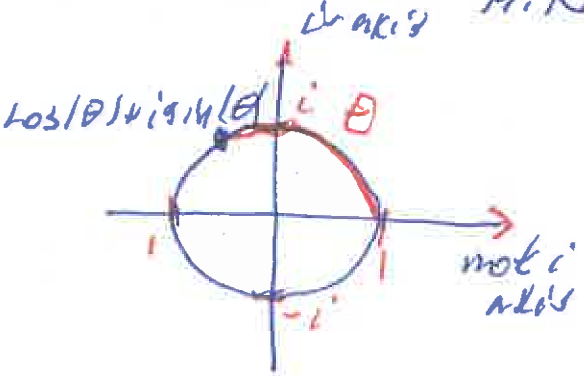
Trig identities

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$



Since $x^2+y^2=1$ on the unit circle

$$\cos(\theta)^2 + \sin(\theta)^2 = 1$$

Because $e^{i(\theta+\psi)} = e^{i\theta}e^{i\psi}$ then

$$\cos(\theta+\psi) = \cos(\theta)\cos(\psi) - \sin(\theta)\sin(\psi)$$
$$\sin(\theta+\psi) = \sin(\theta)\cos(\psi) + \cos(\theta)\sin(\psi)$$

So

$$\cos(2\theta) = \cos(\theta+\theta) = \cos(\theta)^2 - \sin(\theta)^2$$
$$\sin(2\theta) = \sin(\theta+\theta) = 2\sin(\theta)\cos(\theta)$$

From the xy-coordinates on a circle

$$\cos(-\theta) = \cos(\theta), \qquad \cos(0) = 1, \qquad \cos\left(\tfrac{\pi}{2}\right) = 0$$
$$\sin(-\theta) = -\sin(\theta), \qquad \sin(0) = 0, \qquad \sin\left(\tfrac{\pi}{2}\right) = 1.$$

Definitions

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}, \quad \sec(\theta) = \frac{1}{\cos(\theta)}, \quad \csc(\theta) = \frac{1}{\sin(\theta)}$$

$$\cot(\theta) = \frac{1}{\tan(\theta)}$$

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Proof by induction

To show: If $n \in \mathbb{Z}_{>0}$ then $\frac{dx^n}{dx} = nx^{n-1}$.

MUST use the definitions of $\mathbb{Z}_{>0}$ and $\frac{d}{dx}$

$\mathbb{Z}_{>0} = \{1, 2, 3, \ldots\}$ where
$$\begin{aligned} 2 &= 1+1 \\ 3 &= 2+1 \\ 4 &= 3+1 \\ &\vdots \end{aligned}$$

$\frac{d}{dx}$ satisfies

(a) $\frac{d}{dx}(c_1 f_1 + c_2 f_2) = c_1 \frac{df_1}{dx} + c_2 \frac{df_2}{dx}$
if c_1, c_2 are constants

(b) $\frac{d}{dx}(f_1 f_2) = f_1 \frac{df_2}{dx} + \frac{df_1}{dx} f_2$

(c) $\frac{dx}{dx} = 1$.

Having reviewed these we can do the proof.

Base case $n=1$ $\qquad \frac{dx}{dx} = 1 = 1x^0 = 1x^{1-1}$

Base case $n=2$

$$\frac{dx^2}{dx} = \frac{d\,x \cdot x}{dx} = x\frac{dx}{dx} + \frac{dx}{dx} \cdot x = x \cdot 1 + 1 \cdot x = x + x$$
$$= 2x = 2x^{2-1}$$

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Base case $n=3$ (note $3=2+1$)

$$\frac{dx^3}{dx} = \frac{d\, x \cdot x^2}{dx} = x\frac{dx^2}{dx} + \frac{dx}{dx}x^2$$

$$= x\,2x^{2-1} + 1\cdot x^2$$ by the previous case

$$= 2x^2 + x^2$$

$$= 3x^2 = 3x^{3-1}$$

Base case $n=4$

$$\frac{dx^4}{dx} = \frac{d\, x\, x^3}{dx} = x\frac{dx^3}{dx} + \frac{dx}{dx}\cdot x^3$$

$$= x\,3x^{3-1} + 1\cdot x^3$$ by the previous case

$$= 3x^3 + x^3 = 4x^3$$

$$= 4x^{4-1}$$

⋮

Base case $n=6337$

$$\frac{dx^{6337}}{dx} = \frac{d\, x\cdot x^{6336}}{dx} = x\frac{dx^{6336}}{dx} + \frac{dx}{dx}\cdot x^{6336}$$

$$= x\cdot 6336x^{6336-1} + 1\cdot x^{6336}$$ by the previous case

$$= 6336x^{6336} + x^{6336}$$

$$= 6337x^{6336}$$

Do enough base cases to see how the previous case is used and what the pattern is.

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<u>Induction step</u> (i.e. Base case $n \le n$)

$$\frac{dx^n}{dx} = \frac{d\,x x^{n-1}}{dx} = x\frac{dx^{n-1}}{dx} + \frac{dx}{dx}x^{n-1}$$

$$= x\cdot(n-1)x^{n-2} + 1\cdot x^{n-1},$$ by the induction assumption (i.e. by the previous case)

$$= (n-1)x^{n-1} + x^{n-1}$$

$$= n x^{n-1}.$$