620-295 Real Analysis with Applications

Assignment 3: Due 5pm on 4 September

Lecturer: Arun Ram
Department of Mathematics and Statistics
University of Melbourne
Parkville VIC 3010 Australia
aram@unimelb.edu.au

Due 5pm on 4 September in the appropriate assignment box on the ground floor of Richard Berry.

1. Define the following and give an example for each:
   (a) cardinality,
   (b) finite,
   (c) infinite,
   (d) countable,
   (e) uncountable.

2. Prove that $\mathrm{Card}(\mathbb{Z}_{>0}) \neq \mathrm{Card}(\mathbb{R})$.

3. Define the following and give an example for each:
   (a) sequence,
   (b) converges (for a sequence),
   (c) diverges (for a sequence),
   (d) limit (of a sequence),
   (e) sup (of a sequence),
   (f) inf (of a sequence),
   (g) lim sup (of a sequence),
   (h) lim inf (of a sequence),
   (i) bounded (for a sequence),
   (j) increasing (for a sequence),
   (k) decreasing (for a sequence),
   (l) monotone (for a sequence),
   (m) Cauchy sequence.

4. Give an example of a sequence (a_n) such that none of $\inf a_n$, $\liminf a_n$, $\limsup a_n$, and sup a_n are equal.

5. Find the power series expansions and the radius of convergence of e^x, $\log(1+x)$, $\frac{1}{1-x}$, $(1+x)^{1/2}$, $\arctan x$, and $\sinh x$.

6. Let $r \in \mathbb{R}$ with $0 < r < 1$. Find $\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n$, and explain why this limit is important to everyone with a credit card.

(1)(a) Let S and T be sets.

The sets S and T have the same <u>cardinality</u> if there exists a bijection $f: S \to T$.

Example: If $S = \{\forall, \exists, \ni\}$ and $T = \{1, 2, 3\}$ then $\mathrm{Card}(S) = \mathrm{Card}(T)$ since

$$\begin{array}{ccc} S & \longrightarrow & T \\ \forall & \mapsto & 3 \\ \exists & \mapsto & 2 \\ \ni & \mapsto & 1 \end{array}$$

is a bijection.

(b) A set S is <u>finite</u> if there exists $n \in \mathbb{Z}_{>0}$ such that $\mathrm{Card}(S) = \mathrm{Card}(\{1, 2, \ldots, n\})$ or $S = \emptyset$.

<u>Example</u> $S = \{\forall, \exists, \ni\}$ is finite since $\mathrm{Card}(S) = \mathrm{Card}(\{1, 2, 3\})$.

(c) A set S is <u>infinite</u> if S is not finite.

<u>Example</u> $\mathbb{Z}_{>0} = \{1, 2, 3, \ldots\}$ is infinite.

(d) A set S is <u>countable</u> if $S = \emptyset$ or S is finite or $\mathrm{Card}(S) = \mathrm{Card}(\mathbb{Z}_{>0})$.

<u>Example</u> $\mathbb{Q} = \{\frac{a}{b} \mid a, b \in \mathbb{Z} \text{ with } b \neq 0\}$ with $\frac{a}{b} = \frac{c}{d}$ if $ad = bc$ is a countable set.

(e) A set S is <u>uncountable</u> if it is not countable.

<u>Example</u> The set of real numbers, $\mathbb{R}$, is uncountable.

(2) To show: $\mathrm{Card}(\mathbb{Z}_{>0}) \neq \mathrm{Card}(\mathbb{R})$

To show: There does not exist a bijection $f: \mathbb{Z}_{>0} \to \mathbb{R}$

Proof by contradiction.

Assume $f: \mathbb{Z}_{>0} \to \mathbb{R}$ is a bijection.

Let $f(j) = a_{j0}.a_{j1}a_{j2}a_{j3}\cdots$ so that

$$f(1) = a_{10}.a_{11}a_{12}a_{13}\cdots,$$
$$f(2) = a_{20}.a_{21}a_{22}a_{23}\cdots,$$
$$f(3) = a_{30}.a_{31}a_{32}a_{33}\cdots, \quad \text{etc.}$$

Let $s = 0.s_1s_2s_3s_4\ldots$ be such that

$s_1 \neq a_{11}, s_2 \neq a_{22}, s_3 \neq a_{33}, \ldots$

Then $s \neq f(j)$ for all $j \in \mathbb{Z}_{>0}$.

