620-295 Real Analysis with applications, Lect. 19,
Limits and continuity in metric spaces 7 Sept. 2009 ①

Let X and Y be metric spaces.

Let $a \in X$ and $\ell \in Y$.

The limit of $f: X \to Y$ as x approaches a is ℓ if
f satisfies:

if $\varepsilon \in \mathbb{R}_{>0}$ then there exists $\delta \in \mathbb{R}_{>0}$ such that
if $x \in X$ and $d(x,a) < \delta$ then $d(f(x), \ell) < \varepsilon$.

Write $\lim_{x \to a} f(x) = \ell$ if the limit of f as x approaches a is ℓ.

Example $\lim_{x \to 2} \frac{x^2+1}{2x-1}$. We think it is $\frac{5}{3}$.

To show: $\lim_{x \to 2} \frac{x^2+1}{2x-1} = \frac{5}{3}$

To show: If $\varepsilon \in \mathbb{R}_{>0}$ then there exists $\delta \in \mathbb{R}_{>0}$
such that
if $x \in \mathbb{R}$ and $d(x,2) < \delta$ then $d\left(\frac{x^2+1}{2x-1}, \frac{5}{3}\right) < \varepsilon$.

Assume $\varepsilon \in \mathbb{R}_{>0}$

To show: there exists $\delta \in \mathbb{R}_{>0}$ such that
if $x \in \mathbb{R}$ and $d(x,2) < \delta$ then $d\left(\frac{x^2+1}{2x-1}, \frac{5}{3}\right) < \varepsilon$.

Let $\delta = \min\left(2\varepsilon, \frac{1}{10}\right)$

To show: If $x \in \mathbb{R}$ and $d(x, 2) < \delta$ then $d\left(\frac{x^2+1}{2x-1}, \frac{5}{3}\right) < \varepsilon$

Assume $x \in \mathbb{R}$ and $|x-2| < \delta$.

To show: $\left|\frac{x^2+1}{2x-1} - \frac{5}{3}\right| < \varepsilon$

$$\left|\frac{x^2+1}{2x-1} - \frac{5}{3}\right| = \left|\frac{(x-2+2)^2+1}{2((x-2)+2)-1} - \frac{5}{3}\right| = \left|\frac{(x-2)^2+4(x-2)+4+1}{2(x-2)+4-1} - \frac{5}{3}\right|$$

$$= \left|\frac{(x-2)^2+4(x-2)+5}{2(x-2)+3} - \frac{5}{3}\right| = \left|\frac{3(x-2)^2+12(x-2)+15-10(x-2)-15}{3(2(x-2)+3)}\right|$$

$$= \left|\frac{(x-2)(3(x-2)+2)}{3(2(x-2)+3)}\right| < \frac{\delta(3\delta+2)}{3\left(3-\frac{2}{10}\right)}$$

$$< \frac{3\delta}{3\cdot 2} = \frac{\delta}{2} < \varepsilon.$$

So $\lim\limits_{x\to 2} \frac{x^2-1}{2x-1} = \frac{5}{3}$. //

Let $f: \mathbb{R} \to \mathbb{R}$. The function f is continuous at $x = a$ if it doesn't jump at $x = a$:

$$\lim_{x \to a} f(x) = f(a).$$

Let X, Y be metric spaces.

Let $f: X \to Y$ be a function and let $a \in X$.

The function f is <u>continuous</u> <u>at</u> <u>$x=a$</u> if

$$\lim_{x \to a} f(x) = f(a).$$

In other words,

the function f is <u>continuous</u> <u>at</u> <u>$x=a$</u> if

f satisfies:

if $\varepsilon \in \mathbb{R}_{>0}$ then there exists $\delta \in \mathbb{R}_{>0}$ such that

if $x \in X$ and $d(x,a) < \delta$ then $d(f(x), f(a)) < \varepsilon$

The function f is <u>differentiable</u> <u>at</u> <u>$x=a$</u> if

$$\lim_{x \to a} \frac{f(x)-f(a)}{x-a} \quad \text{exists}$$

Write

$$f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}, \quad \text{if } f \text{ is differentiable at } x=a.$$

A function $f: X \to Y$ is <u>continuous</u> if it satisfies:

if $a \in X$ then f is continuous at $x=a$.

In other words:

A function $f: X \to Y$ is <u>continuous</u> if it satisfies

if $a \in X$ and $\varepsilon \in \mathbb{R}_{>0}$ then there exists $\delta \in \mathbb{R}_{>0}$
such that if $x \in X$ and $d(x,a) < \delta$ then $d(f(x), f(a)) < \varepsilon$.

A function $f: X \to Y$ is <u>uniformly</u> <u>continuous</u> if it satisfies:

if $\varepsilon \in \mathbb{R}_{>0}$ then there exists a $\delta \in \mathbb{R}_{>0}$ such that
if $x, y \in X$ and $d(x,y) < \delta$ then $d(f(x), f(y)) < \varepsilon$.

A function $f: X \to Y$ is <u>Lipschitz</u> if it satisfies:

There exists $K \in \mathbb{R}_{>0}$ such that
if $x, y \in X$ then $d(f(x), f(y)) \leq K\, d(x,y)$.

Let X be a metric space and $a \in X$.

Let $\varepsilon \in \mathbb{R}_{>0}$.

The <u>ε-ball</u> at a is

$$B_\varepsilon(a) = \{x \in X \mid d(x,a) < \varepsilon\}.$$

Let (x_n) be a sequence in X. Let $l \in X$.

The sequence (x_n) converges to l if it satisfies:

if $\varepsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that

if $n \in \mathbb{Z}_{>0}$ and $n > N$ then $d(x_n, l) < \varepsilon$.

Let $f: X \to Y$ and let $a \in X$ and $l \in Y$.

The function $f: X \to Y$ has limit l as x approaches a

if f satisfies:

if $\varepsilon \in \mathbb{R}_{>0}$ ~~then~~ there exists $\delta \in \mathbb{R}_{>0}$ such that

if $x \in B_\delta(a)$ then $f(x) \in B_\varepsilon(l)$.

The function $f: X \to Y$ is continuous at a if

f satisfies:

if $\varepsilon \in \mathbb{R}_{>0}$ then there exists $\delta \in \mathbb{R}_{>0}$ such that

if $x \in B_\delta(a)$ then $f(x) \in B_\varepsilon(f(a))$.

The function $f: X \to Y$ is continuous at a if f satisfies:

if $\varepsilon \in \mathbb{R}_{>0}$ then ~~there~~ $f^{-1}(B_\varepsilon(f(a)))$ is a neighborhood of a.

A neighborhood of a is a set N such that there exists ~~$B_\delta(a)$~~ $\delta \in \mathbb{R}_{>0}$ such that $B_\delta(a) \subseteq N$.