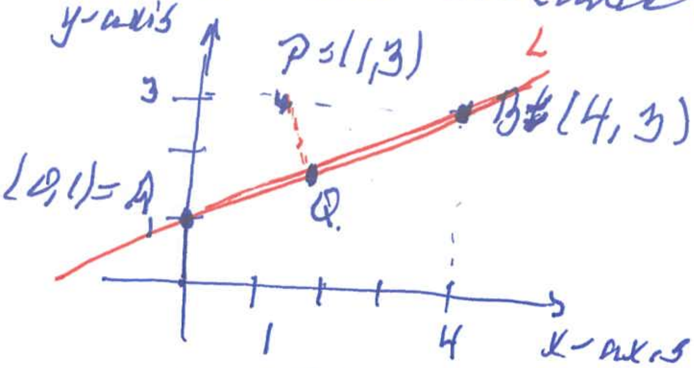
23.05.2025
Calculus Lect (4)
A. Ram

7.24 Let $P=(1,3)$ and

L the line passing through $A=(0,1)$ and $B=(4,3)$.

Find the closest point Q on the line L to P and find the distance from P to L.



$\vec{AP} = P - A = (1,3) - (0,1) = (1,2)$

$\vec{AB} = B - A = (4,3) - (0,1) = (4,2)$

$$\text{proj}_{\vec{AB}}(\vec{AP}) = \frac{1}{\|\vec{AB}\|^2} \cdot \langle \vec{AP}, \vec{AB} \rangle \cdot \vec{AB}$$

$$= \frac{1}{4^2+2^2} \cdot (1\cdot 4 + 2\cdot 2) \cdot (4,2) = \frac{1}{20} \cdot 8 \cdot (4,2)$$

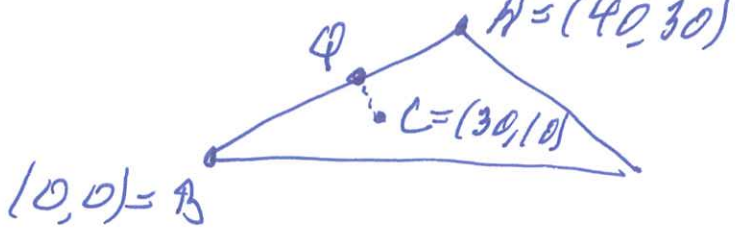
$$= \left(\frac{32}{20}, \frac{16}{20}\right) = \left(\frac{8}{5}, \frac{4}{5}\right)$$

$$Q = A + \left(\frac{8}{5}, \frac{4}{5}\right) = (0,1) + \left(\frac{8}{5}, \frac{4}{5}\right) = \left(\frac{8}{5}, \frac{9}{5}\right)$$

$$\vec{QP} = P - Q = (1,3) - \left(\frac{8}{5}, \frac{9}{5}\right) = \left(-\frac{3}{5}, \frac{6}{5}\right)$$

$$\|\vec{QP}\| = \sqrt{\frac{3^2}{5^2} + \frac{6^2}{5^2}} = \sqrt{\frac{9+36}{25}} = \frac{\sqrt{40}}{5} = \frac{2\sqrt{2}}{\sqrt{5}}.$$

7.25 Lara Croft believes there is a secret chamber in an ancient pyramid at the position C.



$A=(40,30)$
$B=(0,0)$
$C=(30,10)$

Lara plans to dig a shaft from the surface of the pyramid to the chamber. To minimise the length of the shaft

(1) where should she start digging?
(2) How long will the shaft be?

$$\vec{BA} = A - B = (40,30)$$
$$\vec{BC} = C - B = (30,10)$$

$$Q = \mathrm{proj}_{\vec{BA}}(\vec{BC}) = \frac{1}{\|\vec{BA}\|^2}\langle \vec{BA}, \vec{BC}\rangle \vec{BA}$$
$$= \frac{1}{40^2+30^2}\cdot\langle (40,30),(30,10)\rangle\cdot(40,30)$$
$$= \frac{1}{2500}\cdot(1200+300)\cdot(40,30)$$
$$= \frac{15}{25}(40,30) = \frac{3}{5}(40,30) = (24,18)$$

$$\vec{QC} = C - Q = (30,10)-(24,18) = (6,-8)$$

$$\|\vec{QC}\| = \sqrt{6^2+8^2} = \sqrt{2^2\cdot 3^2 + 2^2\cdot 4^2} = 2\cdot 5 = 10.$$

Flies

A <u>fly in $\mathbb{R}^2$</u> is a function $r: \mathbb{R} \to \mathbb{R}^2$

A <u>fly in $\mathbb{R}^3$</u> is a function $r: \mathbb{R} \to \mathbb{R}^3$.

<u>Examples of flies in $\mathbb{R}^2$</u> $r(t) = x(t)\hat{\imath} + y(t)\hat{\jmath}$
$= (x(t), y(t))$.

