620-295 Real Analysis with applications Lecture 11, 18.08.2009

Let S and T be sets.

The sets S and T have the same cardinality if there exists a bijection $f: S \to T$.

Write $\mathrm{Card}(S) = \mathrm{Card}(T)$, if S and T have the same cardinality.

For $n \in \mathbb{Z}_{>0}$, write

$$\mathrm{Card}(S) = n, \text{ if there is a bijection } f: S \to \{1, 2, \ldots, n\}.$$

For example, write

$$\mathrm{Card}(S) = 5, \text{ if there exists a bijection } f: S \to \{1, 2, 3, 4, 5\}.$$

A set S is finite if there exists $n \in \mathbb{Z}_{>0}$ such that $\mathrm{Card}(S) = n$.

A set S is infinite if S is not finite.

A set S is countable if

S is finite or $\mathrm{Card}(S) = \mathbb{Z}$.

A set S is uncountable if S is not countable.

Example $\text{Card}\{a,b,c,d,e\} = 5$ since

$$\{a,b,c,d,e\} \longrightarrow \{1,2,3,4,5\}$$

$$\begin{aligned} a &\longmapsto 3 \\ b &\longmapsto 1 \\ c &\longmapsto 5 \\ d &\longmapsto 2 \\ e &\longmapsto 4 \end{aligned}$$

is a bijection.

Example $\text{Card}(\mathbb{Z}_{>0}) = \text{Card}(\mathbb{Z}_{\geq 0})$ since

$$\mathbb{Z}_{>0} \longrightarrow \mathbb{Z}_{\geq 0}$$

$$\begin{aligned} 1 &\longmapsto 0 \\ 2 &\longmapsto 1 \\ 3 &\longmapsto 2 \\ 4 &\longmapsto 3 \\ \vdots & \quad\;\; \vdots \\ n &\longmapsto n-1 \\ k+1 &\longleftarrow\!\shortmid\; k \end{aligned}$$

is a bijection,
(the inverse function exists).

Example $\text{Card}(\mathbb{Z}) = \text{Card}(\mathbb{Z}_{\geq 0})$ since

$$\mathbb{Z} \longrightarrow \mathbb{Z}_{\geq 0}$$

$$\begin{aligned} 0 &\longmapsto 0 \\ 1 &\longmapsto -1 \\ 2 &\longmapsto +1 \\ 3 &\longmapsto -2 \\ 4 &\longmapsto 2 \\ 5 &\longmapsto -3 \\ 6 &\longmapsto 3 \\ &\;\vdots \end{aligned}$$

is a bijection.

__Example__ $\mathrm{Card}((0,1]_{\mathbb{Q}}) = \mathrm{Card}(\mathbb{Z}_{>0})$

$$(0,1]_{\mathbb{Q}} = \left\{ \tfrac{1}{1}, \tfrac{1}{2}, \tfrac{2}{2}, \tfrac{1}{3}, \tfrac{2}{3}, \tfrac{3}{3}, \tfrac{1}{4}, \tfrac{2}{4}, \tfrac{3}{4}, \tfrac{4}{4}, \ldots \right\}$$

$$\mathbb{Z}_{>0} = \{ 1, 2, 3, 4, 5, 6, 7, 8, \ldots \}$$

is a bijection.

__Theorem__ $\mathrm{Card}((0,1]_{\mathbb{R}}) \neq \mathrm{Card}(\mathbb{Z}_{>0})$

__Proof__ Proof by contradiction

Assume $f : \mathbb{Z}_{>0} \to (0,1]_{\mathbb{R}}$, $k \mapsto r_k$ is a bijection

Then

$$r_1 = 0.r_{11}r_{12}r_{13}r_{14}r_{15}r_{16}\ldots$$
$$r_2 = 0.r_{21}r_{22}r_{23}r_{24}r_{25}r_{26}\ldots$$
$$r_3 = 0.r_{31}r_{32}r_{33}r_{34}r_{35}r_{36}\ldots$$
$$r_4 = 0.r_{41}r_{42}r_{43}r_{44}r_{45}r_{46}\ldots$$
$$\vdots$$

Let

$s = 0.s_1s_2s_3s_4s_5s_6\ldots$ with
$$s_1 \neq r_{11}$$
$$s_2 \neq r_{22}$$
$$s_3 \neq r_{33}$$
$$\vdots \quad \text{etc.}$$

Then $s \neq r_k$, for all $k \in \mathbb{Z}$.

So $s \in (0,1]_{\mathbb{R}}$ and there does not exist $k \in \mathbb{Z}_{>0}$ with $f(k) = s$. So f is not surjective.
This is a contradiction to f being a bijection //