So s is an element of $\mathbb{R}$ such that there does not exist $j \in \mathbb{Z}_{>0}$ with $f(j) = s$.

So $f: \mathbb{Z}_{>0} \to \mathbb{R}$ is not surjective.

This is a contradiction to the assumption that f is a bijection.

So there does not exist a bijection $f: \mathbb{Z}_{>0} \to \mathbb{R}$.

So $\mathrm{Card}(\mathbb{Z}_{>0}) \neq \mathrm{Card}(\mathbb{R})$. //

(3) (a) Let X be a metric space.

A <u>sequence</u> in X is a function $\begin{array}{l}\mathbb{Z}_{>0} \to X \\ n \mapsto a_n.\end{array}$

(b) A sequence (a_n) <u>converges</u> if there exists $l \in X$ such that if $\varepsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{>0}$ and $n > N$ then $d(a_n, l) < \varepsilon$.

(c) A sequence (a_n) <u>diverges</u> if it does not converge.

<u>Examples</u> of (a), (b), (c):

(a) and (b): $X = \mathbb{R}$, $a_n = \frac{1}{n}$ converges to 0.

(a) and (c): $X = \mathbb{R}$, $a_n = (-1)^n$ diverges.

(d) Let (a_n) be a sequence in X. The <u>limit</u> of (a_n) is $l \in X$ such that (a_n) converges to l.

<u>Example</u>: If $X = \mathbb{R}$ and $a_n = \frac{1}{n}$ then (a_n) has limit 0.

(e) Let (a_n) be a sequence in $\mathbb{R}$.

The <u>supremum</u> of (a_n) is

$$\sup a_n = \sup\{a_1, a_2, \ldots\},$$

the least upper bound of the set $\{a_1, a_2, \ldots\}$.

(f) Let (a_n) be a sequence in $\mathbb{R}$.

The <u>infimum</u> of (a_n) is

$$\inf a_n = \inf\{a_1, a_2, \ldots\},$$

the greatest lower bound of the set $\{a_1, a_2, \ldots\}$.

(g) Let (a_n) be a sequence in $\mathbb{R}$.

The <u>upper limit</u> of (a_n) is

$$\limsup a_n = \lim_{k\to\infty} \sup\{a_k, a_{k+1}, \ldots\}.$$

(h) Let (a_n) be a sequence in $\mathbb{R}$.

The <u>lower limit</u> of (a_n) is

$$\liminf a_n = \lim_{k\to\infty} \inf\{a_k, a_{k+1}, \ldots\}.$$

<u>Examples</u> of $\sup a_n$, $\inf a_n$, $\limsup a_n$, $\liminf a_n$ are given in problem 4 below.

(i) Let (a_n) be a sequence in a metric space X.

The sequence (a_n) is <u>bounded</u> if there exists $p \in X$ and $M \in \mathbb{R}_{>0}$ such that

$$\text{if } n \in \mathbb{Z}_{>0} \text{ then } \cancel{d(a_n, 0)}\ d(a_n, p) < M.$$

<u>Examples</u>: $X = \mathbb{R}$, $a_n = \frac{1}{n}$ is bounded.

$X = \mathbb{R}$, $a_n = n$ is not bounded.

(j) Let (a_n) be a sequence in $\mathbb{R}$.

The sequence (a_n) is <u>increasing</u> if it satisfies:

if $n \in \mathbb{Z}_{>0}$ then $a_n \leq a_{n+1}$

(k) The sequence (a_n) is <u>decreasing</u> if it satisfies:

if $n \in \mathbb{Z}_{>0}$ then $a_n \geq a_{n+1}$.

(l) The sequence (a_n) is <u>monotone</u> if it is increasing or decreasing.

(m) Let (a_n) be a sequence in a metric space X.

The sequence (a_n) is <u>Cauchy</u> if it satisfies:

if $\varepsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that if $m, n \in \mathbb{Z}_{>0}$ ~~then~~ and $m > N$ and $n > N$ then $d(a_m, a_n) < \varepsilon$.

<u>Examples</u>: $X = \mathbb{R}$, $a_n = \frac{1}{n}$ is decreasing

$X = \mathbb{R}$, $a_n = -\frac{1}{n}$ is increasing.

In $X = \mathbb{R}$ a sequence is Cauchy if and only if it is convergent, so

$a_n = \frac{1}{n}$ is Cauchy and $a_n = (-1)^n$ is not.

