620-295 Real Analysis with Applications

Assignment 2: Due 5pm on 21 August

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Due 5pm on 21 August in the appropriate assignment box on the ground floor of Richard Berry.

1. Let $f : S \to T$ be a function. Show that the inverse function to f exists if and only if f is bijective.

2. Add up the positive integers from 1 to 100. Then add up the squares 1^2 to 100^2.

3. Let S be a set with an associative operation with identity. Show that the identity is unique. (This tells us that any commutative monoid has only one heart.)

4. Let S be a set with an associative operation with identity. Let $s \in S$ and assume that s has an inverse in S. Show that the inverse of s is unique. (This tells us that any element of an abelian group has only one mate.)

5. Let S be a ring. Show that if $s \in S$ then $s \cdot 0 = 0$.

6. Prove that $\displaystyle\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$.

7. Define the following and give an example for each:
   (a) order,
   (b) maximum,
   (c) minimum,
   (d) upper bound,
   (e) lower bound,
   (f) bounded above,
   (g) bounded below,
   (j) supremum,
   (k) infimum,

8. Prove that if $n \in \mathbb{Z}_{>0}$ then $x - y$ is a factor of $x^n - y^n$.

9. For each of the following subsets of $\mathbb{R}$ find the maximum, the minimum, an upper bound, a lower bound, the supremum, and the infimum:
   (a) $\{2^{-m} - 3^n \mid m, n \in \mathbb{Z}_{\geq 0}\}$,
   (b) $\{x \in \mathbb{R} \mid x^3 - 4x < 0\}$,
   (c) $\{1 + x^2 \mid x \in \mathbb{R}\}$,

10. What is the triangle inequality and how do you justify it?

(1) Let $f: S \to T$ be a function. An inverse function to f exists if and only if f is bijective.

Proof

To show: (a) If an inverse function to f exists then f is bijective

(b) If f is bijective then an inverse function to f exists.

(a) Assume $f: S \to T$ has an inverse function $f^{-1}: T \to S$.

To show: (aa) f is injective

(ab) f is surjective.

(aa) Assume $f(s_1) = f(s_2)$.

To show: $s_1 = s_2$

$$s_1 = f^{-1}(f(s_1)) = f^{-1}(f(s_2)) = s_2.$$

So f is injective.

(ab) Let $t \in T$

To show: There exists $s \in S$ such that $f(s) = t$.

Let $s = f^{-1}(t)$

Then

$$f(s) = f(f^{-1}(t)) = t.$$

So f is surjective

So f is bijective.

(b) Assume $f: S \to T$ is bijective.

To show: f has an inverse function

We need to define a function $\varphi: T \to S$.

Let $t \in T$.

Since f is surjective there exists $s \in S$ such that $f(s) = t$.

Define $\varphi(t) = s$.

To show: (ba) φ is well defined

(bb) φ is an inverse function to f.

(ba) To show: (baa) If $t \in T$ then $\varphi(t) \in S$.

(bab) If $t_1, t_2 \in T$ and $t_1 = t_2$ then $\varphi(t_1) = \varphi(t_2)$

(baa) From the definition of φ, $\varphi(t) \in S$.

(bab) To show: If $t_1, t_2 \in T$ and $t_1 = t_2$ then $\varphi(t_1) = \varphi(t_2)$

Assume $t_1, t_2 \in T$ and $t_1 = t_2$.

Let $s_1, s_2 \in S$ such that $f(s_1) = t_1$ and $f(s_2) = t_2$.

Since $t_1 = t_2$, $f(s_1) = f(s_2)$.

Since f is injective this implies that $s_1 = s_2$.

So $\varphi(t_1) = s_1 = s_2 = \varphi(t_2)$.

So φ is well defined.

(bb) To show: (bba) If $s \in S$ then $\varphi(f(s)) = s$.

(bbb) If $t \in T$ then $f(\varphi(t)) = t$.

(bba) This is part of the definition of φ.

(bbb) Assume $t \in T$.

Let $s \in S$ be such that $f(s) = t$.

Then

$$f(\varphi(t)) = f(s) = t.$$

So $\varphi \circ f$ and $f \circ \varphi$ are the identity functions on S and T respectively.

So φ is an inverse function to f. //.

(2) (a) Compute $1+2+3+4+\cdots+99+100$.

$$\begin{array}{l} 1+2+3+4+\cdots+98+99+100 \\ +100+99+98+97+\cdots+3+2+1 \end{array}$$

$$= 101+101+101+\cdots+101+101+101 = 10100.$$

So $2(1+2+3+4+\cdots+99+100) = 10100.$

So $1+2+3+\cdots+99+100 = \frac{1}{2}(10100) = 5050$.

(b) Compute $1^2+2^2+3^2+4^2+\cdots+99^2+100^2$.

