620-295 Real Analysis with applications

Assignment 1: Due 7 August 2009

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1. Define the following sets and give examples of elements of each:
   (a) the set of rational numbers,
   (b) the set of real numbers,
   (c) the set of complex numbers.

2. Let $\frac{a}{b}, \frac{c}{d}, \frac{e}{f} \in \mathbb{Q}$. Show that $\frac{a}{b} + \left(\frac{c}{d} + \frac{e}{f}\right) = \left(\frac{a}{b} + \frac{c}{d}\right) + \frac{e}{f}$.

3. State and prove the Pythagorean Theorem.

4. Compute and graph the following:
   (a) $\dfrac{-15+i}{4+2i}$,
   (b) $\left(27^{1/3}\right)^4$,
   (c) $27^{(4+1/3)}$.

5. Let $z = x + iy$ with $x, y \in \mathbb{R}$. Compute and graph $\left|\dfrac{(3+4i)(-1+2i)}{(-1-i)(3-i)}\right|$.

6. Define the following and give examples:
   (a) injective,
   (b) surjective,
   (c) composition of functions,
   (d) abelian group.

7. Let $D : \mathbb{Q}[x] \longrightarrow \mathbb{Q}[x]$ be a function such that
   (a) If $f, g \in \mathbb{Q}[x]$ then $D(f+g) = D(f) + D(g)$
   (b) If $c \in \mathbb{Q}$ and $f \in \mathbb{Q}[x]$ then $D(cf) = cD(f)$,
   (c) If $f, g \in \mathbb{Q}[x]$ then $D(fg) = fD(g) + D(f)g$, and
   (d) $D(x) = 1$.

   Compute $D(x^n)$, for $n \in \mathbb{Z}_{\geq 0}$.

8. Write $\dfrac{1-x^n}{1-x}$ as an element of $\mathbb{Q}[x]$.

(1)(a) The rational numbers is the set

$$\mathbb{Q} = \left\{ \frac{a}{b} \;\middle|\; a, b \in \mathbb{Z},\ b \neq 0 \right\}$$

with $\frac{a}{b} = \frac{c}{d}$ if $ad = bc$.

The expression $\frac{9}{-27}$ is an element of $\mathbb{Q}$

The expression $-\frac{1}{3} = \frac{9}{-27}$ as elements of $\mathbb{Q}$.

(b) The real numbers is the set

$$\left\{ a_0 . a_1 a_2 a_3 \ldots,\ -a_0 . a_1 a_2 a_3 \ldots \;\middle|\; \begin{array}{l} a_0 \in \mathbb{Z}_{\geq 0}, \\ a_1, a_2, \ldots \in \{0, 1, \ldots, 9\} \end{array} \right\}$$

with

$$a_0 . a_1 a_2 a_3 \ldots = b_0 . b_1 b_2 b_3 \ldots \quad \text{if}$$

$$a_0 . a_1 a_2 a_3 \ldots - b_0 . b_1 b_2 b_3 \ldots = 0.$$

The expression $-0.999\ldots$ is an element of $\mathbb{R}$

The expression $-1.000\ldots = -0.999\ldots$ as elements of $\mathbb{R}$.

(c) The complex numbers is the set

$$\mathbb{C} = \{ a + bi \mid a, b \in \mathbb{R} \}.$$

The expression $3 + 2i$ is an element of $\mathbb{C}$

The expression $0 + \pi i$ is an element of $\mathbb{C}$.

The sets $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ are quite boring without the operations of addition and multiplication.

The addition and multiplication in $\mathbb{Q}$ are stolen from $\mathbb{Z}$ by defining

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd} \quad \text{and} \quad \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}.$$

The addition and multiplication in $\mathbb{R}$ are stolen from $\mathbb{Q}$ by saying

$$a_0.a_1a_2a_3\ldots = a_0 + \frac{a_1}{10} + \frac{a_2}{10^2} + \frac{a_3}{10^3} + \cdots$$

and

$$-a_0.a_1a_2a_3\ldots = -a_0 - \frac{a_1}{10} - \frac{a_2}{10^2} - \frac{a_3}{10^3} - \cdots$$

The addition and multiplication in $\mathbb{C}$ are stolen from $\mathbb{R}$ by saying $i^2 = -1$ so that

$(a+bi) + (c+di) = (a+c) + (b+d)i$, and

$(a+bi)(c+di) = (ac - bd) + i(ad+bc)$.

