620-295 Real Analysis with applications   Lecture 2, 28.07.2009

A set is a collection of elements.

Let S and T be sets.

The product of S and T is the set

$$S \times T = \{(s,t) \mid s \in S,\ t \in T\}$$

Example: If $S = \{\triangle, \square\}$ and $T = \{\forall, \exists\}$

then

$$S \times T = \left\{ \begin{matrix} (\triangle,\forall), (\triangle,\exists) \\ (\square,\forall), (\square,\exists) \end{matrix} \right\}.$$

Number systems

$\mathbb{Z}_{>0} = \{2,1,4,3,6,5,8,7,\ldots\}$ with $+ : \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$
$(s,t) \mapsto s+t.$

$\mathbb{Z}_{\geq 0} = \{2,1,0,4,3,\ldots\}$ with $+ : \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$
$(s,t) \mapsto s+t$

$\mathbb{Z} = \{1,-1,2,-2,0,3,-3,\ldots\}$ with $+ : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$
$(s,t) \mapsto s+t.$

BUT what if you only want part of the sausage ... and so we discovered ...

the set of rational numbers

$$\mathbb{Q} = \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z},\ b \neq 0 \right\}$$

with $\frac{a}{b} = \frac{c}{d}$ if $ad = bc$.

The clock: $\mathcal{C} = \{10, 11, 12, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ with

$$+ : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$$

$(3, 4) \longmapsto 7$ so that $3 + 4 = 7$

$(12, 5) \longmapsto 5$ $\quad 12 + 5 = 5$

$(10, 7) \longmapsto 5$ $\quad 10 + 7 = 5$

Types of number systems

{commutative monoids possibly without identity}

{commutative monoids}

{abelian groups}

{rings}

{fields}

A **commutative monoid possibly without identity** is a set Q with an operation $+ : Q \times Q \to Q$ such that

(a) If $s_1, s_2, s_3 \in Q$ then $(s_1 + s_2) + s_3 = s_1 + (s_2 + s_3)$

(b) If $s_1, s_2 \in Q$ then $s_1 + s_2 = s_2 + s_1$

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(b) If $s_1, s_2 \in Q$ then $s_1 + s_2 = s_2 + s_1$

(c) There exist an element $\heartsuit \in S$ such that if $s \in Q$ then $\heartsuit + s = s + \heartsuit = s$.

An **abelian group** is a set Q with an operation $+ : Q \times Q \to Q$ such that

(a) If $s_1, s_2, s_3 \in Q$ then $(s_1 + s_2) + s_3 = s_1 + (s_2 + s_3)$,

(b) If $s_1, s_2 \in Q$ then $s_1 + s_2 = s_2 + s_1$,

(c) There exists an element $\heartsuit \in Q$ such that if $s \in Q$ then $\heartsuit + s = s + \heartsuit = s$.

(d) If $s \in Q$ then there exists an element $-s \in Q$ such that

$$s + (-s) = \heartsuit \text{ and } (-s) + s = \heartsuit.$$

A *ring* is a set Q with two operations

$+: Q \times Q \to Q$ and $\cdot: Q \times Q \to Q$ such that

(a) If $s_1, s_2, s_3 \in Q$ then $s_1 + (s_2 + s_3) = (s_1 + s_2) + s_3$,

(b) If $s_1, s_2 \in Q$ then $s_1 + s_2 = s_2 + s_1$

(c) There exists an element $\heartsuit \in Q$ such that

if $s \in Q$ then $\heartsuit + s = s$ and $s + \heartsuit = s$

(d) If $s \in Q$ then there exists an element $-s \in S$ such that

$$s + (-s) = \heartsuit \text{ and } (-s) + s = \heartsuit$$

(e) If $s_1, s_2, s_3 \in Q$ then $(s_1 \cdot s_2) s_3 = s_1 (s_2 s_3)$.

