620-295 Real Analysis with applications

Problem Sheet 4

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1. Sequences and series

1. Define the following and give an example for each:
   (a) metric space,
   (b) complete (for a metric space),
   (c) completion (of a metric space) ,

2. Let (a_n) be a sequence in $\mathbb{R}$. Show that if (a_n) is increasing and bounded then (a_n) converges.

3. Let (a_n) be a sequence in $\mathbb{R}_{\geq 0}$. Show that if $\sum_{n=1}^{\infty} a_n$ is bounded then $\sum_{n=1}^{\infty} a_n$ converges.

4. Let X be a metric space and let (a_n) be a sequence in X. Show that if (a_n) converges then (a_n) is Cauchy.

5. Give an example of a Cauchy sequence that does not converge.

6. Let (a_n) be a sequence in $\mathbb{R}_{\geq 0}$. Show that if $\sum_{n=1}^{\infty} a_n$ converges then $\lim_{n\to\infty} |a_n| = 0$.

7. Let (a_n) be a sequence in $\mathbb{R}_{\geq 0}$. Show that if $\sum_{n=1}^{\infty} |a_n|$ converges then $\sum_{n=1}^{\infty} a_n$ converges.

8. Let $r \in \mathbb{R}$ with $0 < r < 1$. Prove that $\lim_{n\to\infty} \dfrac{\log\left(1+\frac{r}{n}\right)}{\frac{r}{n}} = 1$.

9. Prove that $\lim_{x\to 0} \dfrac{\log(1+x)}{x} = 1$.

10. Prove that $\lim_{x \to 0} \dfrac{\sin x}{x} = 1$.

11. Prove that $\lim_{x \to 0} \dfrac{\cos x - 1}{x} = 0$.

12. Prove that $\lim_{x \to 0} \dfrac{\cos x - 1}{x^2} = -\frac{1}{2}$.

2. Limits

1. Define the following and give an example for each:
   (a) continuous at p,
   (b) $\lim_{x \to a} f(x)$,
   (c) continuous,
   (d) uniformly continuous,
   (d) Lipschiz continuous,
   (e) derivative at p,

2. For each of the following, guess the limit and then prove the guess by using the definition of limit:
   (a) $\lim_{x \to 4} \left(\frac{1}{2} x - 3\right)$,
   (b) $\lim_{x \to 0} \frac{1}{1+x}$,
   (c) $\lim_{x \to 4} \frac{1}{1+x^2}$,
   (d) $\lim_{x \to 1} \frac{x^2-1}{x-1}$,
   (e) $\lim_{x \to 9} \frac{x+1}{x^2+1}$,
   (f) $\lim_{x \to \infty} \frac{\sin x}{x}$,
   (g) $\lim_{x \to 2} \frac{2x^2+3x-8}{x^3-2x^2+x-12}$,
   (h) $\lim_{x \to \infty} \frac{\log x + 2x}{3x-5}$,

3. Evaluate the following limits:
   (a) $\lim_{x \to 0} x \cos \frac{1}{x^2}$,
   (b) $\lim_{x \to 0} \left(\sqrt{5 + x^2} - \sqrt{x^{-2} - 1}\right)$,
   (c) $\lim_{x \to 0} \frac{\sqrt{1+x}-1}{x}$,
   (d) $\lim_{x \to \infty} \frac{x^4+x}{x^4+1}$,
   (e) $\lim_{x \to \infty} \frac{7x-1}{x^2}$,

(f) $\lim_{x \to 0^+} \frac{\sqrt{x}}{\sqrt{7+\sqrt{x+5}}}$,

(g) $\lim_{x \to 1} \frac{|x-1|+1}{x+|x+1|}$,

(h) $\lim_{x \to \infty} \frac{3x^2+1}{2x+1}$,

4. Evaluate the following limits:

(a) $\lim_{x \to 0} \frac{1 - \cos x}{x + x^2}$,

(b) $\lim_{x \to \infty} \frac{\log x}{x}$,

(c) $\lim_{x \to 0+} \sqrt{x} \quad \log x$,

(d) $\lim_{x \to 0+} \frac{\sqrt{x}}{\log x}$,

(e) $\lim_{x \to 0} \frac{\sin x}{x}$,

(f) $\lim_{x \to 0} \left(\frac{1}{\arcsin x} - \frac{1}{\sin x}\right)$.

