620-295 Real analysis with applications Lect 21, 10.09.2009 (1)

The function f is differentiable at $x=c$ if

$$\lim_{x\to c} \frac{f(x)-f(c)}{x-c} \quad \text{exists.}$$

The derivative of f at $x=c$ is

$$f'(c) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c},$$

if f is differentiable at $x=c$.

If $f:[a,b]\to\mathbb{R}$ and $g:[a,b]\to\mathbb{R}$ are functions and $\ell\in\mathbb{R}$ define

$(f+g):[a,b]\to\mathbb{R}$ by $(f+g)(x) = f(x)+g(x)$,

$(\ell f):[a,b]\to\mathbb{R}$ by $(\ell f)(x) = \ell\, f(x)$

$fg:[a,b]\to\mathbb{R}$ by $(fg)(x) = f(x)\cdot g(x)$.

Proposition: If $c\in[a,b]$ and $f'(c)$ exists and $g'(c)$ exists then

(a) $(f+g)'(c) = f'(c)+g'(c)$

(b) $(\ell f)'(c) = \ell f'(c)$

(c) $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$.

Proof (a) To show: $(f+g)'(c) = f'(c) + g'(c)$.

$$(f+g)'(c) = \lim_{x\to c} \frac{(f+g)(x) - (f+g)(c)}{x-c}$$

$$= \lim_{x\to c} \frac{f(x)+g(x) - (f(c)+g(c))}{x-c}$$

$$= \lim_{x\to c} \frac{f(x)-f(c) + g(x)-g(c)}{x-c}.$$

$$= \lim_{x\to c} \frac{f(x)-f(c)}{x-c} + \frac{g(x)-g(c)}{x-c}$$

$$= \lim_{x\to c} \frac{f(x)-f(c)}{x-c} + \lim_{x\to c} \frac{g(x)-g(c)}{x-c}$$

$$= f'(c) + g'(c),$$

if you believe that $\lim\limits_{x\to c} A(x) + B(x) = \lim\limits_{x\to c} A(x) + \lim\limits_{x\to c} B(x)$.

Proposition Let $A:[a,b]\to\mathbb{R}$ and $B:[a,b]\to\mathbb{R}$ and assume $\lim\limits_{x\to c} A(x)$ exists and $\lim\limits_{x\to c} B(x)$ exists

Then

$\lim\limits_{x\to c} A(x)+B(x)$ exists and

$$\lim_{x\to c} A(x)+B(x) = \lim_{x\to c} A(x) + \lim_{x\to c} B(x).$$

Proof Let $l_1 = \lim_{x \to c} A(x)$ and $l_2 = \lim_{x \to c} B(x)$.

To show: $\lim_{x \to c} A(x) + B(x) = l_1 + l_2$.

To show: If $\varepsilon \in \mathbb{R}_{>0}$ then there exists $\delta \in \mathbb{R}_{>0}$

such that if $x \in B_\delta(c)$ then $|A(x) + B(x) - (l_1 + l_2)| < \varepsilon$.

Assume $\varepsilon \in \mathbb{R}_{>0}$.

We know: There exists $\delta_1 \in \mathbb{R}_{>0}$ such that

if $x \in B_{\delta_1}(c)$ then $|A(x) - l_1| < \varepsilon/2$.

We know: There exists $\delta_2 \in \mathbb{R}_{>0}$ such that

if $x \in B_{\delta_2}(c)$ then $|A(x) - l_2| < \varepsilon/2$.

Let $\delta = \min(\delta_1, \delta_2)$.

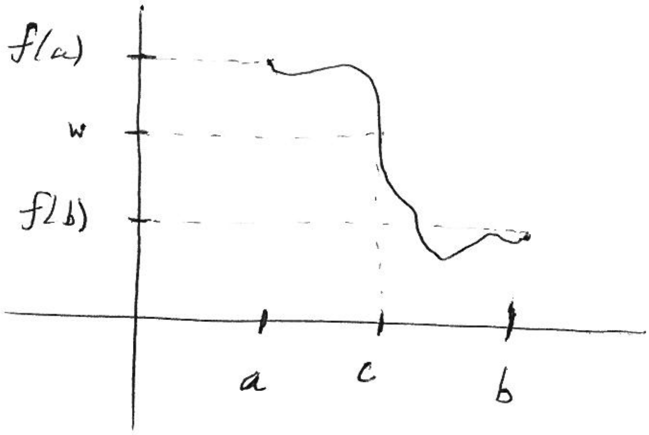
To show: $|A(x) + B(x) - (l_1 + l_2)| < \varepsilon$.

$$
\begin{aligned}
|A(x) + B(x) - (l_1 + l_2)| &= |(A(x) - l_1) + (B(x) - l_2)| \\
&\le |A(x) - l_1| + |B(x) - l_2| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}
$$

So $\lim_{x \to c} A(x) + B(x) = \lim_{x \to c} A(x) + \lim_{x \to c} B(x)$.

