A **field** is a set S with two operations

$$+ : S \times S \to S \quad \text{and} \quad S \times S \to S$$
$$(s,t) \mapsto s+t \qquad\qquad (s,t) \mapsto st$$

with

(a) If $s_1, s_2, s_3 \in S$ then $(s_1+s_2)+s_3 = s_1+(s_2+s_3)$,

(b) If $s_1, s_2 \in S$ then $s_1+s_2 = s_2+s_1$,

(c) There exists $0 \in S$ such that
if $s \in S$ then $s+0 = s$ and $0+s = s$.

(d) If $s \in S$ there exists $-s \in S$ such that
$s+(-s) = 0$ and $(-s)+s = 0$

(e) If $s_1, s_2, s_3 \in S$ then $(s_1 s_2) s_3 = s_1 (s_2 s_3)$

(f) There exists $1 \in S$ such that
if $s \in S$ then $1s = s$ and $s \cdot 1 = s$

(g) If $s_1, s_2, s_3 \in S$ then $s_1(s_2+s_3) = s_1s_2 + s_1s_3$
and $(s_1+s_2)s_3 = s_1s_3 + s_2s_3$.

(h) If $s \in S$ and $s \neq 0$ then there exists $s^{-1} \in S$
such that $ss^{-1} = s^{-1}s = 1$.

(i) If $s_1, s_2 \in S$ then $s_1s_2 = s_2s_1$.

Number systems with addition and multiplication:

$\mathbb{Z}_{>0}$, $\mathbb{Z}_{\geq 0}$, $\mathbb{Z}$ and

$\mathbb{Q} = \{ \frac{a}{b} \mid a, b \in \mathbb{Z},\ b \neq 0 \}$ with

$$\frac{a}{b} = \frac{c}{d} \text{ if } ad = bc \qquad \left(\frac{1}{3} = \frac{641}{2023} \right)$$

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd} \quad \text{and} \quad \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}.$$

The real numbers is the set

$$\mathbb{R} = \{ a_0 . a_1 a_2 a_3 \ldots \mid a_0 \in \mathbb{Z},\ a_i \in \{0,1,2,3,4,5,6,7,8,9\} \text{ for } i \in \mathbb{Z}_{>0} \}$$

Addition:

$$3.1415926\ldots + 3.1415926\ldots = 6.283185\ldots.$$

Multiplication:

$$(3.1415926\ldots)(3.1415926\ldots) = 9.86\ldots$$

since

$$\begin{array}{r} 3.14 \\ 3.14 \\ \hline 1256 \\ 314 \\ 942 \\ \hline 9.8596 \end{array} \quad \text{and} \quad \begin{array}{r} 3.141 \\ 3.141 \\ \hline 3141 \\ 12564 \\ 3141 \\ 9423 \\ \hline 9.865881 \end{array}$$

We steal these operations from $\mathbb{Q}$ using

$$3.14159 = 3 + \frac{1}{10} + \frac{4}{100} + \frac{1}{1000} + \frac{5}{10000} + \frac{9}{100000}$$

Important remark

$$\mathbb{R} = \{a_0.a_1a_2a_3\ldots \mid a_0 \in \mathbb{Z},\ a_i \in \{0,1,2,3,4,5,6,7,8,9\} \text{ for } i \in \mathbb{Z}_{>0}\}$$

with

$$a_0.a_1a_2a_3\ldots = b_0.b_1b_2b_3\ldots \quad \text{if}$$

$$a_0.a_1a_2a_3\ldots + (-b_0.b_1b_2b_3\ldots) = 0.000\ldots$$

Example $1.000\ldots + (-0.999\ldots) = 0.0000\ldots$

So $1.000\ldots = .999\ldots$

Another way to see this is

$$.9999\ldots = \frac{9}{10}\left(1 + \frac{1}{10} + \frac{1}{100} + \cdots\right)$$

$$= \frac{9}{10}\left(\frac{1}{1-\frac{1}{10}}\right) \qquad \left(\text{since } 1+x+x^2+x^3+\cdots = \frac{1}{1-x}\right)$$

$$= \frac{9}{10}\ \frac{1}{9/10} = 1$$

$$= 1.000\ldots$$

We should really define

$$\mathbb{R}_{>0} = \{a_0.a_1a_2a_3\ldots \mid a_0 \in \mathbb{Z}_{\geq 0},\ a_i \in \{0,1,\ldots,9\} \text{ for } i \in \mathbb{Z}_{>0}\}$$

$$\mathbb{R}_{<0} = \{-a_0.a_1a_2a_3\ldots \mid a_0 \in \mathbb{Z}_{\geq 0},\ a_i \in \{0,1,\ldots,9\} \text{ for } i \in \mathbb{Z}_{>0}\}$$

and $\mathbb{R} = \mathbb{R}_{>0} \cup \mathbb{R}_{<0}$ with $a_0.a_1a_2\ldots = -b_0.b_1b_2\ldots$ if

$$a_0.a_1a_2a_3\ldots + b_0.b_1b_2\ldots = 0.000\ldots$$

Complex numbers

$\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$ with $i^2 = -1$.