With these operations, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ are <u>fields</u>.

(2) Let $\frac{a}{b}, \frac{c}{d}, \frac{e}{f} \in \mathbb{Q}$ so that $a, b, c, d, e, f \in \mathbb{Z}$ and b, d, f are not equal to 0.

Show that $\frac{a}{b} + \left(\frac{c}{d} + \frac{e}{f}\right) = \left(\frac{a}{b} + \frac{c}{d}\right) + \frac{e}{f}$.

$$LHS = \frac{a}{b} + \left(\frac{c}{d} + \frac{e}{f}\right) = \frac{a}{b} + \left(\frac{cf + de}{df}\right)$$

$$= \frac{a(df) + b(cf + de)}{b(df)}$$

$$= \frac{a(df) + b(cf) + b(de)}{b(df)}$$, by the distributive ~~law~~ property in $\mathbb{Z}$

$$= \frac{(ad)f + (bc)f + (bd)e}{(bd)f}$$, by associativity for multiplication in $\mathbb{Z}$.

$$RHS = \left(\frac{a}{b} + \frac{c}{d}\right) + \frac{e}{f}$$

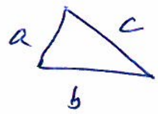
$$= \frac{ad + bc}{bd} + \frac{e}{f}$$

$$= \frac{(ad + bc)f + (bd)e}{(bd)f}$$

$$= \frac{(ad)f + (bc)f + (bd)e}{(bd)f}$$, by the distributive property in $\mathbb{Z}$

So $LHS = RHS$ and $\frac{a}{b} + \left(\frac{c}{d} + \frac{e}{f}\right) = \left(\frac{a}{b} + \frac{c}{d}\right) + \frac{e}{f}$.

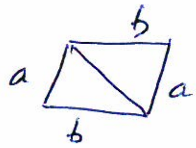
(3) <u>The Pythagorean Theorem</u>

Let 

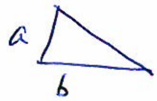
be a right triangle with leg lengths a and b and hypotenuse length c.

Then

$$a^2+b^2=c^2$$

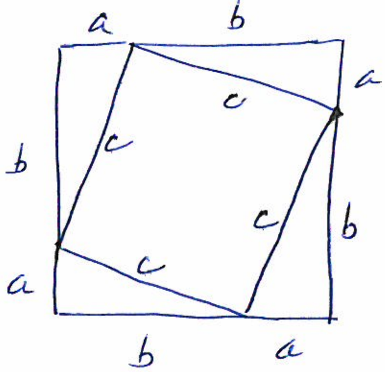
<u>Proof</u> Since (area of 

) $= ab$,

then (area of 

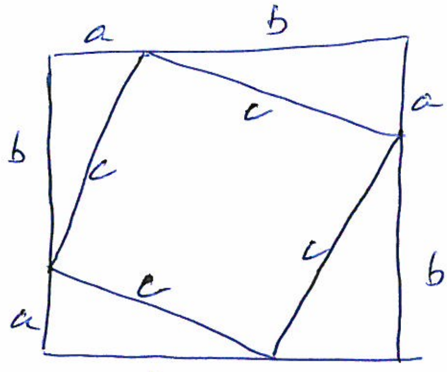
) $= \frac{1}{2}ab$.

Then

area of 

$= (a+b)^2$

and

area of 

$= 4(\frac{1}{2}ab)+c^2$.