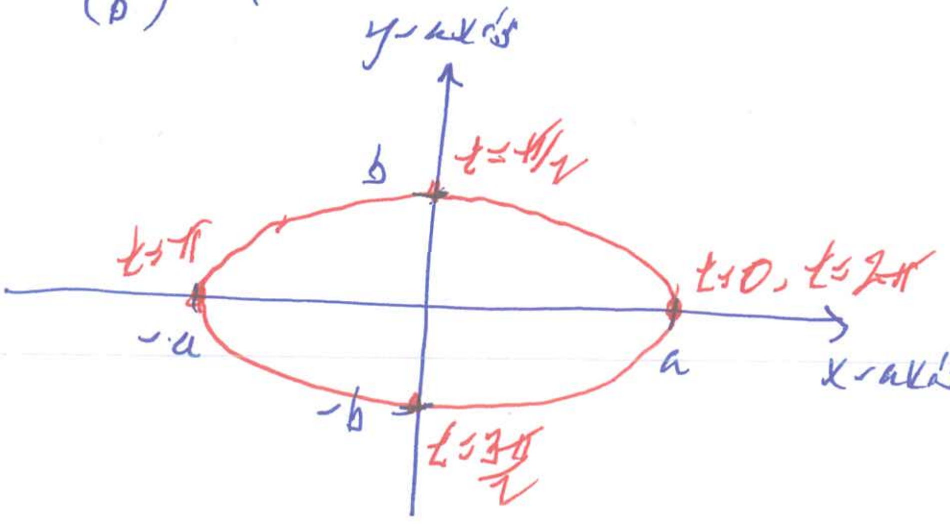
<u>7.26</u> $r(t) = \cos(t)\hat{\imath} + \sin(t)\hat{\jmath}$ (7.26 with $a=1$ and $b=1$)

<u>7.27</u> $r(t) = a\cos(t)\hat{\imath} + b\sin(t)\hat{\jmath}$, with $a, b \in \mathbb{R}_{>0}$.

<u>7.28</u> $r(t) = \cos(t)\hat{\imath} + 2\sin(t)\hat{\jmath}$. (7.26 with $a=1$ and $b=2$)

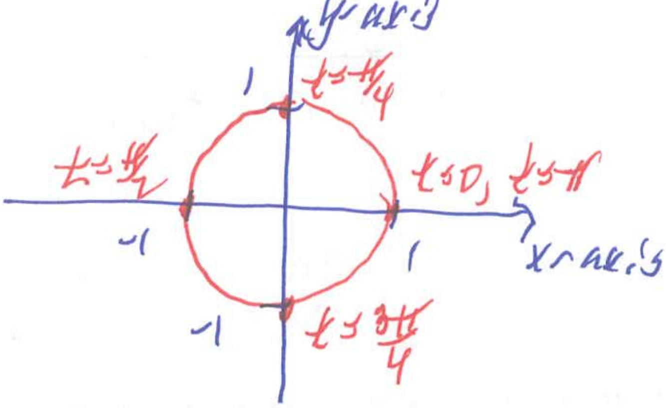
For 7.27, $x(t) = a\cos(t)$, $\frac{x}{a} = \cos(t)$,
$y(t) = b\sin(t)$, $\frac{y}{b} = \sin(t)$, and

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$



7.29 $r(t) = \cos(2t)\hat{i} + \sin(2t)\hat{j}$

is just like $r(t) = \cos(t)\hat{i} + \sin(t)\hat{j}$
except the fly flies twice as fast.



7.30 Let $r_1 : \mathbb{R} \to \mathbb{R}^2$ ~~be given by~~ and $r_2 : \mathbb{R} \to \mathbb{R}^2$
be given by

$$r_1(t) = (t+1)\hat{i} + (t^2 - 4t)\hat{j}$$
$$r_2(t) = 2t\hat{i} + (6t-9)\hat{j}$$

(a) Determine the times and points at which the flies collide

(b) The distance between the flies at $t=4$.

(a) If $r_1(t) = r_2(t)$ then $t+1 = 2t$ and
$t^2 - 4t = 6t - 9$.

So $1 = t$ and $1^2 - 4 = 6 - 9$.

So $t = 1$. and then $r_1(1) = (1+1)\hat{i} + (1^2 - 4\cdot 1)\hat{j}$
$= 2\hat{i} - 3\hat{j} = (2, -3)$
is where they collide

(b) The distance between $r_1(9)$ and $r_2(4)$ is

$\| r_2(9) - r_1(4) \| = \| (2\cdot 9\hat{i} + (6\cdot 9 - 9)\hat{j}) - ((4+1)\hat{i} + (9^2 - 4\cdot 9)\hat{j}) \|$

$= \| (18\hat{i} + 45\hat{j}) - (10\hat{i} + 45\hat{j}) \|$

$= \| (18-10)\hat{i} + (45-45)\hat{j} \| = \| 8\hat{i} \| = \| (8,0) \|$

$= \sqrt{8^2 + 0^2} = 8.$

For $r_1(t)$: $x = t+1$, $y = t^2 - 4t$ and $t = x-1$

$y = (x-1)^2 - 4(x-1)$

$= x^2 - 2x + 1 - 4x + 4$

$= x^2 - 6x + 5 = (x-5)(x-1)$

For $r_2(t)$: $x = 2t$, $y = 6t - 9$ and $t = x/2$

$y = 6\left(\frac{x}{2}\right) - 9 = 3x - 9.$