(f) There exists an element $1 \in Q$ such that

if $s \in Q$ then $1s = s1 = s$

(g) If $s_1, s_2, s_3 \in Q$ then $s_1(s_2 + s_3) = s_1 s_2 + s_1 s_3$

(h) If $s_1, s_2, s_3 \in Q$ then $(s_1 + s_2) s_3 = s_1 s_3 + s_2 s_3$

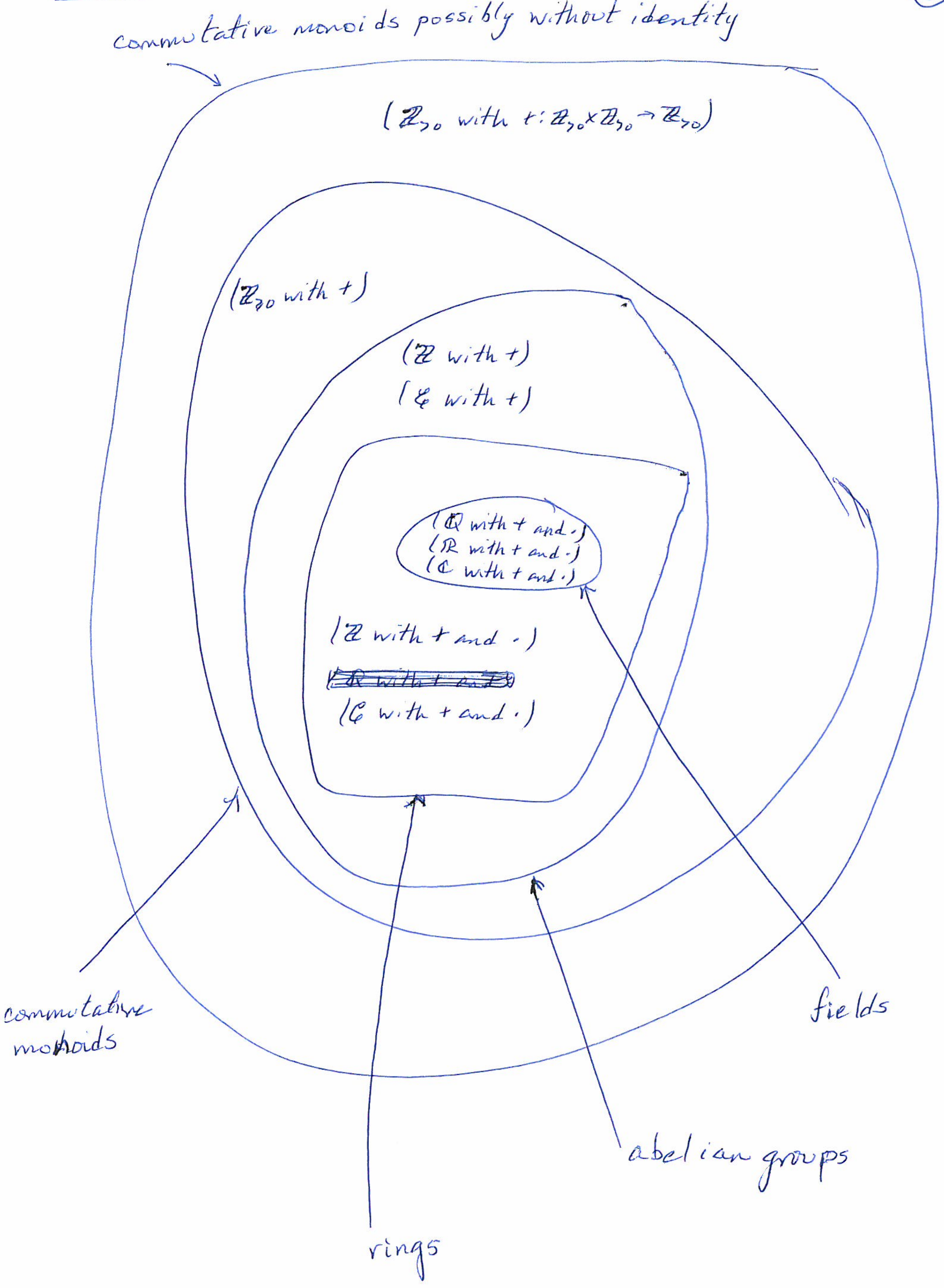
A *field* is a ring Q with $+: Q \times Q \to Q$ and $\cdot: Q \times Q \to Q$ such that

(i) If $s_1, s_2 \in Q$ then $s_1 s_2 = s_2 s_1$

(j) If $s \in Q$ and $s \neq \heartsuit$ then there exists

$s^{-1} \in Q$ such that $s \cdot s^{-1} = 1$ and $s^{-1} s = 1$.

The Zoo



commutative monoids possibly without identity
($\mathbb{Z}_{\geq 0}$ with $+ : \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$)
($\mathbb{Z}_{\geq 0}$ with $+$)
($\mathbb{Z}$ with $+$)
($\mathbb{C}$ with $+$)
($\mathbb{Q}$ with $+$ and $\cdot$)
($\mathbb{R}$ with $+$ and $\cdot$)
($\mathbb{C}$ with $+$ and $\cdot$)
($\mathbb{Z}$ with $+$ and $\cdot$)
~~($\mathbb{R}$ with $+$ and $\cdot$)~~
($\mathbb{C}$ with $+$ and $\cdot$)
commutative monoids
fields
abelian groups
rings