So $(a+b)^2 = 4(\frac{1}{2}ab)+c^2$

So $a^2+b^2+2ab = 2ab+c^2$

So $a^2+b^2 = c^2$. //

(4) Compute and graph

(a) $\dfrac{-15+i}{4+2i}$ (b) $(27^{\frac{1}{3}})^4$ (c) $27^{4+\frac{1}{3}}$

(a) $$\frac{-15+i}{4+2i} = \frac{(-15+i)}{(4+2i)}\frac{(4-2i)}{(4-2i)} = \frac{-60+30i+4i-2i^2}{16-8i+8i-4i^2}.$$

$$= \frac{-60+2+34i}{16+4} = \frac{-62+34i}{20} = \frac{-31}{10}+\frac{17}{10}i.$$

(b) $$(27^{\frac{1}{3}})^4 = \begin{cases} ((3^3(e^{2i\pi/3})^3)^{\frac{1}{3}})^4 \\ ((3^3(e^{4i\pi/3})^3)^{\frac{1}{3}})^4 \\ ((3^3(e^{0i\pi/3})^3)^{\frac{1}{3}})^4 \end{cases} = \begin{cases} (3e^{2\pi i/3})^4 \\ (3e^{4\pi i/3})^4 \\ (3e^0)^4 \end{cases}$$

$$= \begin{cases} 81e^{8\pi i/3} \\ 81e^{16\pi i/3} \\ 81 \end{cases} = \begin{cases} 81e^{2\pi i/3} \\ 81e^{\pi i/3} \\ 81 \end{cases} = \begin{cases} 81\cos(2\pi/3) + i\,81\sin(2\pi/3) \\ 81\cos(\frac{\pi}{3}) + i\,81\sin(\pi/3) \\ 81 \end{cases}$$

$$= \begin{cases} 81(-\frac{1}{2} - \frac{\sqrt{3}}{2}i) \\ 81(\frac{-1}{2}+\frac{\sqrt{3}}{2}i) \\ 81 \end{cases}$$

since $1 = \begin{cases} e^0 \\ e^{2\pi i} \\ e^{4\pi i} \end{cases} = \begin{cases} (e^{0/3})^3 \\ (e^{2\pi i/3})^3 \\ (e^{4\pi i/3})^3 \end{cases}$ and $e^{i\theta} = \cos\theta + i\sin\theta$.

(c) Since $e^{2\pi i} = 1$ and $e^{i\theta} = \cos\theta + i\sin\theta$

$$1 = \begin{cases} e^{0} \\ e^{2\pi i} \\ e^{4\pi i} \end{cases} = \begin{cases} (e^{0/3})^3 \\ (e^{2\pi i/3})^3 \\ (e^{4\pi i/3})^3 \end{cases}.$$

