Define

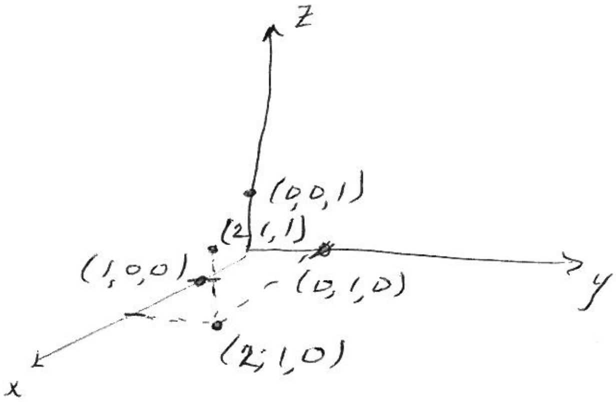
$$\mathbb{R}^n = \{(x_1, x_2, \cdots, x_n) \mid x_1, \dots, x_n \in \mathbb{R}\}.$$

so that

$$\mathbb{R}^1 = \mathbb{R}, \qquad \mathbb{R}^2 = \{(x,y) \mid x, y \in \mathbb{R}\}$$

$$\mathbb{R}^3 = \{(x,y,z) \mid x,y,z \in \mathbb{R}\}.$$

Graph points in $\mathbb{R}^3$:



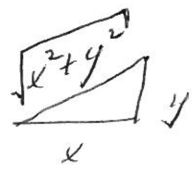
The absolute value on $\mathbb{R}^n$ is the function

$|\cdot| : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ given by

$$|(x_1, \dots, x_n)| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

For example:

If $n=2$, $\quad |(x,y)| = \sqrt{x^2+y^2} \in \mathbb{R}_{\geq 0}$



If $n=1$, $\quad |x| = \sqrt{x^2} \in \mathbb{R}_{\geq 0}$.

Lagrange's identity

$$\left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right) - \left(\sum_{i=1}^{n} x_i y_i\right)^2 = \frac{1}{2}\sum_{i,j}\left(x_i y_j - x_j y_i\right)^2.$$

Proof

$$\frac{1}{2}\sum_{i,j=1}^{n}\left(x_i y_j - x_j y_i\right)^2 = \frac{1}{2}\sum_{i,j=1}^{n} x_i^2 y_j^2 - 2x_i y_j x_j y_i + x_j^2 y_i^2$$

$$= \frac{1}{2}\sum_{i,j=1}^{n} x_i^2 y_j^2 + \frac{1}{2}\sum_{i,j=1}^{n} x_j^2 y_i^2 - \sum_{j,i=1}^{n} x_i y_i x_j y_j$$

$$= \sum_{i,j=1}^{n} x_i^2 y_j^2 - \left(\sum_{i=1}^{n} x_i y_i\right)^2.$$

$$= \left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{j=1}^{n} y_j^2\right) - \left(\sum_{i=1}^{n} x_i y_i\right)^2. \quad /\!/$$

If $n=2$

$$\frac{1}{2}\left((x_1 y_1 - x_1 y_1)^2 + (x_1 y_2 - x_2 y_1)^2 + (x_2 y_1 - x_1 y_2)^2 + (x_2 y_2 - x_2 y_2)^2\right).$$

$= \ldots\ldots$

$$= (x_1^2 + x_2^2)(y_1^2 + y_2^2) - (x_1 y_1 + x_2 y_2)^2.$$

The <u>inner product</u> on $\mathbb{R}^n$ is the function

$$\mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$$
$$(x,y) \longmapsto \langle x,y\rangle \quad \text{given by}$$

$$\langle x,y\rangle = (x_1,\dots,x_n)\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = x_1y_1 + \cdots + x_ny_n = \sum_{i=1}^{n} x_iy_i .$$

<u>Note:</u>
$$|x| = \sqrt{x_1^2 + \cdots x_n^2} = \sqrt{\langle x,x\rangle}$$

<u>Theorem</u> (The Schwarz inequality).

$$\langle x,y\rangle \leq |x||y| .$$

<u>Proof</u> Lagranges identity tells us

$$|x|^2|y|^2 - \langle x,y\rangle^2 \geq 0 .$$

So $(|x||y|)^2 \geq \langle x,y\rangle^2$.

So $|x||y| \geq \langle x,y\rangle$. //.

<u>Theorem</u> (The triangle inequality). Let $x,y \in \mathbb{R}^n$

Then $|x+y| \leq |x| + |y|$.

<u>Proof</u>

<u>Proof</u>

$$\langle x+y, x+y\rangle = \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle$$
$$= |x|^2 + 2\langle x,y\rangle + |y|^2$$
$$\leq |x|^2 + 2|x||y| + |y|^2$$
$$= (|x|+|y|)^2$$

So $|x+y|^2 \leq (|x|+|y|)^2$.

So $|x+y| \leq |x|+|y|$. //

An <u>ordered field</u> is a field S with a total order $\leq$ such that

(a) if $x,y,z \in S$ ~~then~~ and $x \leq y$ then $x+z \leq y+z$.

(b) if $x,y \in S$ and $x \geq 0$ and $y \geq 0$ then $xy \geq 0$.

<u>Claim</u>: Let S be an ordered field and $x,y \in S$ with $x \geq 0$ and $y \geq 0$. Then

$$x \leq y \text{ if and only if } x^2 \leq y^2.$$

<u>Proof</u> Assume $x,y \in S$ and $x \geq 0$ and $y \geq 0$.

To show: (a) If $x \leq y$ then $x^2 \leq y^2$

(b) If $x^2 \leq y^2$ then $x \leq y$.

(b) Assume $x^2 \le y^2$.

Then $y^2 + (-x^2) \geqslant x^2 + (-x^2) = 0$.

So $y^2 - x^2 \geq 0$.

So $(y-x)(y+x) \geq 0$.

Since $x \geq 0$ and $y \geq 0$ then $x+y \geq 0$

and $(x+y)^{-1} > 0$ (or $x=0$ and $y=0$).

So $(y-x)(y+x)(x+y)^{-1} \geq 0$.

So $y - x \geq 0$.

(a) You do this. //

Note: Actually, the proof above skips <u>lots</u> of steps.

(like $y + (-x^2) = (y+(-x))(y+x)$).

You should justify these.

Why is $(-x)y = -(xy)$??