3. Continuous functions

1. Let $f : \mathbb{R} \to \mathbb{R}$ be such that f is continuous at $x = 0$ and if $x, y \in \mathbb{R}$ then $f(x + y) = f(x)f(y)$. Show that if $a \in \mathbb{R}$ then f is continuous at $x = a$.

2. Let $f : \mathbb{R}_{>0} \to \mathbb{R}$ be such that f is continuous at $x = 1$ and if $x, y \in \mathbb{R}_{>0}$ then $f(xy) = f(x) + f(y)$. Show that if $a \in \mathbb{R}_{>0}$ then f is continuous at $x = a$.

3. Let I be an interval in $\mathbb{R}$. Let $f : I \to \mathbb{R}$ be continous. Show that the function $|f| : I \to \mathbb{R}$ given by $|f|(x) = |f(x)|$ is continuous.

4. Let I be an interval in $\mathbb{R}$ and let $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be continous. Show that the function $\max(f, g) : I \to \mathbb{R}$ given by $\max(f, g)(x) = \max(f(x), g(x))$ is continuous.

5. (Thomae's function) Let $f : [0, 1] \to \mathbb{R}$ be given by $f(x) = \begin{cases} \frac{1}{n}, & \text{if } \frac{m}{n} \in \mathbb{Q} \text{ is reduced,} \\ 0, & \text{if } x \notin \mathbb{Q}. \end{cases}$

   Show that

   (a) If $a \notin \mathbb{Q}$ then f is continuous at $x = a$, and

   (b) If $a \in \mathbb{Q}$ then f is not continuous at $x = a$.

6. Let I be an interval in $\mathbb{R}$ and let $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be continous. Show that the function $\min(f, g) : I \to \mathbb{R}$ given by $\min(f, g)(x) = \min(f(x), g(x))$ is continuous.

7. Let $a \in \mathbb{R}$ and let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \begin{cases} ax, & \text{if } x \le 0, \\ \sqrt{x}, & \text{if } x > 0. \end{cases}$

   Show that f is continuous.

8. Is the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x$ uniformly continuous?

9. Is the function $f : (0, 1) \to \mathbb{R}$ given by $f(x) = \frac{1}{x}$ uniformly continuous?

10. Is the function $f : (10^{-4}, 1) \to \mathbb{R}$ given by $f(x) = \frac{1}{x}$ uniformly continuous?

11. Is the function $f : (0, 1) \to \mathbb{R}$ given by $f(x) = x^2$ uniformly continuous?

12. Is the function $f : [-1, 1] \to \mathbb{R}$ given by $f(x) = \sqrt{1 - x^2}$ uniformly continuous?

13. Is the function $f : (1, \infty) \to \mathbb{R}$ given by $f(x) = \log x$ uniformly continuous?

14. Is the function $f : (0, \infty) \to \mathbb{R}$ given by $f(x) = \log x$ uniformly continuous?

15. Let $f : \mathbb{R} \to \mathbb{R}$ be the function given by $f(x) = \frac{x}{(1+|x|)}$. Show that
    (a) f is continuous,
    (b) f is uniformly continuous,
    (c) $\sup(f(\mathbb{R})) = 1$,
    (d) There does not exist $x \in \mathbb{R}$ such that $f(x)) = 1$,
    (e) $\inf(f(\mathbb{R})) = -1$,
    (d) There does not exist $y \in \mathbb{R}$ such that $f(y)) = -1$.

16. Show that the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3 - 6x + 3$ has exactly 3 roots.

17. Let I be an interval in $\mathbb{R}$ and let $f : I \to \mathbb{R}$ be a continuous function. Prove that $f(I)$ is an interval.

18. Let I and J be intervals in $\mathbb{R}$ and let $f : I \to J$ be a surjective strictly monotonic continuous function. Prove that the inverse function $g : J \to I$ exists and is strictly monotonic and continuous.

4. Differentiability

1. Let $a, b \in \mathbb{R}$ and let $f : [a, b] \to \mathbb{R}$ be a function. Let $c \in [a, b]$ and carefully define $f'(c)$.