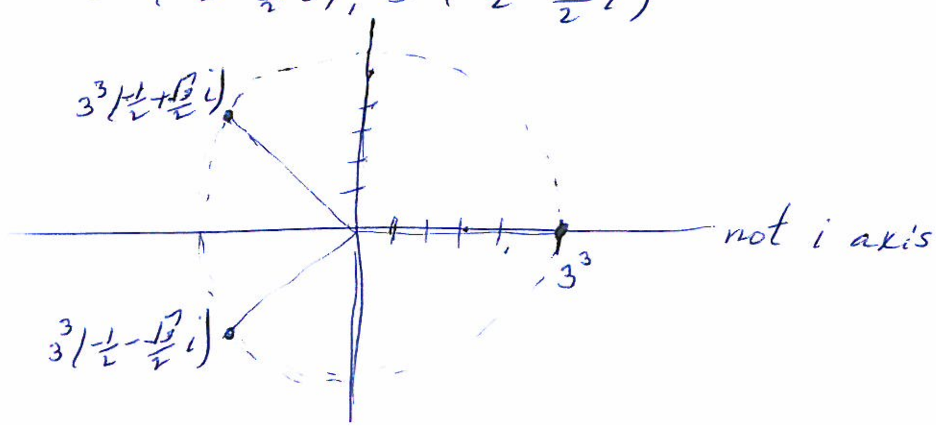
So

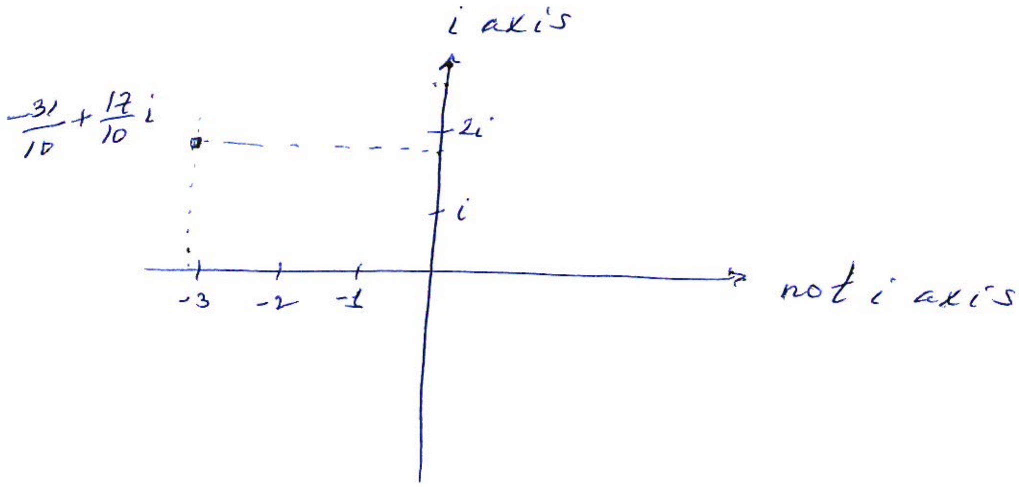
$$(27)^{4+\frac{1}{3}} = 27^4 \cdot 27^{\frac{1}{3}} = \begin{cases} (3^3)^4 \cdot ((3e^{0/3})^3)^{\frac{1}{3}} \\ (3^3)^4 \cdot ((3e^{2\pi i/3})^3)^{\frac{1}{3}} \\ (3^3)^4 \cdot ((3e^{4\pi i/3})^3)^{\frac{1}{3}} \end{cases}$$

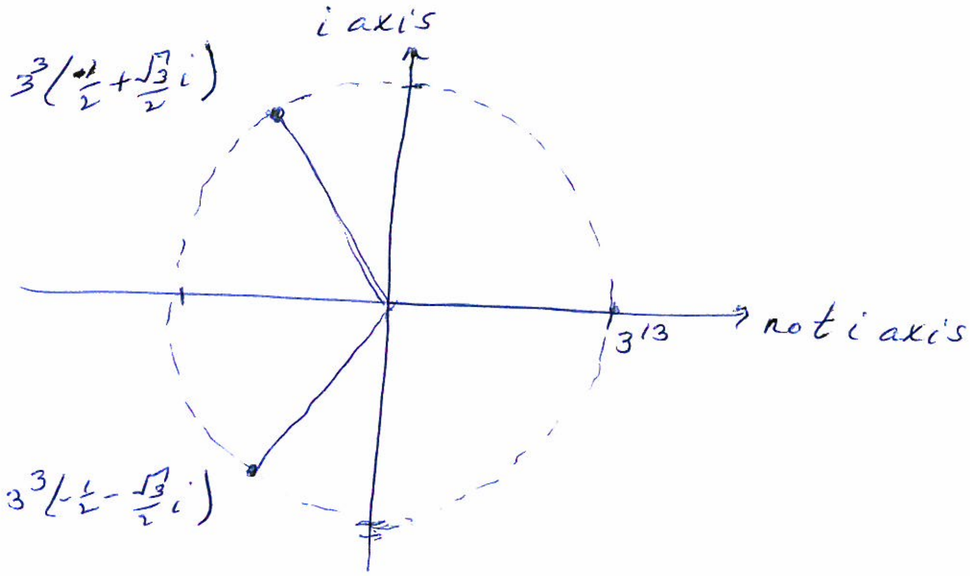
$$= \begin{cases} (3^3)^4 \cdot 3 e^{0/3} \\ (3^3)^4 \cdot 3 e^{2\pi i/3} \\ (3^3)^4 \cdot 3 e^{4\pi i/3} \end{cases} = \begin{cases} 3^{13} \\ 3^{13} \cdot (\cos 2\pi/3 + i\sin 2\pi/3) \\ 3^{13} (\cos 4\pi/3 + i\sin 4\pi/3) \end{cases}$$

$$= \begin{cases} 3^{13} \\ 3^{13}(-\frac{1}{2} + \frac{\sqrt{3}}{2} i) \\ 3^{13}(-\frac{1}{2} - \frac{\sqrt{3}}{2} i) \end{cases}$$