By question 6,

$$\sum_{k=1}^{n} k^2 = \frac{1}{6} n(n+1)(2n+1)$$

So

$$1^2+2^2+3^2+4^2+\cdots+99^2+100^2$$

$$= \sum_{k=1}^{100} k^2 = \frac{1}{6}\, 100\cdot(100+1)(2\cdot 100+1)$$

$$= \frac{1}{3\cdot 2}\cdot 100\cdot 101\cdot 201 = 50\cdot 101\cdot 67$$

$$= 5050\cdot 67 = \frac{10100}{2}\cdot 67 = \frac{676700}{2}$$

$$= 338350.$$

(3) Let S be a set with an operation

$$\begin{aligned} S \times S &\to S \\ (s,t) &\mapsto s+t \end{aligned}$$ which is associative and

has an identity.

To show: The identity is unique.

Assume Δ and ∇ are both identities for the operation.

To show: $\Delta = \nabla$.

Since Δ is an identity $\Delta + \nabla = \nabla$

Since ∇ is an identity $\Delta + \nabla = \Delta$.

So $\nabla = \Delta + \nabla = \Delta$ //.

(4) Let S be a set with an operation

$$\begin{aligned} S \times S &\to S \\ (s,t) &\mapsto s+t \end{aligned}$$ such that

if $s_1, s_2, s_3 \in S$ then $(s_1+s_2)+s_3 = s_1+(s_2+s_3)$.

Assume that there exists $0 \in S$ such that if $s \in S$ then $0+s=s$ and $s+0=s$.

Let $s \in S$ and assume s has an inverse in S.

To show: The inverse of s is unique.

Assume t_1 and t_2 are both inverses of s.

To show: $t_1 = t_2$.

Since t_1 is an inverse of s, $t_1+s=0$.

Since t_2 is an inverse of s, $s+t_2=0$

So

$$\begin{aligned} t_1 = t_1+0 &= t_1+(s+t_2) \\ &= (t_1+s)+t_2 \quad \text{(by associativity)} \\ &= 0+t_2 \\ &= t_2. \end{aligned}$$

So $t_1 = t_2$.

So the inverse of s is unique. //

(5) A ring is a set S with operations $\begin{array}{l} S\times S \to S \\ (s,t)\mapsto s+t \end{array}$

and $\begin{array}{l} S\times S \to S \\ (s,t)\mapsto st \end{array}$ such that

(a) If $s_1, s_2, s_3 \in S$ then $(s_1+s_2)+s_3 = s_1+(s_2+s_3)$,

(b) If $s_1 s_2 \in S$ then $s_1+s_2 = s_2+s_1$,

(c) There exists $0\in S$ such that

if $s\in S$ then $0+s=s$ and $s+0=s$,

(d) If $s\in S$ then there exists $-s\in S$ such that

$s+(-s)=0$ and $(-s)+s=0$,

(e) If $s_1, s_2, s_3 \in S$ then $(s_1 s_2) s_3 = s_1 (s_2 s_3)$,

(f) There exists $1\in S$ such that

if $s\in S$ then $1\cdot s = s$ and $s\cdot 1 = s$.

(g) If $s_1, s_2, s_3 \in S$ then $s_1(s_2+s_3) = s_1 s_2 + s_1 s_3$

and $(s_1+s_2)s_3 = s_1 s_3 + s_2 s_3$.

To show: If $s\in S$ then $s\cdot 0 = 0$.

Assume $s\in S$.

To show: $s\cdot 0 = 0$.

$s\cdot 0 = s(0+0)$ (since $0+0=0$ by (c))

$= s\cdot 0 + s\cdot 0$ by (g).

By (d) there exists $-(s\cdot 0)\in S$.

Since $s \cdot 0 = s \cdot 0 + s \cdot 0$,

$$s \cdot 0 + -(s \cdot 0) = s \cdot 0 + s \cdot 0 + -(s \cdot 0)$$

So, by (d),

$$0 = s \cdot 0 + 0.$$
$$= s \cdot 0 \quad (\text{by (c)}).$$

So $0 = s \cdot 0$. //.

(6)

Show that $\sum_{k=1}^{n} k^2 = \frac{1}{2} n(n+1)(2n+1)$.

<u>Proof</u>

<u>Base case</u>: $n=1$. To show $1^2 = \frac{1}{6} 1\cdot(1+1)(2\cdot 1+1)$.

$RHS = \frac{1}{6} 1\cdot(1+1)(2\cdot 1+1) = \frac{2\cdot 3}{6} = \frac{3}{3} = 1$.

<u>Base case</u>; $n=2$. $1^2+2^2 = \frac{1}{6} 2(2+1)(2\cdot 2+1)$.

$LHS = 1^2+2^2 = 1+4 = 5$.

$RHS = \frac{1}{6}\cdot 2\cdot 3\cdot 5 = \frac{6\cdot 5}{6} = 5$.