Addition:

$$(3+4i) + (7+9i) = 10 + 13i.$$

$$(3+4i) + (0+0i) = 3+4i.$$

$$(3+4i) + (-3+(-4)i) = 0+0i.$$

Multiplication

$$(3+4i)(7+9i) = 21 + 28i + 27i + 36i^2$$

$$= 21 - 36 + 55i$$

$$= -15 + 55i.$$

$$(3+4i)(1+0i) = 3 + 0i + 4i + 0i^2 = 3+4i.$$

$$(3+4i)\left(\frac{3}{25} - \frac{4}{25}i\right) = \frac{9}{25} + \frac{12}{25}i - \frac{12}{25}i - \frac{16}{25}i^2$$

$$= \frac{9}{25} + \frac{16}{25} + 0i = 1 + 0i.$$

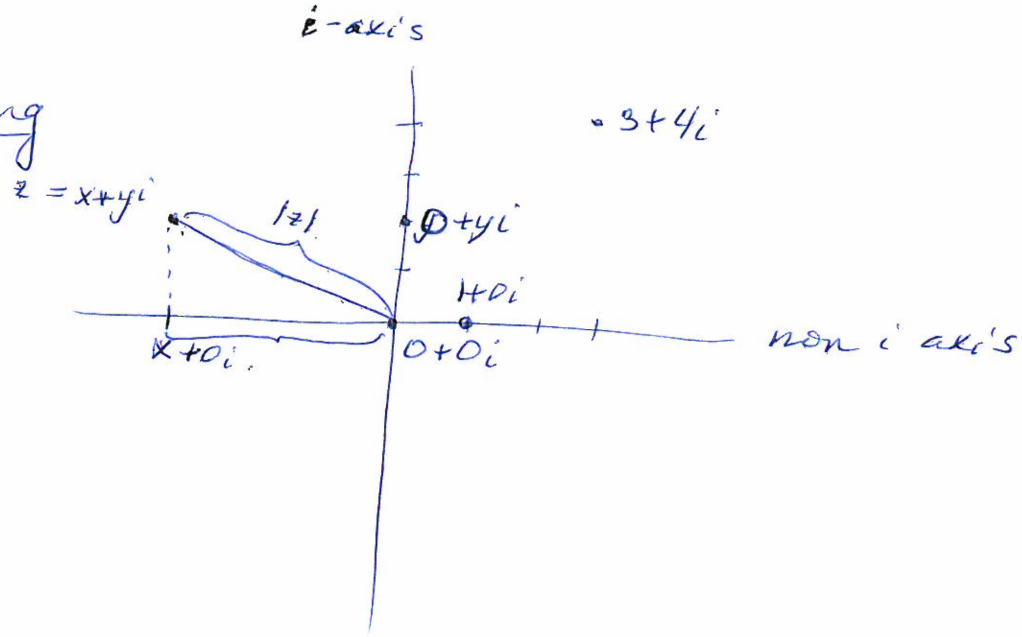
Let $z = x + yi$, with $x, y \in \mathbb{R}$.

The conjugate of z is $\bar{z} = x - yi$.

The absolute value of z is

$$|z| = \sqrt{x^2 + y^2}.$$

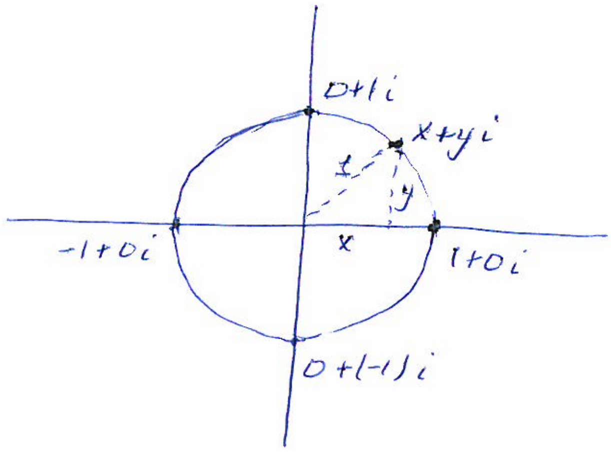
Graphing



Example: $\{z \in \mathbb{C} \mid |z| = 1\}$

$= \{x+yi \mid x,y \in \mathbb{R} \text{ and } \sqrt{x^2+y^2} = 1\}$

is



a circle of radius 1,

since

$\{x+yi \mid x,y \in \mathbb{R} \text{ and } x^2+y^2 = 1\}$

$= \{x+yi \mid x+yi \text{ is distance } 1 \text{ from } 0+0i\}$.