Prove that if $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ are functions then $(fg)'(c) = f(c)g'(c) + f'(c)g(c)$, whenever $f'(c)$ and $g'(c)$ exist.

2. Let $f : \mathbb{R}_{>0} \to \mathbb{R}$ be such that f is differentiable at $x = 1$ and if $x, y \in \mathbb{R}_{>0}$ then $f(xy) = f(x) + f(y)$. Show that
   (a) if $c \in \mathbb{R}_{>0}$ then f is differentiable at $x = c$,
   (b) if $c \in \mathbb{R}_{>0}$ then $f'(c) = f'(1)/c$,
   (c) Show that f is infinitely differentiable.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be such that f is differentiable at $x = 0$ and if $x, y \in \mathbb{R}$ then $f(x + y) = f(x)f(y)$. Show that
   (a) if $c \in \mathbb{R}$ then f is differentiable at $x = c$,
   (b) if $c \in \mathbb{R}_{>0}$ then $f'(c) = f'(0)f(c)$,
   (c) Show that f is infinitely differentiable.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \begin{cases} -x^2, & \text{if } x \le 0, \\ x, & \text{if } x > 0. \end{cases}$

   Is f continuous at $x = 0$? Is f differentiable at $x = 0$?

5. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \begin{cases} -x^2, & \text{if } x \le 0, \\ x^3, & \text{if } x > 0. \end{cases}$

   Is f continuous at $x = 0$? Is f differentiable at $x = 0$?

6. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \begin{cases} \frac{\sin x}{x}, & \text{if } x < 0, \\ 1 + x^2, & \text{if } x \ge 0. \end{cases}$

   Is f continuous at $x = 0$? Is f differentiable at $x = 0$?

7. Let $a, b \in \mathbb{R}$ and assume that $f : [a, b) \to \mathbb{R}$ is differentiable on (a, b) and continuous on $[a, b)$. Assume that the limit $\lim_{x \to a+} f'(x) = L$ exists. Prove that the right derivative $f_+'(a)$ exists and that $f_+'(a) = L$.

8. Let $a, b \in \mathbb{R}$ and assume that $f : (a, b) \to \mathbb{R}$ is differentiable at c. Show that $\lim_{h \to 0+} \dfrac{f(c + h) - f(c - h)}{2h}$ exists and equals $f'(c)$. Is the converse true?

9. Prove that $\frac{d}{dx}\sqrt{x} = \dfrac{1}{2\sqrt{x}}$.

10. Prove that $\frac{d}{dx}\arcsin x = \dfrac{1}{\sqrt{1-x^2}}$.

5. Mean value theorem

1. Use the mean value theorem to prove the following inequalities:
   (a) $|\sin x - \sin y| \le |x - y|$ for all $x,\ y \in \mathbb{R}$.
   (b) $|\log x - \log y| \le \frac{1}{2}|x - y|$ for all $x,\ y \in [2,\ \infty)$,
   (c) $|(x+1)^{1/5} - x^{1/5}| \le \left(5x^{4/5}\right)^{-1}$ for all $x \in \mathbb{R}_{>0}$.

2. Use the mean value theorem to show that if a function $f : (a,\ b) \to \mathbb{R}$ is differentiable with $f'(x) > 0$ for all x then f is strictly increasing.

3. Use the mean value theorem to show that if a function $f : (a,\ b) \to \mathbb{R}$ is twice differentiable with $f''(x) > 0$ then f is strictly convex. (f is strictly convex if $f(tx + (1-t)y) < tf(x)$ $+(1-t)f(y)$ for all $x,\ y \in (a,\ b)$ and $t,\ y \in (0,\ 1)$.

6. Picard and Newton iteration

1. Let $f : (0,\ \frac{1}{2}\pi) \to \mathbb{R}$ is given by $f(x) = \frac{1}{2}\tan x$. Estimate numerically the solution to $x = f(x)$ with $x \in (0,\ \frac{1}{2}\pi)$ using Picard iteration.