(4) Let $a_n = (-1)^n(1+\frac{1}{n})$.

Then $a_1 = (-1)(1+1) = -2$,

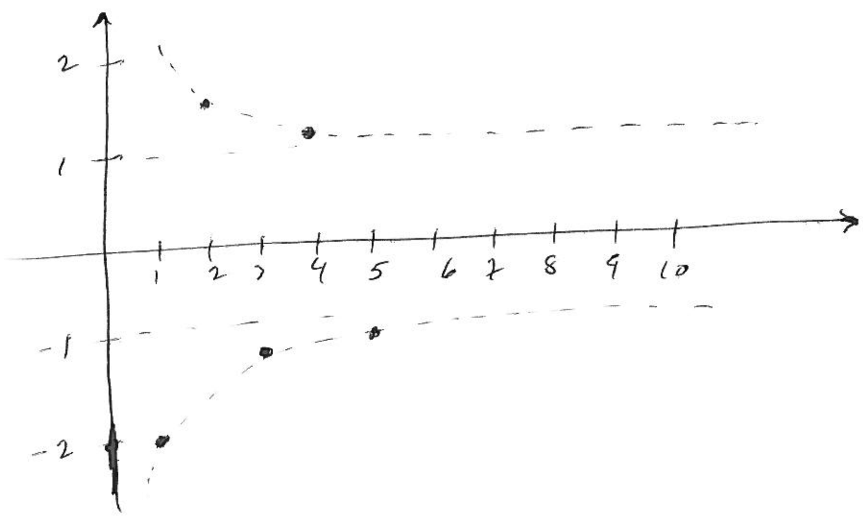
$a_2 = (1+\frac{1}{2}) = 3/2$,

$a_3 = -(1+\frac{1}{3}) = -4/3$,

$a_4 = (1+\frac{1}{4}) = 5/4$, etc

Then $\sup a_n = 3/2$, $\inf a_n = -2$,

$\limsup a_n = 1$ and $\liminf a_n = -1$.



(5) (a) Since $\frac{d}{dx} e^x = e^x$ and $e^0 = 1$

$$\frac{1}{k!}\left(\frac{d^k}{dx^k} e^x\right)\Big|_{x=0} = \frac{1}{k!} e^x\Big|_{x=0} = \frac{1}{k!} e^0 = \frac{1}{k!},$$

and, by Taylor's theorem,

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots$$

Let $r \in \mathbb{R}_{\geq 0}$. Then

$$\lim_{n\to\infty} \left| \frac{\frac{r^{n+1}}{(n+1)!}}{\frac{r^n}{n!}} \right| = \lim_{n\to\infty} \left| \frac{r}{n+1} \right| = 0$$

So, by the ratio test,

$$1 + r + \frac{1}{2!}r^2 + \frac{1}{3!}r^3 + \cdots \quad \text{converges,}$$

since $\lim_{n\to\infty} \left| \frac{\frac{r^{n+1}}{(n+1)!}}{r^n/n!} \right| < 1$.

So $C = \{ r \in \mathbb{R}_{\geq 0} \mid \sum_{n=0}^{\infty} \frac{r^n}{n!} \text{ converges} \} = \mathbb{R}_{\geq 0}$.

The <u>radius</u> <u>of</u> <u>convergence</u> of e^x is $\sup C$ $= \sup \mathbb{R}_{\geq 0}$ which does not exist since C has no upper bound.

(b) Since $\frac{d}{dx}\log(1+x) = \frac{1}{1+x}$, $\frac{d}{dx}\left(\frac{1}{1+x}\right) = (-1)\frac{1}{(1+x)^2}$

and $\frac{d^k}{dx^k}\log(1+x) = (-1)^{k-1}\frac{1}{(1+x)^k}(k-1)!$,

then $$\frac{1}{k!}\left(\frac{d^k}{dx^k}\log(1+x)\right)\Bigg|_{x=0} = \frac{(k-1)!}{k!}(-1)^{k-1}\frac{1}{(1+x)^k}\Bigg|_{x=0}$$

$$= \frac{1}{k}(-1)^{k-1}\frac{1}{1} = (-1)^{k-1}\frac{1}{k}.$$

So $$\log(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \tfrac{1}{5}x^5 - \cdots$$
$$= \sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^k}{k}.$$

