620-695 Real Analysis with Applications Lecture 10 17.08.2009 1

Let S be a set.

A partial order on S is a relation $\le$ on S

such that

(a) If $x, y, z \in S$ and $x \le y$ and $y \le z$ then $x \le z$,

(b) If $x, y \in S$ and $x \le y$ and $y \le x$ then $x = y$.

Let S be a set with a partial order.

Write $x < y$ if $x \le y$ and $x \ne y$.

The intervals is S are

$$[a,b] = \{x \in S \mid a \le x \le b\},$$
$$(a,b] = \{x \in S \mid a < x \le b\},$$
$$[a,b) = \{x \in S \mid a \le x < b\},$$
$$(a,b) = \{x \in S \mid a < x < b\},$$
$$[a,\infty) = \{x \in S \mid a \le x\},$$
$$(-\infty,b) = \{x \in S \mid x < b\},$$

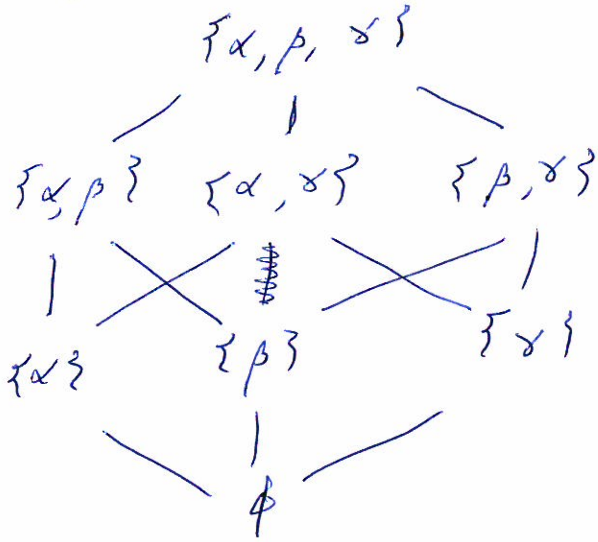
for $a, b \in S$.

Let $E \subseteq S$.

A maximum of E is $m \in E$ such that

if $x \in E$ then $x \not> m$.

<u>Example</u> Let S be the set of subsets of $\{\alpha, \beta, \gamma\}$ ordered by inclusion



Let $E = \{ \{\alpha\}, \{\beta\}, \{\gamma\}, \{\beta, \gamma\} \}$

Then $\{\alpha\}$ and $\{\beta, \gamma\}$ are both maximums of E.

$(-\infty, \{\beta, \gamma\}) = \{ \phi, \{\beta\}, \{\gamma\} \}$.

$(-\infty, \{\beta, \gamma\}] = \{ \phi, \{\beta\}, \{\gamma\}, \{\beta, \gamma\} \}$.

Let S be a set with a partial order $\le$

Let E be a subset of S.

An <u>upper bound of E</u> is an element $b \in S$ such that

$$\text{if } x \in E \text{ then } x \le b.$$

The set E is <u>bounded above</u> if it has an upper bound in S.

A <u>supremum</u>, or <u>least upper bound</u>, of E in S is an element $\sup E \in S$ such that

(a) $\sup E$ is an upper bound of E,

(b) If ~~b is an~~ $b \in S$ is an upper bound of E then $b \le \sup S$.

<u>Example</u>: Subsets of $\mathbb{R}$.

(a) $[0,1)$ has many upper bounds in $\mathbb{R}$:

$$2,\ 3,\ \pi,\ 7.1,\ 2.78,\ \ldots$$

$[0,1)$ does not have a maximum

$\sup [0,1) = 1$

(b) $[0,1]$ has many upper bounds in $\mathbb{R}$.

$[0,1]$ has maximum 1

$\sup [0,1] = 1$.

<u>Proposition</u> Let S be a set with a partial order $\leq$. Let $E \subseteq S$. If the supremum of E exists in S then it is unique.

<u>Proof</u> Assume that a supremum of E exists in S.

To show: The supremum of E is unique.

Assume $s_1, s_2 \in S$ are supremums of E.

To show: $s_1 = s_2$

Since s_1 is an upper bound of E and
s_2 is a supremum of E,

$s_2 \leq s_1$

Since s_2 is an upper bound of E and
s_1 is a supremum of E

$s_1 \leq s_2$.

By part (b) in the definition of partial order,

$s_1 = s_2$

So $\sup E$ is unique if it exists.

<u>Proposition</u> As a subset of $\mathbb{R}$, $\mathbb{Z}_{>0}$ is not bounded above.

<u>Proof</u> Proof by contradiction.

Assume $b \in \mathbb{R}$ is an upper bound of $\mathbb{Z}_{>0}$,

Then $b = b_0 . b_1 b_2 b_3 \ldots$, with $b_0 \in \mathbb{Z}$.

<u>Case 1</u> $b_0 \le 0$. Then $1 \in \mathbb{Z}_{>0}$ and $1 > b$.

This is a contradiction to b being an upper bound.

<u>Case 2</u> $b_0 \in \mathbb{Z}_{>0}$. Then $b_0 + 2 \in \mathbb{Z}_{>0}$ and $b_0 + 2 > b$.

This is a contradiction to b being an upper bound.

So $\mathbb{Z}_{>0}$ is not bounded above in $\mathbb{R}$.

<u>Example</u> Find $\sup\{a \in \mathbb{Q} \mid a^2 < 2\}$ in $\mathbb{R}$.

i.e. $\{a \in \mathbb{Q} \mid a^2 < 2\} \subseteq \mathbb{R}$. What is its supremum?

Lets see

$2^2 = 4$, so 2 is an upper bound of E.

Is it the least upper bound?

$(1.5)^2 = 2.25$, so 1.5 is an upper bound of E.

Is it the least upper bound?

$(1.42)^2 = 2.0164$, so 1.42 is an upper bound of E.

$(1.415)^2 =$