac{dx}{x^p} = \lim_{t\to 0}\left(\int_t^1 x^{-p}\,dx\right) = \lim_{t\to 0}\left(\frac{x^{-p+1}}{-p+1}\Big|_{x=t}^{x=1}\right)$$

$$= \lim_{t\to 0}\left(\frac{1}{1-p}\,x^{1-p}\Big|_{x=t}^{x=1}\right) = \lim_{t\to 0}\left(\frac{1}{1-p} - \frac{t^{1-p}}{1-p}\right) = \frac{1}{1-p}.$$

If $p=1$ then

$$\int_0^1 \frac{dx}{x^p} = \lim_{t\to 0} \int_t^1 \frac{dx}{x} = \lim_{t\to 0}\left(\log x\Big|_{x=t}^{x=1}\right) = \lim_{t\to 0}(\log 1 - \log t)$$

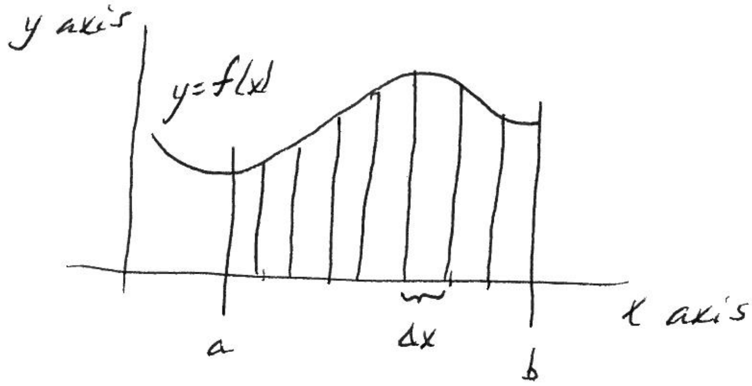
$$= \lim_{t\to 0}(-\log t).$$

So $\int_0^1 \frac{dx}{x}$ diverges.

If $p\in\mathbb{R}$ and $p>1$ then

$$\int_0^1 \frac{dx}{x^p} > \int_0^1 \frac{dx}{x}$$ so that $\int_0^1 \frac{dx}{x^p}$ diverges.

Trapezoidal integral



Area of

is $f(l)\cdot\Delta x + \frac{1}{2}\left(f(r)-f(l)\right)\Delta x$

$$= \frac{\Delta x}{2}\left(f(r)+f(l)\right)$$

So

$$\int_a^b f(x)\,dx = \lim_{\Delta x \to 0} \frac{\Delta x}{2}\Big(f(a)+f(a+\Delta x) + f(a+\Delta x)+f(a+2\Delta x)+\dots$$
$$\dots + \underbrace{f(b-2\Delta x)+f(b-\Delta x)}_{\substack{\text{contribution}\\ \text{from next to last}\\ \text{trapezoid}}} + \underbrace{f(b-\Delta x)+f(b)}_{\substack{\text{contribution}\\ \text{from last}\\ \text{trapezoid}}}\Big)$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x}{2}\Big(f(a)+2f(a+\Delta x)+2f(a+2\Delta x)+\dots$$
$$\dots + 2f(b-\Delta x) + f(b)\Big)$$