Let X be a partially ordered set (think $\mathbb{R}$) with partial order $\le$.

The <u>intervals</u> in X are

$(a,b) = \{x \in X \mid a < x < b\}$ open, not closed

$[a,b] = \{x \in X \mid a \le x \le b\}$ closed, not open

$[a,b) = \{x \in X \mid a \le x < b\}$ not open, not closed

$(a,b] = \{x \in X \mid a < x \le b\}$

$(a,\infty) = \{x \in X \mid a < x\}$ open, not closed

$(-\infty,b) = \{x \in X \mid x < b\}$ open, not closed.

A <u>topological space</u> is a set X with a specification of the <u>open subsets</u> of X where it is required that

(a) $\emptyset$ is open and X is open,

(b) Unions of open sets are open,

(c) Finite intersections of open sets are open.

A <u>topology</u> on X is a set $\mathcal{T}$ of subsets of X such that (a) $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$,

(b) If $U_i \in \mathcal{T}$ then $(\bigcup U_i) \in \mathcal{T}$

(c) If $U_1, \dots, U_n$ is a finite collection of elements of $\mathcal{T}$ then $U_1 \cap \cdots \cap U_n \in \mathcal{T}$

Open sets and closed sets

A topological space is a set X with a topology $\mathcal{T}$

An open set in X is a set in $\mathcal{T}$.

A closed set is a subset E of X such that

E^c is open,

where $E^c = \{x \in X \mid x \notin E\}$.

Let X be a topological space.

Let $E \subseteq X$.

The interior of E is the subset E° of E such that

(a) E° is open in X

(b) If U is an open set of X and $U \subseteq E$ then $U \subseteq E^\circ$.

The closure of E is the subset $\bar{E}$ of X such that

(a) $\bar{E}$ is closed in X and $\bar{E} \supseteq E$.

(b) If C is a closed set in X and $C \supseteq E$ then $C \supseteq \bar{E}$.

Neighborhoods

Let X be a topological space

Let $x \in X$.

A neighborhood of x is a subset N of X such that there exists an open set U of X with $x \in U \subseteq N$.

Let $E \subseteq X$.

An interior point of E is a point $x \in X$ such that there exists a neighborhood N of x with $N \subseteq E$

A close point to E is a point $x \in X$ such that

if N is a neighborhood of x then N contains a point of E.

Examples
.9 is an interior point of $[0,1]$ in $\mathbb{R}$

1 is not an interior point of $[0,1]$ in $\mathbb{R}$.

0 is a close point to $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$ in $\mathbb{R}$.

<u>Proposition</u> Let X be a topological space. Let $E \subseteq X$.

(a) The interior of E is the set of interior points of E

(b) The closure of E is the set of close points of E.

<u>Proof</u> (a) Let $I = \{x \in E \mid x \text{ is an interior point of } E\}$.

To show: $E^\circ = I$.

To show: (aa) $I \subseteq E^\circ$

(ab) $E^\circ \subseteq I$.

(aa) Let $x \in I$. Then there exists a neighborhood N of x with $N \subseteq E$.

So there exists an open set $\mathcal{U}$ with $x \in \mathcal{U} \subseteq N \subseteq E$.

Since $\mathcal{U} \subseteq E$ and $\mathcal{U}$ is open $\mathcal{U} \subseteq E^\circ$.

So $x \in E^\circ$.

So $I \subseteq E^\circ$.

(ab) To show: If $x \in E^\circ$ then $x \in I$.

Assume $x \in E^\circ$.

Then E° is open and $x \in E^\circ \subseteq E$.

So x is an interior point of E.

So $x \in I$

So $E^\circ \subseteq I$

So $I = E^\circ$.

Let X be a topological space.
Let $x \in X$.
A <u>fundamental system of neighborhoods of x</u>
is a set $\mathcal{N}$ of neighborhoods of x such that
if V is a neighborhood of x then there exists
$N \in \mathcal{N}$ with $N \subseteq V$.

A function $f: X \to Y$ is <u>continuous</u> if f satisfies:
if V is open in Y then $f^{-1}(V)$ is open in X.

A <u>base</u> of the topology of X is a collection $\mathcal{B}$
of open set such that
(a) If $B \in \mathcal{B}$ then B is open,
(b) If U is open in X then U is a union of
elements of $\mathcal{B}$.

<u>Proposition</u> Let X be a topological space and let
$\mathcal{B}$ be a collection of subsets of X. Then $\mathcal{B}$ is
a base if and only if $\mathcal{B}$ satisfies
if $x \in X$ and $\mathcal{B}(x) = \{ B \in \mathcal{B} \mid x \in B \}$ then
$\mathcal{B}(x)$ is a fundamental system of neighborhoods
of x.

Let X be a metric space.

The topology on X is the collection of unions of sets in

$$\mathcal{B} = \{ B_\varepsilon(a) \mid \varepsilon \in \mathbb{R}_{>0} \text{ and } a \in X \}.$$

where

$$B_\varepsilon(a) = \{ x \in X \mid d(x,a) < \varepsilon \}$$