Department of Mathematics and Statistics
620–295
Real analysis with applications

Laboratory Class 1: Numbers, Sequences and Series

Aims: To explore some of the properties of rational numbers; to explore sequences in order to enhance your understanding of convergence and to illustrate the dependence of $N(\varepsilon)$ on ε; also to explore sequences of partial sums, which produce infinite series. In all examples, examine the MATLAB code and analyze the effect of various commands.

Before starting, copy the folder `Lab1` from the lab server `M&S Lab Materials\620-295` to `D:MATLAB` and set the path to `D:MATLAB` including subfolders.

1 The decimal expansion of rational numbers

One amazing property of rational numbers is that their decimal expansions eventually show periodicity, i.e. a repeating pattern of digits. We make this precise by the mathematical statement:

In the decimal expansion of a rational number

$$x = d_0.d_1d_2d_3\cdots$$

there exist $n, k \in \mathbb{Z}_{>0}$ such that, for all $\ell \in \mathbb{Z}_{>0}$ with $\ell > k, d_{\ell+n} = d_l$

Your task is to decode this statement and use it. Are terminating expansions covered by the statement above?

Enter the following commands into the Matlab command window:

```
digits(100);
vpa(35/29)
```

What are the values of k, n for the rational number $x = 35/29$?

`vpa` stands for Variable Precision Arithmetic, which allows you to increase the number of decimal digits the computer uses, beyond the usual 16 or so. If you don't see a period, try increasing the number of digits from 100.

Now try: $x = 35/28, 35/27, 35/26, 35/24, 35/23$ — what are k, n in each case?

2 Euler's number: code Lab1Ex2

Start with the standard sequence $a_n = (1 + \frac{1}{n})^n$.

The limit of this sequence is $e \approx 2.718281828459045...$

`Lab1Ex2.m` is a function M-file which takes a postive integer N as input computes and plots the sequence to the Nth term. It also computes the error, that is, $|a_N - e|$ as output: type `error = Lab1Ex2(100)` at the command line, then hit Return to run with $N = 100$. Use $\uparrow$ to recall the command and edit N.

Let $N(\varepsilon)$ be the minimal positive integer such that $|a_n - e| < \varepsilon$ for all $n > N(\varepsilon)$. To explore the dependence of $N(\varepsilon)$ on ε consider (i) $\varepsilon = 0.1$, (ii) $\varepsilon = 0.01$, (iii) $\varepsilon = 0.001$ and (iv) $\varepsilon = 0.0001$ and determine by experiment how large N needs to be to get an approximation within the prescribed error ε. Determine $N(\varepsilon)$ in each case:

Answers: (i) (ii) (iii) (iv)

3 Partial sums: code Lab1Ex3

A common class of sequences are formed from partial sums. Here are two examples also related to irrational numbers.

$$a_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \pm \frac{1}{n}, \qquad n = 1, 2, 3, ...$$

$$b_n = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \pm \frac{1}{2n-1}, \qquad n = 1, 2, 3, ...$$

where the last term is positive or negative depending on whether n is odd or even.

Amazingly, the a_n sequence gets closer and closer to $\log 2 \approx 0.69314718055994530941723212145817 65$and the b_n sequence gets closer and closer to $\pi/4 \approx 0.78539816339744830961566084581987572104929 23$.

For each of the four values $\varepsilon = 0.1$, $\varepsilon = 0.01$, $\varepsilon = 0.001$ and $\varepsilon = 0.0001$ use `Lab1Ex3` to experimentally determine the minimal positive integer $N(\varepsilon)$ such that $|a_n - \log 2| < \varepsilon$ for $n > N(\varepsilon)$.

Answers: (i) (ii) (iii) (iv)

Repeat for b_n.

Answers: (i) (ii) (iii) (iv)

`Lab1Ex3.m` is a function M-file which computes and plots both sequences up to $n = N$ with N as input and both errors as output: type `[error1,error2] = Lab1Ex3(100)` at the command line to run with $N = 100$.

4 Compare the partial sums: code Lab1Ex4

We explore three sequences

$$a_n = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}}, \qquad n = 1, 2, 3, ...$$

$$b_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}, \qquad n = 1, 2, 3, ...$$

and

$$c_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}, \qquad n = 1, 2, 3, ...$$

One possible way to determine if the series converges to a limit or not is to compare with another series whose behaviour we know.

What do a_n, b_n and c_n get closer and closer to? Run `Lab1Ex4` e.g. by `Lab1Ex4(100)` — it demonstrates that

$$a_n > b_n > c_n, \quad \text{for all } 1 \le n \le 100$$

Prove these inequalities for all $n > 1$.

Later you will use inequalities like this to prove convergence or divergence of complicated series.

5 A complex series: code Lab1Ex5

Sequences and series can just as easily be constructed from complex numbers.

The geometric series familiar from school

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots \qquad \text{for } x \in \mathbb{R}, |x| < 1$$

applies equally well for complex values:

$$\frac{1}{1-z} = 1 + z + z^2 + \cdots \qquad \text{for } z \in \mathbb{C}, |z| < 1$$

The code `Lab1Ex5` computes and plots the partial sums of the geometric series, with the complex number z as input. Try the following and compute the modulus of z for each:

```
>>Lab1Ex5(-0.4+sqrt(3)/2*1i)
>>Lab1Ex5(-0.6+sqrt(3)/2*1i)
```