620-295 Real Analysis with applications, Lect. 16, 31. Aug. 2009.

The ratio test and the root test

Proposition Let (a_n) be a sequence in $\mathbb{R}_{\geq 0}$.

(a) Assume $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = a$ exists and $a<1$.

Then $\sum_{n=1}^{\infty} a_n$ converges.

(b) Assume $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = a$ exists and $a>1$

Then $\sum_{n=1}^{\infty} a_n$ diverges.

(c) Assume $\lim_{n\to\infty} a_n^{1/n} = b$ exists and $b<1$.

Then $\sum_{n=1}^{\infty} a_n$ converges

(d) Assume $\lim_{n\to\infty} a_n^{1/n} = b$ exists and $b>1$.

Then $\sum_{n=1}^{\infty} a_n$ diverges.

Proof (a) Assume $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = a$ exists and $a<1$.

Let $\varepsilon \in \mathbb{R}_{>0}$ so that $a+\varepsilon<1$.

Since $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = a$ there exists $N \in \mathbb{Z}_{>0}$

with $\frac{a_{n+1}}{a_n} < a+\varepsilon$ if $n \in \mathbb{Z}_{>0}$ with $n>N$.

Then

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_N + a_{N+1} + a_{N+2} + a_{N+3} + \cdots$$

$$= a_1 + \cdots + a_N + a_{N+1} + \frac{a_{N+2}}{a_{N+1}} a_{N+1} + \frac{a_{N+3}}{a_{N+2}} \frac{a_{N+2}}{a_{N+1}} a_{N+1} + \cdots$$

$$= a_1 + \cdots + a_N + a_{N+1}\left(1 + \frac{a_{N+2}}{a_{N+1}} + \frac{a_{N+3}}{a_{N+2}} \frac{a_{N+2}}{a_{N+1}} + \cdots\right)$$

$$< a_1 + \cdots + a_N + a_{N+1}\left(1 + (a+\varepsilon) + (a+\varepsilon)^2 + (a+\varepsilon)^3 + \cdots\right)$$

$$= a_1 + \cdots + a_N + a_{N+1}\left(\frac{1}{1-(a+\varepsilon)}\right)$$

So $\sum_{n=1}^{\infty} a_n$ converges.

(b) Assume $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ exists and $a > 1$.

Let $\varepsilon \in \mathbb{R}_{>0}$ such that $a - \varepsilon > 1$.

Let $N \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{>0}$ and $n > N$ then

$$\frac{a_{n+1}}{a_n} > a - \varepsilon.$$

Then

$$\sum_{n=1}^{\infty} a_n = a_1 + \cdots + a_N + a_{N+1} + a_{N+2} + a_{N+3} + \cdots$$

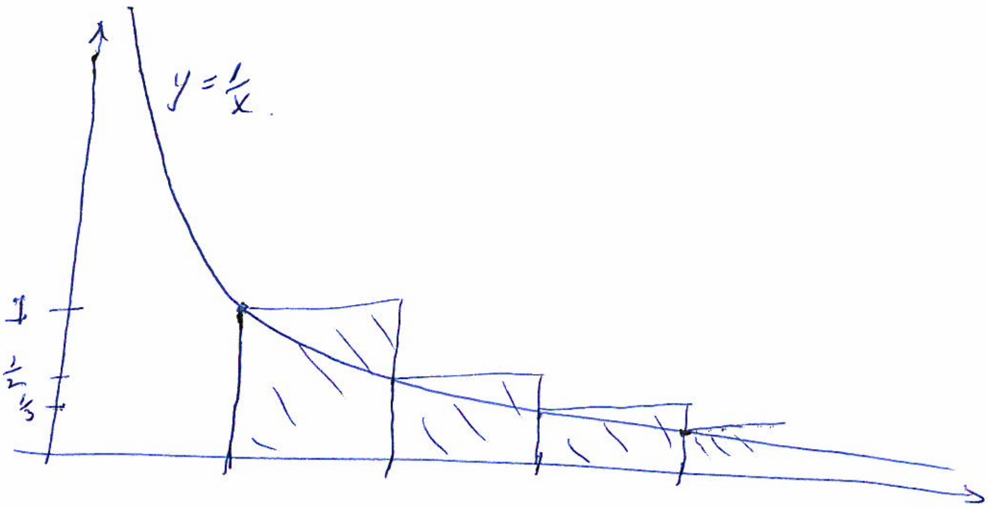
$$= a_1 + \cdots + a_N + a_{N+1}\left(1 + \frac{a_{N+2}}{a_{N+1}} + \frac{a_{N+3}}{a_{N+2}} \frac{a_{N+2}}{a_{N+1}} + \cdots\right)$$

