Examples of power series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots$$

Question: For which $x \in \mathbb{R}$ does $\sum_{n=0}^{\infty} a_n x^n$ converge?

A series $\sum_{n=0}^{\infty} a_n$ is absolutely convergent if $\sum_{n=0}^{\infty} |a_n|$ converges.

A series $\sum_{n=0}^{\infty} a_n$ is conditionally convergent if $\sum_{n=0}^{\infty} |a_n|$ diverges and $\sum_{n=0}^{\infty} a_n$ converges.

Proposition (a) Let (a_n) be a sequence in $\mathbb{R}$.
If $\sum_{n=0}^{\infty} |a_n|$ converges then $\sum_{n=0}^{\infty} a_n$ converges.

(b) If $\sum_{n=0}^{\infty} a_n$ converges then $\lim_{n\to\infty} |a_n| = 0$.

Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series.

Let
$$A = \{ r \in \mathbb{R}_{\geq 0} \mid \sum_{n=0}^{\infty} |a_n r^n| \text{ converges} \}.$$
$$C = \{ r \in \mathbb{R}_{\geq 0} \mid \sum_{n=0}^{\infty} a_n r^n \text{ converges} \}.$$

<u>Claim:</u> If $s \in C$ then $[0, s) \subseteq C$.

<u>Proof</u> Assume $s \in C$.

Then $\sum_{n=0}^{\infty} a_n s^n$ converges.

So $\lim_{n \to \infty} |a_n s^n| = 0$.

Let $\varepsilon \in \mathbb{R}_{>0}$. Then there exists $N \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{>0}$ and $n > N$ then $|a_n s^N| < \varepsilon$.

~~Then~~ Let $r \in [0, s)$. Then

$$\sum_{n=0}^{\infty} |a_n r^n| = |a_0| + |a_1 r| + \cdots + |a_N r^N| + |a_{N+1} r^{N+1}| + \cdots$$
$$= |a_0| + \cdots + |a_N r^N| + |a_{N+1} s^{N+1}| \left| \frac{r^{N+1}}{s^{N+1}} \right| + |a_{N+2} s^{N+2}| \frac{r^{N+2}}{s^{N+2}} + \cdots$$
$$< |a_0| + \cdots + |a_N r^N| + \varepsilon \frac{r^{N+1}}{s^{N+1}} + \varepsilon \frac{r^{N+2}}{s^{N+2}} + \varepsilon \frac{r^{N+3}}{s^{N+3}} + \cdots$$
$$= |a_0| + \cdots + |a_N r^N| + \varepsilon \frac{r^{N+1}}{s^{N+1}} \left(1 + \frac{r}{s} + \left(\frac{r}{s}\right)^2 + \cdots \right)$$
$$= |a_0| + \cdots + |a_N r^N| + \varepsilon \frac{r^{N+1}}{s^{N+1}} \left(\frac{1}{1 - \frac{r}{s}} \right).$$

So $\sum_{n=0}^{\infty} |a_n r^n|$ converges.

So $[0, s) \subseteq A$.

By Proposition (a), $A \subseteq C$.

So $[0, s) \subseteq C$ //.

~~Let~~ The <u>radius of convergence</u> of $\sum_{n=0}^{\infty} a_n x^n$

is
$$\sup \{ r \in \mathbb{R}_{\geq 0} \mid \sum_{n=0}^{\infty} a_n r^n \text{ converges} \}.$$

If $R = \sup C$ then $C = [0, R)$ or $C = [0, R]$.

<u>Example</u> $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

Then $1 \in C$, since $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

converges.

So $[0, 1) \subseteq A \subseteq C$.

Let $s \in \mathbb{R}$, slightly larger than 1. Then

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} s^n \right| = |s| + \frac{1}{2}|s^2| + \frac{1}{3} s^3 + \cdots$$
$$> 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

so that $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} s^n \right|$ diverges.

So $A \subseteq [0, 1]$ and $C \subseteq [0, 1]$.

Here $[0, 1) = A \subseteq C = [0, 1]$.

Thinking about rearrangements

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\cdots = \log(2)$$

$$-\frac{1}{2}-\frac{1}{4}-\frac{1}{6}-\frac{1}{8}-\cdots = -\frac{1}{2}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots\right)$$

$$= -\frac{1}{2}(\text{VERY VERY LARGE})$$

$$1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\cdots > \frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\cdots = \text{VERY VERY LARGE}.$$

Pick a number $\ell \in \mathbb{R}_{>0}$.
If we take

$1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\cdots+\frac{1}{299}$ just until we get larger than ℓ

then add

$-\frac{1}{2}-\frac{1}{4}-\frac{1}{6}-\cdots-\frac{1}{70}$ just until we get smaller than ℓ

then add

$\frac{1}{301}+\frac{1}{303}+\cdots$ just until we get larger than ℓ

then add

$-\frac{1}{72}-\frac{1}{74}-\cdots$ just until we get smaller than ℓ

$\vdots$

This will create a series that converges to ℓ.