620-295 Real Analysis with applications Lect. 18, 03.09.2009 (1)

A *metric space* is a set X with a function

$d: X \times X \to \mathbb{R}_{\geq 0}$ such that

(a) If $p \in X$ then $d(p,p) = 0$,

(b) If $p, q \in X$ and $p \neq q$ then $d(p,q) \neq 0$,

(c) If $p, q \in X$ then $d(p,q) = d(q,p)$,

(d) If $p, q, r \in X$ then $d(p,r) \leq d(p,q) + d(q,r)$.

Let X be a metric space.

A *sequence* in X is a function
$$\begin{aligned} \mathbb{Z}_{>0} &\to X. \\ n &\mapsto a_n \end{aligned}$$

Let (a_n) be a sequence in X.

The sequence (a_n) *converges* if there exists $\ell \in X$ such that if $\varepsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that

if $n \in \mathbb{Z}_{>0}$ and $n > N$ then $d(a_n, \ell) < \varepsilon$.

Example $X = \mathbb{R}$ with $d(p,q) = |q - p|$. and

$$a_n = \frac{n^2 - 1}{2n^2 + 3} = \frac{\frac{1}{n^2}(n^2-1)}{\frac{1}{n^2}(2n^2+3)} = \frac{1 - \frac{1}{n^2}}{2 + \frac{3}{n^2}}.$$

<u>Claim</u> $\lim_{n\to\infty} a_n = \frac{1}{2}$.

<u>Proof</u> To show: If $\varepsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that

if $n \in \mathbb{Z}_{>0}$ and $n > N$ then $|a_n - \frac{1}{2}| < \varepsilon$.

Assume $\varepsilon \in \mathbb{R}_{>0}$.

To show: There exists $N \in \mathbb{Z}_{>0}$ such that

if $n \in \mathbb{Z}_{>0}$ and $n > N$ then $|a_n - \frac{1}{2}| < \varepsilon$.

Let $N \geq \cancel{\frac{10}{\varepsilon^2}}\ \sqrt{\frac{5}{4\varepsilon}}$

To show: If $n \in \mathbb{Z}_{>0}$ and $n > N$ then $\left|\frac{n^2-1}{2n^2+3} - \frac{1}{2}\right| < \varepsilon$.

Assume $n \in \mathbb{Z}_{>0}$ and $n > N$.

To show: $\left|\frac{n^2-1}{2n^2+3} - \frac{1}{2}\right| < \varepsilon$.

$$\left|\frac{n^2-1}{2n^2+3} - \frac{1}{2}\right| = \left|\frac{2n^2-2-2n^2-3}{2(2n^2+3)}\right|$$

$$= \left|\frac{-5}{4n^2+6}\right| = \left|\frac{5}{4n^2+6}\right|$$

$$< \frac{5}{4N^2+6} < \frac{5}{4N^2}$$

$$< \varepsilon$$

So $\lim_{n\to\infty} \frac{n^2-1}{2n^2+3} = \frac{1}{2}$ //.

A sequence (a_n) is Cauchy if it satisfies

if $\varepsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that

if $m, n \in \mathbb{Z}_{>0}$ and $m > N$ and $n > N$ then $d(a_m, a_n) < \varepsilon$.

Theorem Let (a_n) be a sequence in X.

(a) If (a_n) converges then (a_n) is Cauchy.

(b) If $X = \mathbb{R}$ and (a_n) is Cauchy then (a_n) converges.

Proof (a) Assume (a_n) converges.

Then there exists $L \in X$ such that

if $\varepsilon \in \mathbb{R}_{>0}$ ~~then~~ there exists $N \in \mathbb{Z}_{>0}$ such that

if $n \in \mathbb{Z}_{>0}$ and $n > N$ then $d(a_n, L) < \varepsilon$.

To show: (a_n) is Cauchy.

To show: If $\delta \in \mathbb{R}_{>0}$ then there exists $P \in \mathbb{Z}_{>0}$ such that if $m, p \in \mathbb{Z}_{>0}$ and $m > P$ and $p > P$ then $d(a_m, a_p) < \delta$.

Assume $\delta \in \mathbb{R}_{>0}$.

Let $\varepsilon = \delta/2$. Then there exists $N \in \mathbb{Z}_{>0}$ such that

if $n \in \mathbb{Z}_{>0}$ and $n > N$ then $d(a_n, L) < \varepsilon$.

To show: There exists $P \in \mathbb{Z}_{>0}$ such that

if $m, p \in \mathbb{Z}_{>0}$ and $m > P$ and $n > P$ then $d(a_m, a_p) < \delta$.

Let $P = N$.

To show: If $m, p \in \mathbb{Z}_{>0}$ and $m > P$ and $n > P$

then $d(a_m, a_n) < \delta$

Assume $m, p \in \mathbb{Z}_{>0}$ and $m > P$ and $n > P$.

Then $m > N$ and $n > N$.

So $d(a_m, L) < \varepsilon$ and $d(a_n, L) < \varepsilon$.

To show: $d(a_m, a_n) < \delta$.

$$d(a_m, a_n) \leq d(a_m, L) + d(L, a_n)$$
$$< \varepsilon + \varepsilon = \delta/2 + \delta/2 = \delta.$$

If you use Taylor's theorem you will find:

$$(1+x)^{\frac{1}{2}} = 1 + \tfrac{1}{2}x + \frac{\frac{1}{2}(-\frac{1}{2})}{2!}x^2 + \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{3!}x^3 + \cdots$$

$$= 1 + \tfrac{1}{2}x - \frac{1}{2^2 2!}x^2 + \frac{1 \cdot 3}{2^3 3!}x^3 - \frac{1 \cdot 3 \cdot 5}{2^4 \cdot 4!}x^4 + \cdots$$

So

$$(1+1)^{\frac{1}{2}} = 1 + \tfrac{1}{2} - \frac{1}{2 \cdot 4} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 6} - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10} - \cdots$$

Let $(s_1, s_2, \ldots)$ be given by

$s_1 = 1$,

$s_2 = 1 + \frac{1}{2}$,

$s_3 = 1 + \frac{1}{2} - \frac{1}{2 \cdot 4}$,

$s_4 = 1 + \frac{1}{2} - \frac{1}{2 \cdot 4} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 6}$

$s_5 = 1 + \frac{1}{2} - \frac{1}{2 \cdot 4} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 6} - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8}$

$\vdots$

Then (s_n) is a sequence in $\mathbb{R}$ that converges to $\sqrt{2}$,

$$\lim_{n\to\infty} s_n = \sqrt{2}. \quad \text{in } \mathbb{R}.$$

Of course,

(s_n) is a sequence in $\mathbb{Q}$ but

$$\lim_{n\to\infty} s_n \quad \text{does not exist.}$$

The sequence (s_n) is Cauchy (since ~~it~~ converges in $\mathbb{R}$).

The Cauchy sequence $(s_1, s_2, \ldots)$ in $\mathbb{Q}$ does not converge in $\mathbb{Q}$.

Let X be a metric space.

The <u>completion</u> of X is the smallest metric space $\hat{X}$ such that

(a) $\hat{X} \supseteq X$

(b) every Cauchy sequence in $\hat{X}$ converges in X.

A metric space X is <u>complete</u> if it satisfies if (a_n) is a Cauchy sequence in X then (a_n) converges in X.