$$> a_1 + \cdots + a_N + a_{N+1}\left(1 + (a-\varepsilon) + (a-\varepsilon)^2 + \cdots\right)$$

If $F(x) = \log(x)$ then $\frac{dF}{dx} = \frac{1}{x}$ and

$$\int_a^b \frac{1}{x}\, dx = \log(b) - \log(a).$$

$$= (\text{area under } y = \tfrac{1}{x} \text{ from } x = a \text{ to } x = b)$$



So $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$

= area of the shaded boxes

> area under $y = \frac{1}{x}$ from $x = 1$ to $x = 1000000000000$

$= \log(1000000000000000) - \log(1)$

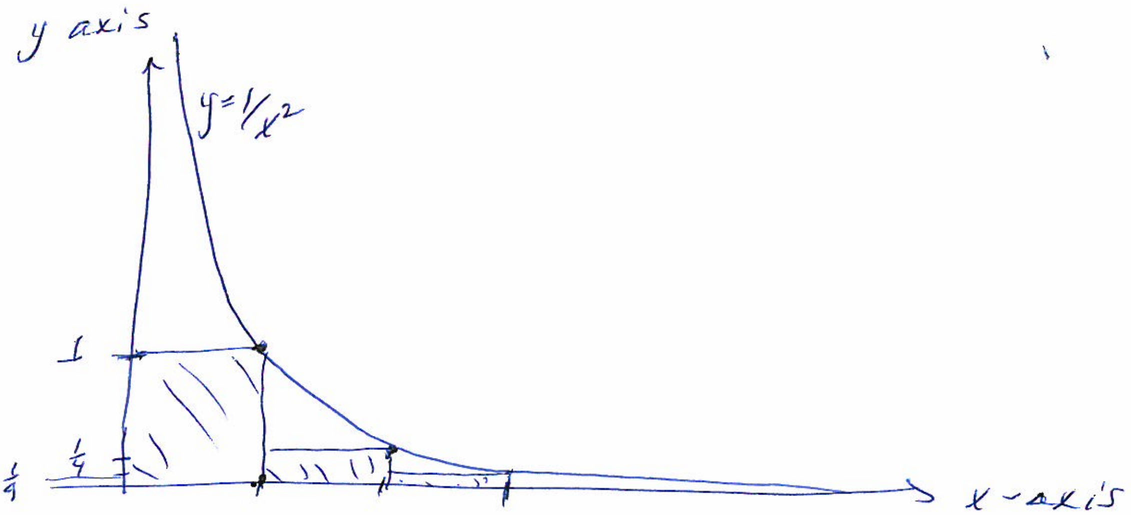
= VERY LARGE.

So $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Example $(a_n) = \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

If $F(x) = \frac{-1}{x}$ then $\frac{dF}{dx} = \frac{1}{x^2}$. and

$\int_a^b \frac{1}{x^2}\,dx = \frac{-1}{b} - \left(\frac{-1}{a}\right) =$ area under $y = \frac{1}{x^2}$ between $x = a$ and $x = b$



Then
$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

= area of shaded boxes

$<$ 1 + (area under $y = 1/x^2$ from $x = 1$ to $x = 1000000000000$)

$= 1 + \left(\frac{-1}{1000000000000} - \frac{-1}{1} \right)$

$= 1 + 1 - \frac{1}{mmmmm}$ is very close to 2.

So $\sum_{n=1}^{\infty} \frac{1}{n^2} < 2$.