<u>Graphs of</u> $\frac{-31}{10} + \frac{17}{10} i$, 3^3, $3^3(-\frac{1}{2} - \frac{\sqrt{3}}{2} i)$, $3^3(-\frac{1}{2} + \frac{\sqrt{3}}{2} i)$, 3^{13},
$3^{13}(-\frac{1}{2} + \frac{\sqrt{3}}{2} i)$, $3^{13}(-\frac{1}{2} - \frac{\sqrt{3}}{2} i)$




i axis
$-\frac{31}{10}+\frac{17}{10}i$
2i
i
-3
-2
-1
not i axis


i axis
$3^3\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)$
3^{13}
not i axis
$3^3\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)$

(5) Let $a = a_1 + i a_2$ and $b = b_1 + i b_2$, with $a_1, a_2, b_1, b_2 \in \mathbb{R}$.

Claim: (a) $\overline{ab} = \bar{a}\,\bar{b}$

(b) $|ab| = |a|\,|b|$

(c) $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$

Proof (a) To show: $\overline{ab} = \bar{a}\,\bar{b}$.

$$\overline{ab} = \overline{(a_1 + i a_2)(b_1 + i b_2)} = \overline{(a_1 b_1 - a_2 b_2) + i(a_1 b_2 + a_2 b_1)}$$

$$= (a_1 b_1 - a_2 b_2) - i(a_1 b_2 + a_2 b_1).$$

$$\bar{a}\,\bar{b} = (a_1 - i a_2)(b_1 - i b_2) = (a_1 b_1 - a_2 b_2) + i(-a_1 b_2 - a_2 b_1)$$

$$= (a_1 b_1 - a_2 b_2) - i(a_1 b_2 + a_2 b_1).$$

So $\overline{ab} = \bar{a}\,\bar{b}$.

(b) To show: $|ab| = |a|\,|b|$.

$$|ab| = \sqrt{ab\,\overline{ab}} = \sqrt{ab\,\bar{a}\,\bar{b}} = \sqrt{a\bar{a}\,b\bar{b}}$$

$$= \sqrt{a\bar{a}}\,\sqrt{b\bar{b}} = |a|\,|b|.$$

(c) To show: $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$.

Note that $\frac{a}{b}$ really means $a \cdot b^{-1}$, since a, b are complex numbers ($\frac{a}{b} \notin \mathbb{R}$ in this case).

(6) (a) A function $f: S \to T$ is <u>injective</u> if it satisfies:

If $s_1, s_2 \in S$ and $f(s_1) = f(s_2)$ then $s_1 = s_2$.

As an example: The long division function

$$\mathbb{Q} \longrightarrow \mathbb{R}$$
$$\frac{a}{b} \longmapsto (\text{decimal expansion of } \tfrac{a}{b})$$

is an injective function

(b) A function $f: S \to T$ is <u>surjective</u> if it satisfies

If $t \in T$ then there exists $s \in S$ such that $f(s) = t$.

Order the rationals $\{\frac{a}{b} \mid \frac{a}{b} > 0 \text{ and } \frac{a}{b} \leq 1\}$ by

$$\{\tfrac{1}{1}, \tfrac{1}{2}, \cancel{\tfrac{2}{2}}, \tfrac{1}{3}, \tfrac{2}{3}, \cancel{\tfrac{3}{3}}, \tfrac{1}{4}, \cancel{\tfrac{2}{4}}, \tfrac{3}{4}, \cancel{\tfrac{4}{4}}, \dots\}$$

and let $\mathbb{Z}_{>0} \longrightarrow [0,1]_{\mathbb{Q}}$

$$n \longmapsto (n\text{th term in the sequence}).$$

This is a sujective function.

