620-295 Real Analysis with applications Lect. 14, 25.08.2009. (1)

Let (a_n) be a sequence in $\mathbb{R}$ or $\mathbb{C}$

The series $\sum_{n=1}^{\infty} a_n$ converges to L if the sequence $(s_1, s_2, s_3, \ldots)$ converges to L, where

$$s_k = a_1 + a_2 + \cdots + a_k, \quad \text{for } k \in \mathbb{Z}_{>0}.$$

Write $\sum_{n=1}^{\infty} a_n = L$ if $\sum_{n=1}^{\infty} a_n$ converges to L.

Examples (a) $(a_n) = (-1)^n$. Then

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^n \text{ has } (s_1, s_2, s_3 \ldots) = (-1, 0, -1, 0, \ldots)$$

and $\sum_{n=1}^{\infty} (-1)^n$ diverges.

(b) Geometric series Let $x \in \mathbb{C}$, $|x| < 1$.

Think $x = \frac{1}{2}$. Then

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots$$

has
$$s_0 = 1 = \frac{1-x}{1-x}$$
$$s_1 = 1 + x = \frac{1-x^2}{1-x}$$
$$s_2 = 1 + x + x^2 = \frac{1-x^3}{1-x}$$
$$s_3 = 1 + x + x^2 + x^3 = \frac{1-x^4}{1-x}.$$

So
$$\sum_{n=1}^{\infty} x^n = \lim_{k \to \infty} \frac{1-x^k}{1-x}.$$

When $|x| < 1$ then $\lim\limits_{n \to \infty} x^n = 0$

and $$\sum_{n=1}^{\infty} x^n = \lim_{k \to \infty} \frac{1 - x^k}{1-x} = \frac{1}{1-x}.$$

For example
$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = \frac{1}{1-\frac{1}{2}} = 2.$$

If $|x| > 1$ then $|x^n|$ gets larger and larger and the sequence $s_k = \dfrac{1-x^k}{1-x}$ diverges.

So $\sum\limits_{n=1}^{\infty} x^n$ diverges if $|x| > 1$.

Then
$$\sum_{n=1}^{\infty} 1^n = 1 + 1 + 1 + 1 + \cdots \quad \text{diverges}$$
and $$\sum_{n=1}^{\infty} (-1)^n = -1 + 1 + -1 + 1 + -1 + \cdots \quad \text{diverges}.$$

(c) <u>Harmonic series</u>

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}} + \cdots$$
$$> 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots$$

So $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ diverges.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \frac{1}{8^2} + \cdots$$

$$< 1 + \frac{2}{2^2} + \frac{4}{4^2} + \frac{8}{8^2} + \cdots$$

$$= 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$

$$= 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots = \frac{1}{1-\frac{1}{2}} = 2$$

In fact, according to Wolfram alpha,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

More generally, if $k > 1$

$$\sum_{n=1}^{\infty} \frac{1}{n^k} = 1 + \frac{1}{2^k} + \frac{1}{3^k} + \frac{1}{4^k} + \frac{1}{5^k} + \frac{1}{6^k} + \frac{1}{7^k} + \frac{1}{8^k}$$

$$< 1 + \frac{2}{2^k} + \frac{4}{4^k} + \frac{8}{8^k} + \cdots$$

$$= 1 + \frac{1}{2^{k-1}} + \frac{1}{4^{k-1}} + \frac{1}{8^{k-1}} + \cdots$$

$$= 1 + \frac{1}{2^{k-1}} + \left(\frac{1}{2^{k-1}}\right)^2 + \left(\frac{1}{2^{k-1}}\right)^3 + \cdots = \frac{1}{1-\frac{1}{2^k}} = \frac{2^k}{2^k-1}.$$

and $\sum_{n=1}^{\infty} \frac{1}{n^k}$ converges.

If $k < 1$ then

$$\sum_{n=1}^{\infty} \frac{1}{n^k} = 1 + \frac{1}{2^k} + \frac{1}{3^k} + \frac{1}{4^k} + \cdots$$

$$> 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

so that $\sum_{n=1}^{\infty} \frac{1}{n^k}$ diverges.

Let $s \in \mathbb{C}$. The <u>Riemann zeta function at s</u> is

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

So $\zeta(2) = \frac{\pi^2}{6}$.

<u>Example</u> $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$

Since,

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots$$

then $\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \cdots$

and $\log(1+x) = \int \frac{1}{1+x}\,dx = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots$

So $\log(1+1) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$

$$= \log 2 =$$

A series $\sum_{n=1}^{\infty} a_n$ is <u>conditionally convergent</u> if $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges.

A series $\sum_{n=1}^{\infty} a_n$ is <u>absolutely convergent</u> if $\sum_{n=1}^{\infty} |a_n|$ converges.