If $r=1$ then

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \cdots = \log(2) \text{ converges}$$

and $$1 + \left|\tfrac{-1}{2}\right| + \left|\tfrac{1}{3}\right| + \left|\tfrac{-1}{4}\right| + \cdots = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \cdots \text{ diverges.}$$

So $1 \in C$ and $1 \notin A$, where

$$A = \left\{ r \in \mathbb{R}_{\geq 0} \;\middle|\; \sum_{k=1}^{\infty}\left|(-1)^{k-1}\frac{r^k}{k}\right| \text{ converges} \right\}, \text{ and}$$
$$C = \left\{ r \in \mathbb{R}_{\geq 0} \;\middle|\; \sum_{k=1}^{\infty}\frac{r^k}{k} \text{ converges} \right\}.$$

So $[0,1) \subseteq A \subseteq C$ and $[0,1] = C$.

So the radius of convergence of $\log(1+x)$ is $\sup C = 1$.

(c) $\frac{1}{1-x} = 1+x+x^2+x^3+x^4+x^5+\cdots$

since

$$(1-x)(1+x+x^2+x^3+\cdots) = 1+x+x^2+x^3+x^4+\cdots$$
$$-x-x^2-x^3-x^4-\cdots$$
$$= 1$$

Then $1 = \frac{x-1}{x-1}$, $1+x = \frac{x^2-1}{x-1}$, $1+x+x^2 = \frac{1-x^3}{1-x}$

and $1+x+x^2+\cdots+x^{k-1} = \frac{1-x^k}{1-x}$,

so that, if $r \in \mathbb{R}_{\geq 0}$ then

$$1+r+r^2+r^3+\cdots = \lim_{k\to\infty} \frac{1-r^k}{1-r}$$

So

$$1+r+r^2+r^3+\cdots = \begin{cases} \frac{1-0}{1-r}, & \text{if } r<1 \\ \text{diverges,} & \text{if } r \geq 1. \end{cases}$$

So $C = \{r \in \mathbb{R}_{\geq 0} \mid 1+r+r^2+r^3+\cdots \text{ converges}\}$
$= [0,1)$.

So the radius of convergence of $\frac{1}{1-x}$ is
$\sup C = \sup [0,1) = 1$.

(d) Since $\frac{d}{dx}(1+x)^{\frac{1}{2}} = \frac{1}{2}(1+x)^{-\frac{1}{2}}$,

$$\frac{d^2}{dx^2}(1+x)^{\frac{1}{2}} = \frac{1}{2}\left(\frac{-1}{2}\right)(1+x)^{-3/2},$$

$$\frac{d^3}{dx^3}(1+x)^{\frac{1}{2}} = \frac{1}{2}\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)(1+x)^{-5/2},$$

$$\frac{d^4}{dx^4}(1+x)^{\frac{1}{2}} = \frac{1}{2}\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{-5}{2}\right)(1+x)^{-7/2}, \ldots$$

then $\frac{1}{k!}\frac{d^k}{dx^k}(1+x)^{\frac{1}{2}} = (-1)^{k-1}\frac{1\cdot 3\cdot 5\cdots(2k-3)}{k!\,2^k}(1+x)^{-(2k-1)/2}$.

and $\frac{1}{k!}\left(\frac{d^k}{dx^k}(1+x)^{\frac{1}{2}}\right)\Big|_{x=0} = (-1)^{k-1}\frac{1\cdot 3\cdot 5\cdots(2k-3)}{2^k(1\cdot 2\cdot 3\cdots k)}$

So $(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x - \frac{1}{2^2}x^2 + \frac{1\cdot 3}{2^3}x^3 - \frac{1\cdot 3\cdot 5}{2^4}x^4 + \frac{1\cdot 3\cdot 5\cdot 7}{2^5}x^5 - \cdots$

Let $r \in \mathbb{R}_{\geq 0}$. Then

$$\lim_{k\to\infty}\left|\frac{(-1)^{k+1-1}r^{k+1}\dfrac{1\cdot 3\cdot 5\cdots(2(k+1)-3)}{2^{k+1}\,1\cdot 2\cdot 3\cdots(k+1)}}{(-1)^{k-1}r^k\dfrac{1\cdot 3\cdot 5\cdots(2k-3)}{2^k\cdot 1\cdot 2\cdot 3\cdots k}}\right| = \lim_{k\to\infty}\left|(-1)r\frac{2(k+1)-3}{2\cdot(k+1)}\right|$$

$$= \lim_{k\to\infty}\left|(-1)r\cdot\frac{2k-1}{2k+2}\right| = \lim_{k\to\infty} r\frac{\left(1-\frac{1}{2k}\right)}{\left(1+\frac{2}{2k}\right)} = r.$$

So $1+\frac{1}{2}r - \frac{1}{2^2}r^2 + \frac{1\cdot 3}{2^3}r^3 - \cdots$ converges if $r<1$ and diverges if $r>1$. (by the ratio test).