2. Let $f : (0,\ \frac{1}{2}\pi) \to \mathbb{R}$ is given by $f(x) = \frac{1}{2}\tan x$. Estimate numerically the solution to $x = f(x)$ with $x \in (0,\ \frac{1}{2}\pi)$ using Newton iteration (let $F(x) = x - f(x)$).

3. Show that the equation $g(x) = x^3 + x - 1 = 0$ has a solution between 0 and 1. Transform the equation to the form $x = f(x)$ for a suitable function $f : [0,\ 1] \to [0,\ 1]$. Use Picard iteration to find the solution to 3 decimal places. (Try $f(x) = 1/(x^2 + 1)$).

4. Show that the equation $g(x) = x^4 - 4x^2 - x + 4 = 0$ has a solution between $\sqrt{3}$ and 2. Transform the equation to the form $x = f(x)$ for a suitable function $f : [\sqrt{3},\ 2] \to [\sqrt{3},\ 2]$. Use Picard iteration to find the solution to 3 decimal places. (Try $f(x) = \sqrt{2 + \sqrt{x}}$).

7. Topology

1. Define the following and give an example for each:
   (a) metric space,

(b) limit of f as x approaches a,

(c) limit of (x_n) as $n \to \infty$,

(j) continuous at $x = a$,

(c) continuous,

(d) uniformly continuous,

(e) Lipschitz,

(f) ε-ball,

2. Define the following and give an example for each:
   (a) topology,
   (b) topological space,
   (c) open set,
   (d) closed set,
   (e) interior,
   (f) closure,
   (g) interior point,
   (h) close point,
   (i) neighborhood,
   (j) fundamental system of neighborhoods,
   (k) continuous at $x = a$,
   (l) continuous,

3. Define the following and give an example for each:
   (a) topological space,
   (b) Hausdorff,
   (b) fundamental system of neighborhoods,
   (b) basis,
   (c) connected set,
   (d) compact set,

4. Prove that $\mathscr{B}$ and is a basis of $\mathscr{T}$ if and only if $\mathscr{B}$ satisfies: if $x \in X$ then $\mathscr{B}(x) = \{B \in \mathscr{B} \mid x \in B\}$ is a fundamental system of neighborhoods of x.

5. Let X and Y be metric spaces. Define the topology on X and Y. Prove that $f : X \to Y$ is continuous as a function between metric spaces if and only if $f : X \to Y$ is continuous as a function between topological spaces.

6. Define the following and give an example for each:
   (b) filter,
   (c) finer,
   (b) filter base,
   (b) neighborhood filter,
   (d) limit of f as x approaches a,
   (b) Fréchet filter,
   (d) limit of (x_n) as $n \to \infty$.

7. Define the following and give an example for each:
   (c) ultrafilter,
   (d) quasicompact,
   (d) Hausdorff,
   (d) compact,

8. Let X be a Hausdorff topological space and let K be a compact subset of X. Show that K is closed.

9. Let X be a metric space. Show that X is Hausdorff and has a countable basis.

10. Let X be a metric space and let K be a compact subset of X. Show that K is closed and bounded.

11. Let X be a metric space and let E be a subset of X. Show that E is compact if and only if every infinite subset of E has a limit point in E. (What is the definition of limit point???)

12. Let K be a subset of $\mathbb{R}^n$. Show that K is compact if and only if K is closed and bounded.

13. Let X and Y be topological spaces and let $f : X \to Y$ be a continuous function. Show that if X is connected then $f(X)$ is connected.

14. Let $E \subseteq \mathbb{R}$. Show that E is connected if and only if the set E satisfies if $x,\ y \in E$ and $z \in \mathbb{R}$ and $x < z < y$ then $z \in E$.

15. (Intermediate Value Theorem) Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Show that if $z \in \mathbb{R}$ and $f(a) < z < f(b)$ then there exists $c \in (a, b)$ such that $f(c) = z$.

16. Let X and Y be topological spaces and let $f : X \to Y$ be a continuous function. Show that if X is compact then $f(X)$ is compact.

17. Let D be a closed bounded subset of $\mathbb{R}$ and let $f : D \to \mathbb{R}$ be a continuous function.
    (a) f is a bounded function,
    (b) f attains its maximum and minimum on D,
    (a) f is uniformly continuous.