Let $a(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \in \mathbb{Q}[[x]]$.

Then

$$\frac{d^k}{dx^k}(a(x)) = k!\,a_k + (k+1)!\,a_{k+1}x + \frac{(k+2)!}{2!}a_{k+2}x^2 + \frac{(k+3)!}{3!}a_{k+3}x^3 + \cdots$$

So

$$\left.\frac{d^k}{dx^k}(a(x))\right|_{x=0} = k!\,a_k.$$

So

$$a_k = \frac{1}{k!}\left(\left.\frac{d^k a}{dx^k}\right|_{x=0}\right).$$

Example If you know

(a) $e^0 = 1$

(b) $\frac{d}{dx}(e^x) = e^x$

Then you know $\frac{d^2}{dx^2}(e^x) = e^x$, $\frac{d^3}{dx^3}(e^x) = e^x$, ...

so that

$$\frac{d^k}{dx^k}e^x = e^x \quad \text{and} \quad \left.\frac{d^k}{dx^k}(e^x)\right|_{x=0} = 1.$$

So

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \cdots$$

<u>Example</u> If you know

(a) $\sin 0 = 0$ and $\cos 0 = 1$, and

(b) $\dfrac{d}{dx}(\sin x) = \cos x$ and $\dfrac{d}{dx}(\cos x) = -\sin x$

Then

$$\frac{d^2}{dx^2}(\sin x) = -\sin x, \quad \frac{d^3}{dx^3}(\sin x) = -\cos x,$$

$$\frac{d^4}{dx^4}(\sin x) = \sin x, \quad \frac{d^5}{dx^5}(\sin x) = \cos x$$

and

$$\sin x = 0 + 1\cdot x - \frac{1}{2!}\cdot 0x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}\cdot 0x^4 + \frac{1}{5!}\cdot 1\cdot x^5 - \cdots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$$

Since

$$\frac{d}{dx}(\cos x) = -\sin x, \quad \frac{d^2}{dx^2}(\cos x) = -\cos x,$$

$$\frac{d^3}{dx^3}(\cos x) = \sin x, \quad \frac{d^4}{dx^4}(\cos x) = \cos x, \ \ldots$$

then

$$\cos x = 1 - 0x - \frac{1}{2!}\cdot 1\cdot x^2 + \frac{1}{3!}\cdot 0\cdot x^3 + \frac{1}{4!}\cdot 1\cdot x^4 - \cdots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

Then

$$e^{ix} = 1 + (ix) + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \cdots$$

$$= 1 + \frac{i^2x^2}{2!} + \frac{i^4x^4}{4!} + \frac{i^6x^6}{6!}$$
$$+ ix + \frac{i^3x^3}{3!} + \frac{i^5x^5}{5!}$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$$
$$+ ix - \frac{ix^3}{3!} + \frac{ix^5}{5!} + \cdots$$

$$= \cos x + i \sin x.$$

<u>Example</u> If you believe

$$\frac{d}{dx}(1+x)^{\frac{1}{2}} = \frac{1}{2}(1+x)^{-\frac{1}{2}}, \quad \frac{d^2}{dx^2}(1+x)^{\frac{1}{2}} = \frac{-1}{2\cdot 2}(1+x)^{-3/2}$$

$$\frac{d^3}{dx^3}(1+x)^{\frac{1}{2}} = \frac{3}{2^3}(1+x)^{-5/2}, \quad \frac{d^4}{dx^4}(1+x)^{\frac{1}{2}} = \frac{-3\cdot 5}{2^4}(1+x)^{-7/2}$$

$$\frac{d^5}{dx^5}(1+x)^{\frac{1}{2}} = \frac{3\cdot 5\cdot 7}{2^5}(1+x)^{-9/2}, \quad \frac{d^6}{dx^6}(1+x)^{\frac{1}{2}} = \frac{-3\cdot 5\cdot 7\cdot 9}{2^6}(1+x)^{-11/2}.$$

Then

$$(1+x)^{\frac{1}{2}} = 1 + \frac{1}{1!}\cdot\frac{1}{2}x + \frac{1}{2!}\left(\frac{-1}{2^2}\right)x^2 + \frac{1}{3!}\left(\frac{3}{2^3}\right)x^3 + \frac{1}{4!}\left(\frac{-3\cdot 5}{2^4}\right)x^4 + \frac{1}{5!}\left(\frac{3\cdot 5\cdot 7}{2^5}\right)x^5$$
$$+ \cdots$$

So

$$(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x - \frac{1}{2^2}\frac{1}{2!}x^2 + \frac{3}{2^3}\frac{1}{3!}x^3 - \frac{3\cdot 5}{2^4}\frac{1}{4!}x^4 + \ldots$$

$$= 1 + \frac{1}{2}x - \frac{1}{3}\cdot\frac{3}{2\cdot 4}x^2 + \frac{1}{5}\frac{3\cdot 5}{2\cdot 4\cdot 6}x^3 - \frac{1}{7}\cdot\frac{3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}x^4 + \ldots$$

So

$$2^{\frac{1}{2}} = 1 + \frac{1}{2} - \frac{1}{3}\cdot\frac{3}{8} + \frac{1}{5}\cdot\frac{15}{48} - \frac{1}{7}\frac{105}{384} + \cdots$$

Example

$$\frac{d(1+x)^7}{dx} = 7(1+x)^6, \quad \frac{d^2}{dx^2}(1+x)^7 = 7\cdot 6(1+x)^5,$$

$$\frac{d^3}{dx^3}(1+x)^7 = 7\cdot 6\cdot 5(1+x)^4, \quad \frac{d^4}{dx^4}(1+x)^7 = 7\cdot 6\cdot 5\cdot 4(1+x)^3$$

$$\frac{d^5}{dx^5}(1+x)^7 = 7\cdot 6\cdot 5\cdot 4\cdot 3(1+x)^2, \quad \frac{d^6}{dx^6}(1+x)^7 = 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2(1+x)$$

$$\frac{d^7}{dx^7}(1+x)^7 = 7!(1+x)^0, \quad \frac{d^8}{dx^8}(1+x)^7 = 0, \quad \frac{d^9}{dx^9}(1+x)^7 = 0, \ldots$$

So

$$(1+x)^7 = 1 + 7x + \frac{1}{2!}7\cdot 6x^2 + \frac{1}{3!}7\cdot 6\cdot 5x^3 + \frac{1}{4!}7\cdot 6\cdot 5\cdot 4x^4$$

$$+ \frac{1}{5!}7\cdot 6\cdot 5\cdot 4\cdot 3x^5 + \frac{1}{6!}7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2x^6 + \frac{1}{7!}7!x^7 + 0 + 0 + \cdots$$

$$= 1 + 7x + 21x^2 + 35x^3 + 35x^4 + 21x^5 + 7x^6 + x^7.$$

Alternatively

$$x+y = x+y$$

$$(x+y)^2 = x^2 + 2xy + y^2$$

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4x^3y^3 + y^4$$

$$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

$$(x+y)^6 = x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$$

$$(x+y)^7 = x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + y^7$$

so that

$$(1+x)^7 = x^7 + 7x^6 + 21x^5 + 35x^4 + 35x^3 + 21x^2 + 7x + 1.$$

<u>Pascal's triangle</u>

$$\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1.
\end{array}$$