So the radius of convergence of $(1+x)^{\frac{1}{2}}$ is 1

(e) Since $\frac{1}{1-x} = 1+x+x^2+x^3+x^4+x^5+\cdots$,

$$\frac{1}{1+x} = 1-x+x^2-x^3+x^4-x^5+\cdots$$

$$\frac{1}{1+x^2} = 1-x^2+x^4-x^6+x^8-x^{10}+\cdots$$

and

$$\arctan x = \int \frac{1}{1+x^2}\,dx = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + C$$

where C is a constant.

Since $\arctan 0 = 0$, the constant $C=0$ and

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots$$

Let $r \in \mathbb{R}_{\geq 0}$. Then

$$\lim_{k\to\infty} \left| \frac{\frac{(-1)^{k+1-1} r^{2(k+1)-1}}{2(k+1)-1}}{\frac{(-1)^{k-1} r^{2k-1}}{2k-1}} \right| = \lim_{k\to\infty} \left| (-1) \frac{r^{2k+1}(2k-1)}{r^{2k-1}(2k+1)} \right|$$

$$= \lim_{k\to\infty} \left| (-1) r^2 \frac{\left(1-\frac{1}{2k}\right)}{\left(1+\frac{1}{2k}\right)} \right| = \lim_{k\to\infty} r^2 \frac{\left(1-\frac{1}{2k}\right)}{\left(1+\frac{1}{2k}\right)} = r^2$$

So $r - \frac{r^3}{3} + \frac{r^5}{5} - \frac{r^7}{7} + \frac{r^9}{9} - \cdots$ converges if $r^2 < 1$

and diverges if $r^2 > 1$.

If $r=1$ then $1 - \frac{1}{3} + \frac{1}{5} - \cdots = \arctan 1 = \frac{\pi}{4}$.

So $C = \{r \in \mathbb{R}_{\geq 0} \mid r - \frac{r^3}{3} + \frac{r^5}{5} - \frac{r^7}{7} + \frac{r^9}{9} - \cdots \text{ converges}\}$

$= \{r \in \mathbb{R}_{\geq 0} \mid r^2 \leq 1\} = [0,1]$

So the radius of convergence of $\arctan x$ is 1.

(f) $\sinh x = \dfrac{e^x - e^{-x}}{2} = \dfrac{\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right) - \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots\right)}{2}$

$$= \frac{2x + 2\frac{x^3}{3!} + 2\frac{x^5}{5!} + \cdots}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

If $r \in \mathbb{R}_{\geq 0}$ then

$$\lim_{k \to \infty} \frac{\frac{r^{2(k+1)-1}}{(2(k+1)-1)!}}{\frac{r^{2k-1}}{(2k-1)!}} = \lim_{k \to \infty} \frac{r^{2k+1}(2k-1)!}{r^{2k-1}(2k+1)!}$$

$$= \lim_{k \to \infty} \frac{r^2}{2k \cdot 2k+1} = 0.$$

So $r + \frac{r^3}{3!} + \frac{r^5}{5!} + \cdots$ converges for all $r \in \mathbb{R}_{\geq 0}$

So the radius of convergence of $\sinh x$ does not exist (or is "infinity").

(6) Let $r \in \mathbb{R}$ with $0 < r < 1$.

$$\lim_{n\to\infty} \left(1+\frac{r}{n}\right)^n = \lim_{n\to\infty} e^{\log\left(1+\frac{r}{n}\right)^n}$$

$$= \lim_{n\to\infty} e^{n\log\left(1+\frac{r}{n}\right)} = \lim_{n\to\infty} e^{r\cdot\frac{\log(1+r/n)}{r/n}}$$

$$= e^r, \quad \text{since } \lim_{n\to\infty} \frac{\log\left(1+\frac{r}{n}\right)}{r/n} = 1$$

(this will be justified below).

The limit $\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n$ is the amount owed on a loan of \$1 at interest rate r after 1 year if the interest is compounded continuously.