(c) Let $f: S \to T$ and $g: T \to U$ be functions. The <u>composition of f and g</u> is the function $g \circ f: S \to U$ given by

$$(g \circ f)(s) = g(f(s)).$$

Let
$$\begin{array}{rcl} f: \mathbb{Z} & \longrightarrow & \mathbb{Q} \\ a & \longmapsto & \frac{a}{1} \end{array} \quad \text{and} \quad \begin{array}{rcl} g: \mathbb{Q} & \longrightarrow & \mathbb{R} \\ \frac{a}{b} & \longmapsto & \begin{pmatrix} \text{decimal expansion} \\ \text{of } \frac{a}{b} \end{pmatrix}. \end{array}$$

The composition $g \circ f: \mathbb{Z} \to \mathbb{R}$ is the function

$$\begin{array}{rcl} \mathbb{Z} & \longrightarrow & \mathbb{R} \\ a & \longmapsto & a.0000\ldots \end{array}$$

(d) An <u>abelian group</u> is a set S with an operation
$$\begin{array}{rcl} S \times S & \longrightarrow & S \\ (s,t) & \longmapsto & s+t \end{array}$$
such that

(a) If $s_1, s_2, s_3 \in S$ then $(s_1+s_2)+s_3 = s_1+(s_2+s_3)$,

(b) If $s_1, s_2 \in S$ then $s_1+s_2 = s_2+s_1$,

(c) There exists $0 \in S$ such that if $s \in S$ then
$0+s = s$ and $s+0 = s$.

(d) If $s \in S$ then there exists $-s \in S$ such that
$s+(-s) = 0$ and $(-s)+s = 0$.

As an example, a favourite abelian group is the abelian group $\mathbb{Z}$ generated by 1 so that

$$\mathbb{Z} = \left\{ \begin{array}{l} 1,\ 1+1,\ 1+1+1,\ 1+1+1+1,\ \ldots \\ 0, \\ -1,\ (-1)+(-1),\ (-1)+(-1)+(-1),\ \ldots \end{array} \right\}$$

This group contains only 1, 0, ~~the~~ inverse of 1 (i.e. -1) and things that are obtained from them by applying the operation $+ : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$.

(7) $D : \mathbb{Q}[x] \to \mathbb{Q}[x]$ is a function such that

(a) If $f, g \in \mathbb{Q}[x]$ then $D(f+g) = D(f) + D(g)$,

(b) If $c \in \mathbb{Q}$ and $f \in \mathbb{Q}[x]$ then $D(cf) = cD(f)$,

(c) If $f, g \in \mathbb{Q}[x]$ then $D(fg) = fD(g) + D(f)g$,

(d) $D(x) = 1$.

Claim: If $n \in \mathbb{Z}_{>0}$ then $D(x^n) = nx^{n-1}$.

Proof Proof by induction:

Base case: $n=1$. To show: $D(x^1) = 1$.

$D(x) = 1$, by (d).

Base case: $n=2$. To show: $D(x^2) = 2x$

$$\begin{aligned} D(x^2) = D(x \cdot x) &= xD(x) + D(x) \cdot x, \quad \text{by (c)} \\ &= x \cdot 1 + 1 \cdot x, \quad \text{by (d)} \\ &= 2x. \end{aligned}$$

Induction step: Assume that $D(x^r) = rx^{r-1}$ for $r < n$.

To show: $D(x^n) = nx^{n-1}$.

(8) $$\frac{1-x^n}{1-x} = 1+x+x^2+x^3+\cdots+x^{n-2}+x^{n-1}$$

since

$$(1-x)(1+x+x^2+\cdots+x^{n-2}+x^{n-1})$$

$$\begin{aligned} &= 1+x+x^2+\cdots+x^{n-2}+x^{n-1} \\ &\quad -x-x^2-\cdots-x^{n-2}-x^{n-1}-x^n \\ &= 1-x^n. \end{aligned}$$