A topological space is a set X with a specification of the open sets of X, i.e. a set X with a collection $\mathcal{T}$ of subsets of X such that

(a) $\phi \in \mathcal{T}$ and $X \in \mathcal{T}$,

(b) if $U_i \in \mathcal{T}$ then $\bigcup_i U_i \in \mathcal{T}$

(c) if $n \in \mathbb{Z}_{>0}$ and $U_1, \ldots, U_n \in \mathcal{T}$ then $U_1 \cap \cdots \cap U_n \in \mathcal{T}$.

Let X and Y be topological spaces.

A function $f: X \to Y$ is continuous if it satisfies

if V is an open set of Y then $f^{-1}(V)$ is an open set of X.

Let X be a topological space.

A subspace of X is a subset E of X with the topology given by making

$U \cap E$ open in E if U is open in X.

Example: $X = \mathbb{R}$ and $E = [0,1]$.

Then

$[0, \frac{1}{2})$ is not open in $X = \mathbb{R}$

but $[0, \frac{1}{2}) = (-1, \frac{1}{2}) \cap [0,1]$ is open in $[0,1]$.

A topological space X is <u>connected</u> if X satisfies:

There exist open set A, B of X such that

(a) $A \neq \phi$ and $B \neq \phi$

(b) $A \cup B = X$ and $A \cap B = \phi$.

Let X be a topological space.
A <u>connected subset</u> of X is a subset $E \subseteq X$ such that the subspace E of X is a connected topological space.

<u>Theorem</u> Let $f: X \to Y$ be a continous function. If X is connected then $f(X)$ is connected.

<u>Proposition</u> ~~If~~ Let $E \subseteq \mathbb{R}$ be a seubset of $\mathbb{R}$. E is connected if and only if E satisfies: if $x, y \in E$ and $z \in \mathbb{R}$ and $x < z < y$ then $z \in E$.

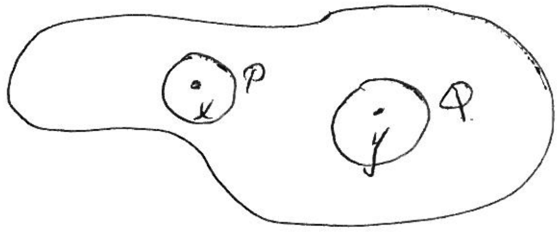
<u>Corollary</u> (Intermediate Value Theorem). Let $a, b \in \mathbb{R}$ and let $f: [a,b] \to \mathbb{R}$ be a continuous function. If $w \in \mathbb{R}$ and w is between $f(a)$ and $f(b)$ then there exists $z \in (a,b)$ such that $f(z) = w$.

A topological space X is Hausdorff if X satisfies:

if $x, y \in X$ and $x \neq y$ then there exist open sets P and Q such that $x \in P$ and $x \in Q$ and

$$P \cap Q = \phi.$$

In half English: X is Hausdorff if any two points can be separated



A topological space X is compact if X satisfies

if $\mathcal{P} \subseteq \mathcal{T}$ and $\bigcup_{U \in \mathcal{P}} U = X$ then

there exists $n \in \mathbb{Z}_{>0}$ and $U_1, \ldots, U_n \in \mathcal{P}$ such that

$$U_1 \cup U_2 \cup \cdots \cup U_n.$$

In half English: X is compact if every open cover has a finite subcover.

Examples: $[0,1]$ is compact

$(0,1)$ is not compact

$\mathbb{R}$ is not compact.

<u>Theorem</u> Let X be a topological space and let $E \subseteq X$.

(a) If X is compact and E is closed then E is compact.

(b) If X is Hausdorff and E is compact then E is closed.

(c) If X is a metric space and E is compact then E is closed and bounded.

(d) If $X = \mathbb{R}^n$ then E is compact if and only if E is closed and bounded.

(e) If X is a metric space then E is compact if and only if E satisfies:
if $S \subseteq E$ and S is infinite then there exists $e \in E$ such that e is a close point to S.

Part (e) in half English:
If X is a metric space then E is compact if and only if every infinite subset of E has a close point in E.