

620-295 Real Analysis with applications Lecture 11, 18.08.2009 ⁽¹⁾

Let S and T be sets.

The sets S and T have the same cardinality

if there exists a bijection $f: S \rightarrow T$.

Write $\text{Card}(S) = \text{Card}(T)$, if S and T have the same cardinality.
For $n \in \mathbb{Z}_{>0}$, write

$\text{Card}(S) = n$, if there is a bijection
 $f: S \rightarrow \{1, 2, \dots, n\}$.

For example, write

$\text{Card}(S) = 5$, if there exists a bijection
 $f: S \rightarrow \{1, 2, 3, 4, 5\}$.

A set S is finite if there exists $n \in \mathbb{Z}_{>0}$ such
that $\text{Card}(S) = n$.

A set S is infinite if S is not finite.

A set S is countable if

S is finite or $\text{Card}(S) = \aleph$.

A set S is uncountable if S is not countable.

Example $\text{Card}\{a, b, c, d, e\} = 5$ since

$$\{a, b, c, d, e\} \rightarrow \{1, 2, 3, 4, 5\}$$

$$a \mapsto 3$$

$$b \mapsto 1$$

$$c \mapsto 5$$

$$d \mapsto 2$$

$$e \mapsto 4$$

is a bijection.

Example $\text{Card}(\mathbb{Z}_{70}) = \text{Card}(\mathbb{Z}_{70})$ since

$$\mathbb{Z}_{70} \rightarrow \mathbb{Z}_{70}$$

$$1 \mapsto 0$$

$$2 \mapsto 1$$

$$3 \mapsto 2$$

$$4 \mapsto 3$$

$$\vdots \quad \vdots$$

$$n \mapsto n-1$$

$$k+1 \longleftarrow k$$

is a bijection,

(the inverse function)
exists

Example $\text{Card}(\mathbb{Z}) = \text{Card}(\mathbb{Z}_{70})$ since

$$\mathbb{Z} \rightarrow \mathbb{Z}_{70}$$

$$0 \mapsto 0$$

$$1 \mapsto -1$$

$$2 \mapsto +1$$

$$3 \mapsto -2$$

$$4 \mapsto 2$$

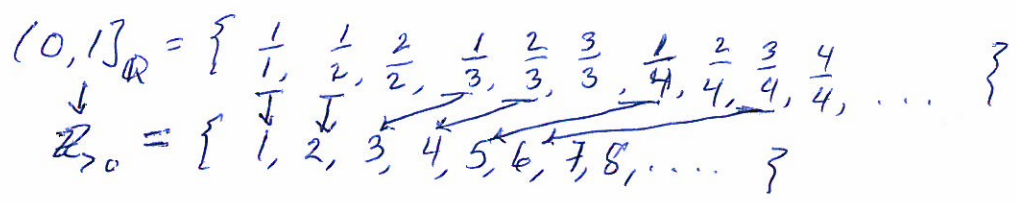
$$5 \mapsto -3$$

$$6 \mapsto 3$$

\vdots

is a bijection.

Example $\text{Card}((0,1]_{\mathbb{R}}) = \text{Card}(\mathbb{Z}_{>0})$



is a bijection.

Theorem $\text{Card}((0,1]_{\mathbb{R}}) \neq \text{Card}(\mathbb{Z}_{>0})$

Proof Proof by contradiction

Assume $f: \mathbb{Z}_{>0} \rightarrow (0,1]_{\mathbb{R}}$ is a bijection
 $k \mapsto r_k$

Then

- $r_1 = 0. r_{11} r_{12} r_{13} r_{14} r_{15} r_{16} \dots$
- $r_2 = 0. r_{21} r_{22} r_{23} r_{24} r_{25} r_{26} \dots$
- $r_3 = 0. r_{31} r_{32} r_{33} r_{34} r_{35} r_{36} \dots$
- $r_4 = 0. r_{41} r_{42} r_{43} r_{44} r_{45} r_{46} \dots$
- \vdots

Let

$s = 0. s_1 s_2 s_3 s_4 s_5 s_6 \dots$ with

- $s_1 \neq r_{11}$
- $s_2 \neq r_{22}$
- $s_3 \neq r_{33}$
- \vdots
- etc.

Then $s \neq r_k$, for all $k \in \mathbb{Z}$.

So $s \in (0,1]_{\mathbb{R}}$ and there does not exist $k \in \mathbb{Z}_{>0}$

with $f(k) = s$. So f is not surjective.

This is a contradiction to f being a bijection.