<u>Induction step</u>: Assume that $\sum_{k=1}^{r} k^2 = \frac{1}{6} r(r+1)(2r+1)$

for $r \in \mathbb{Z}_{>0}$ with $r<n$.

To show: $1^2+2^2+\cdots+(n-1)^2+n^2 = \frac{1}{6} n(n+1)(2n+1)$.

$LHS = 1^2+2^2+\cdots+(n-1)^2+n^2$

$= \frac{1}{6}(n-1)(n-1+1)(2(n-1)+1)+n^2$, by the induction hypothesis,

$= \frac{1}{6}(n-1)n(2n-1)+n^2$

$= \frac{1}{6} n\left((n-1)(2n-1)+6n\right) = \frac{1}{6}\cdot n(2n^2-n-2n+1+6n)$

$= \frac{1}{6} n(2n^2+3n+1) = \frac{1}{6} n(n+1)(2n+1) = RHS$. //

(7) (a) The word "order" could mean either partial order or total order.

Let S be a set.

A <u>partial order</u> on S is a relation $@$ on S such that

(a) If $x,y,z \in S$ and $x@y$ and $y@z$ then $x@z$

(b) If $x,y \in S$ and $x@y$ and $y@x$ then $y=x$.

A <u>total order</u> on S is a relation $@$ on S such that

(a) If $x,y,z \in S$ and $x@y$ and $y@z$ then $x@z$,

(b) If $x,y \in S$ and $x@y$ and $y@x$ then $y=x$,

(c) If $x,y \in S$ then $x@y$ or $y@x$.

Let K be a set and let

$S = \{$subsets of $K\}$ with the partial order given by

$A \subseteq B$ if A is a subset of B.

If $x,y \in K$ then $\{x\} \not\subseteq \{y\}$ and $\{y\} \not\subseteq \{x\}$

so this is a partial order which is not a total order.

The partial order $\le$ on $\mathbb{R}$ given by

$x \le y$ if $y-x \in \mathbb{R}_{\ge 0}$ is a total order.

An *infimum* of E in S is an element $\inf E \in S$ such that

(a) $\inf E$ is a lower bound of E in S,

(b) If l is a lower bound of E in S then $\inf E \geqslant l$.

Examples of these are provided by question 9.

(8) Prove that if $n \in \mathbb{Z}_{>0}$ then $x-y$ is a factor of $x^n - y^n$.

<u>Proof</u> Assume $n \in \mathbb{Z}_{>0}$.

To show: $x-y$ is a factor of $x^n - y^n$.

Since

$$(x-y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \cdots + x^2y^{n-3} + xy^{n-2} + y^{n-1})$$

$$\begin{aligned} &= x^n + x^{n-1}y + x^{n-2}y^2 + \cdots + x^3y^{n-3} + x^2y^{n-2} + xy^{n-1} \\ &\quad - x^{n-1}y - x^{n-2}y^2 - \quad - x^3y^{n-3} - x^2y^{n-2} - xy^{n-1} - y^n \\ &= x^n - y^n, \end{aligned}$$

$x-y$ is a factor of $x^n - y^n$. //

(9) (a) $E = \{2^{-m} - 3^n \mid m, n \in \mathbb{Z}_{\geq 0}\}$

$$= \left\{ \begin{array}{l} 1-1, \ \frac{1}{2}-1, \ \frac{1}{2^2}-1, \ \frac{1}{2^3}-1, \ \ldots \\ 1-3, \ \frac{1}{2}-3, \ \frac{1}{2^2}-3, \ \frac{1}{2^3}-3, \ \ldots \\ 1-3^2, \ \frac{1}{2}-3^2, \ \frac{1}{2^2}-3^2, \ \frac{1}{2^3}-3^2, \ \ldots \\ 1-3^3, \ \frac{1}{2}-3^3, \ \frac{1}{2^2}-3^3, \ \frac{1}{2^3}-3^3, \ \ldots \\ \vdots \quad \vdots \quad \vdots \quad \vdots \end{array} \right\}$$

The maximum of E is $1-1=0$.

E has no minimum.

The elements of the set $[0, \infty)$ are upper bounds of E in $\mathbb{R}$.

E has no lower bounds in $\mathbb{R}$.

$\inf E$ does not exist since E has no lower bounds in $\mathbb{R}$.

$\sup E = 0$, since 0 is the least upper bound of E in $\mathbb{R}$.

(9)(a) $E = \{1 + x^2 \mid x \in \mathbb{R}\} = \{1 + y \mid y \in \mathbb{R}_{\geq 0}\}$

$= [1, \infty) = \{z \in \mathbb{R} \mid z \geq 1\}$.

The minimum of E is 1.

E has no maximum.

All elements in the set $(-\infty, 1]$ are lower bounds of E in $\mathbb{R}$.