More generally, if

L_0 is the initial loan amount,

T is the time (in years) of the loan,

r is the interest rate,

then $\lim_{n\to\infty} L_0\left(1+\frac{r}{n}\right)^{n\cdot T}$ is the amount owed (after T years) if the interest is compounded continuously.

Assume $r \in \mathbb{R}$ and $0 < r < 1$.

To show: $\displaystyle\lim_{n\to\infty} \frac{\log(1+\frac{r}{n})}{r/n} = 1$.

To show: If $\varepsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{>0}$ and $n > N$ then

$$\left| \frac{\log(1+\frac{r}{n})}{r/n} - 1 \right| < \varepsilon$$

Assume $\varepsilon \in \mathbb{R}_{>0}$.

Let $N > \frac{2r}{\varepsilon}$. Assume $n \in \mathbb{Z}_{>0}$ and $n > N$.

To show: $\left| \frac{\log(1+\frac{r}{n})}{r/n} - 1 \right| < \varepsilon$.

$$\left| \frac{\log(1+\frac{r}{n})}{r/n} - 1 \right| = \left| \frac{\frac{r}{n} - \frac{1}{2}\left(\frac{r}{n}\right)^2 + \frac{1}{3}\left(\frac{r}{n}\right)^3 - \cdots}{r/n} - 1 \right|$$

$$= \left| \left(1 - \tfrac{1}{2}\left(\tfrac{r}{n}\right) + \tfrac{1}{3}\left(\tfrac{r}{n}\right)^2 - \cdots\right) - 1 \right|$$

$$= \left| -\tfrac{1}{2}\left(\tfrac{r}{n}\right) + \tfrac{1}{3}\left(\tfrac{r}{n}\right)^2 - \cdots \right|$$

$$\leq \left| -\tfrac{1}{2}\left(\tfrac{r}{n}\right) \right| + \tfrac{1}{3}\left| \left(\tfrac{r}{n}\right)^2 \right| + \left| -\tfrac{1}{4}\left(\tfrac{r}{n}\right)^3 \right| + \cdots$$

$$= \tfrac{1}{2}\left(\tfrac{r}{n}\right) + \tfrac{1}{3}\left(\tfrac{r}{n}\right)^2 + \tfrac{1}{4}\left(\tfrac{r}{n}\right)^3 + \cdots$$

$$< \tfrac{r}{n} + \left(\tfrac{r}{n}\right)^2 + \left(\tfrac{r}{n}\right)^3 + \cdots = \tfrac{r}{n}\left(1 + \tfrac{r}{n} + \left(\tfrac{r}{n}\right)^2 + \cdots\right)$$

$$= \frac{r}{n}\left(\frac{1}{1-\frac{r}{n}}\right) < \frac{r}{n} \cdot 2 < \varepsilon.$$

So $\displaystyle\lim_{n\to\infty} \frac{\log(1+\frac{r}{n})}{r/n} = 1$.

(7) $$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}} + \cdots$$

$$> 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots$$

Which is unbounded.

So $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \underbrace{\frac{1}{2^2} + \frac{1}{3^2}} + \underbrace{\frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2}}$$

$$< 1 + \frac{2}{2^2} + \frac{4}{4^2} + \frac{8}{8^2} + \cdots$$

$$= 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

$$= 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots = \frac{1}{1 - \frac{1}{2}} = 2.$$

So $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a monotone increasing sequence bounded by 2.

So $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

(8) Since $1 = \frac{1-r}{1-r}$, $1+r = \frac{1-r^2}{1-r}$, $1+r+r^2 = \frac{1-r^3}{1-r}$,

and
$$1+r+\cdots+r^{k-1} = \frac{1-r^k}{1-r},$$

then
$$\sum_{n=1}^{\infty} r^n = r+r^2+r^3+\cdots$$
$$= -1+(1+r+r^2+r^3+\cdots)$$
$$= \lim_{k\to\infty}\left(-1+\frac{1-r^k}{1-r}\right)$$
$$= \begin{cases} -1+\frac{1}{1-r}, & \text{if } |r|<1 \\ \text{diverges}, & \text{if } |r|\geq 1 \end{cases}$$

So $\sum_{n=1}^{\infty} r^n = -1+\frac{1}{1-r} = \frac{r-1+1}{1-r} = \frac{r}{1-r}$, if $|r|<1$.

and $\sum_{n=1}^{\infty} r^n$ diverges if $|r|\geq 1$.