Theorem (Intermediate value theorem)

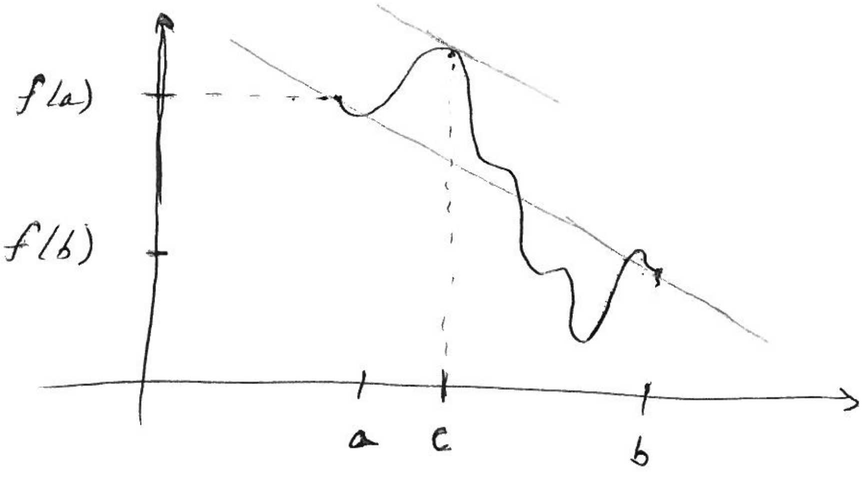
If $f:[a,b] \to \mathbb{R}$ is continuous and w is between $f(a)$ and $f(b)$ then there exists $c \in [a,b]$ such that $f(c) = w$.



Theorem (Mean value theorem)

If $f:[a,b] \to \mathbb{R}$ is continuous for $x \in [a,b]$ and differentiable for $x \in (a,b)$ then there exists $c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$



Functions $f: [a,b] \to \mathbb{R}$

Let $a, b \in \mathbb{R}$. Let $c \in [a,b]$.

The function f is <u>continuous</u> at $x = c$ if f satisfies:

if $\varepsilon \in \mathbb{R}_{>0}$ then there exists $\delta \in \mathbb{R}_{>0}$ such that

if $x \in B_\delta(c)$ then $f(x) \in B_\varepsilon(f(c))$,

where $B_\delta(c) = \{x \in [a,b] \mid |x-c| < \delta\} = (c-\delta, c+\delta)$.

The function f is <u>continuous</u> at $x = c$ if f satisfies

if $\varepsilon \in \mathbb{R}_{>0}$ then there exists $\delta \in \mathbb{R}_{>0}$ such that

$B_\delta(c) \subseteq f^{-1}(B_\varepsilon(f(c)))$.

The function f is <u>continuous</u> at $x = c$ if f satisfies

if $\varepsilon \in \mathbb{R}_{>0}$ and $V = B_\varepsilon(f(c))$ then

c is an interior point of $f^{-1}(V)$.

A set $E \subseteq [a,b]$ is <u>connected</u> if there do not exist open sets $A, B \subseteq [a,b]$ with

(a) $A \neq \emptyset$ and $B \neq \emptyset$

(b) $A \cup B = [a,b]$

(c) $A \cap B = \emptyset$.

<u>Theorem</u> If $f: X \to Y$ is continuous and X is connected then $f(X)$ is connected.

<u>~~Pfoof~~</u> ~~Proof~~ Proof by contradiction.

Assume $f(X)$ is not connected.

Let A, B be open in $f(X)$ such that $A \cup B = f(X)$ and $A \cap B = \phi$.

Then let $C = f^{-1}(A)$ and $D = f^{-1}(B)$.

Then $C \cup D = f^{-1}(A) \cup f^{-1}(B) = f^{-1}(A \cup B) = f^{-1}(f(X)) = X$.

$C \cap D = f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B) = f^{-1}(\phi) = \phi$.

$C \neq \phi$ since $A \neq \phi$ and $A \subseteq f(X)$.

$D \neq \phi$ since $B \neq \phi$ and $B \subseteq f(X)$.

So X is not connected. Contradiction.

So $f(X)$ is connected.