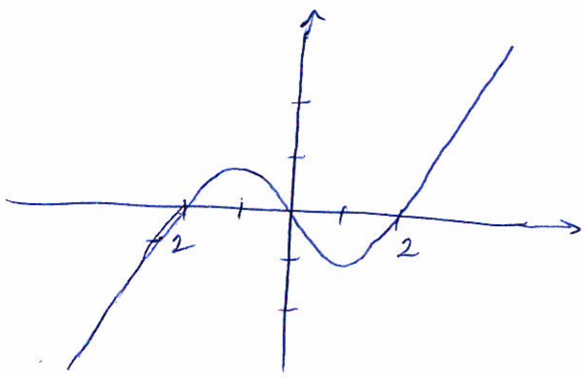
E has no upper bounds in $\mathbb{R}$.

$\sup E$ does not exist.

$\inf E = 1$ since 1 is the greatest lower bound of E in $\mathbb{R}$.

(9)(b) Let $E = \{x \in \mathbb{R} \mid x^3 - 4x < 0\}$

The graph of $y = x^3 - 4x$ is



So

$E = (-\infty, 2) \cup (0, 2)$

and

E has no minimum,

The maximum of E does not exist,

E has no lower bounds in $\mathbb{R}$

All elements of the set $[2, \infty)$ are upper bounds of E in $\mathbb{R}$

$\sup E = 2$, since 2 is the least upper bound of E in $\mathbb{R}$.

$\inf E$ does not exist.

(10) Consider the triangle inequality for $\mathbb{R}^2$:

$$|x+y| \le |x|+|y|$$

where if $x=(x_1,x_2)\in\mathbb{R}^2$ then $|x|=\sqrt{x_1^2+x_2^2}$

Lagrange's identity says

$$\tfrac{1}{2}\left((x_1y_1-x_1y_1)^2+(x_1y_2-x_2y_1)^2+(x_2y_1-x_1y_2)^2+(x_2y_2-x_2y_2)^2\right)$$

$$=\tfrac{1}{2}\begin{pmatrix} x_1^2y_1^2-2x_1y\,x_1y_1+x_1^2y_1^2+x_1^2y_2^2-2x_1y_2x_2y_1+x_2^2y_1^2 \\ +x_2^2y_1^2-2x_2y_1x_1y_2+x_1^2y_2^2+x_2^2y_2^2-2x_2y_2x_2y_2+x_2^2y_2^2 \end{pmatrix}$$

$$=\tfrac{1}{2}\left(x_1^2y_1^2+x_1^2y_2^2+x_2^2y_1^2+x_2^2y_2^2\right)$$

$$+\tfrac{1}{2}\left(x_1^2y_1^2+x_1^2y_2^2+x_2^2y_1^2+x_2^2y_2^2\right)$$

$$-\left(x_1y_1x_1y_1+x_1y_2x_2y_1+x_2y_1x_1y_2+x_2y_2x_2y_2\right)$$

$$=x_1^2y_1^2+x_1^2y_2^2+x_2^2y_1^2+x_2^2y_2^2$$

$$-\left(x_1y_1x_1y_1+x_1y_1x_2y_2+x_1y_1x_2y_2+x_2y_2x_2y_2\right)$$

$$=(x_1^2+x_2^2)(y_1^2+y_2^2)-(x_1y_1+x_2y_2)^2$$

So

$$(x_1^2+x_2^2)(y_1^2+y_2^2)-(x_1y_1+x_2y_2)^2\ge 0$$

So $(x_1^2+x_2^2)(y_1^2+y_2^2) \geq (x_1y_1+x_2y_2)^2$.

So $\sqrt{x_1^2+x_2^2}\sqrt{y_1^2+y_2^2} \geq x_1y_1+x_2y_2$.

So $|x||y| \geq x_1y_1+x_2y_2$.

Then

$$
\begin{aligned}
|x+y|^2 &= |(x_1,x_2)+(y_1,y_2)|^2 = |(x_1+y_1, x_2+y_2)|^2 \\
&= (x_1+y_1)^2+(x_2+y_2)^2 \\
&= x_1^2+2x_1y_1+y_1^2+x_2^2+2x_2y_2+y_2^2 \\
&= x_1^2+x_2^2+2(x_1y_1+x_2y_2)+y_1^2+y_2^2 \\
&\leq x_1^2+x_2^2+2|x||y|+y_1^2+y_2^2 \\
&= |x|^2+2|x||y|+|y|^2 \\
&= (|x|+|y|)^2.
\end{aligned}
$$

So $|x+y| \leq |x|+|y|$.

In this proof we have used (more than once)

If $a,b \in \mathbb{R}_{\geq 0}$ then

$a \geq b$ if and only if $a^2 \geq b^2$.

This is proved in the Lecture notes for Lecture 12, on 20.08.2009