14.05.2015 (1)

Calculus Lect.
A. Ram

Vectors

$\mathbb{C}^n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in \mathbb{C}\}$

$\mathbb{R}^n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in \mathbb{R}\}$

$\mathbb{Z}^n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in \mathbb{Z}\}$

vectors or $1\times n$ matrices.

Examples: $(3, 1, 0) \in \mathbb{Z}^3$

$(\sqrt{3}, 1, \pi, 2) \in \mathbb{R}^4$

$(\sqrt{3}, 1, \pi, 2, 3+\sqrt{2}i, 7e^{i\pi/9}, \sqrt{2}-\sqrt{3}i) \in \mathbb{C}^7$.

Addition is the function $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ given by

$$(a_1, a_2, \ldots, a_n) + (b_1, \ldots, b_n) = (a_1+b_1, \ldots, a_n+b_n).$$

Scalar multiplication is the function $\mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ given by

$$c\cdot(a_1, \ldots, a_n) = (ca_1, ca_2, \ldots, ca_n).$$

7.3 Let $v = (3,-2)$ and $w = (1,5)$. Then

$v + w = (3,-2) + (1,5) = (3+1, -2+5) = (4,3)$

$3w = 3\cdot(1,5) = (3\cdot 1, 3\cdot 5) = (3, 15)$

$v - w = v + (-1)w = (3,-2) + (-1)\cdot(1,5)$

$= (3,-2) + (-1,-5) = (3-1, -2-5) = (2,-7)$

$w - v = w + (-1)v = (1,5) + (-1)(3,-2) = (1,5) + (-3,2) = (-2,7)$

The inner product is the function $\langle\, , \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ given by

(also called scalar product)

$$\langle (a_1, \ldots, a_n), (b_1, \ldots, b_n) \rangle = a_1 b_1 + \cdots + a_n b_n$$

The norm is the function $\| \; \| : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ given by

$$\| (a_1, \ldots, a_n) \| = \sqrt{a_1^2 + \cdots + a_n^2}$$

(also called length)

The metric is the function $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ given by

$$d((a_1, \ldots, a_n), (b_1, \ldots, b_n)) = \| (b_1, \ldots, b_n) - (a_1, \ldots, a_n) \|$$
$$= \sqrt{(b_1 - a_1)^2 + \cdots + (b_n - a_n)^2}$$

also called distance between a and b.

7.18 Let $u = (2, 3, -1)$ and $v = (4, 5, 0)$.

Then

$$u \cdot v = \langle u, v \rangle = \langle (2, 3, -1), (4, 5, 0) \rangle$$
$$= 2 \cdot 4 + 3 \cdot 5 + (-1) \cdot 0 = 8 + 15 + 0 = 23.$$

7.12 Let $P = (-2, 1, 0)$ and $Q = (3, -1, 1)$

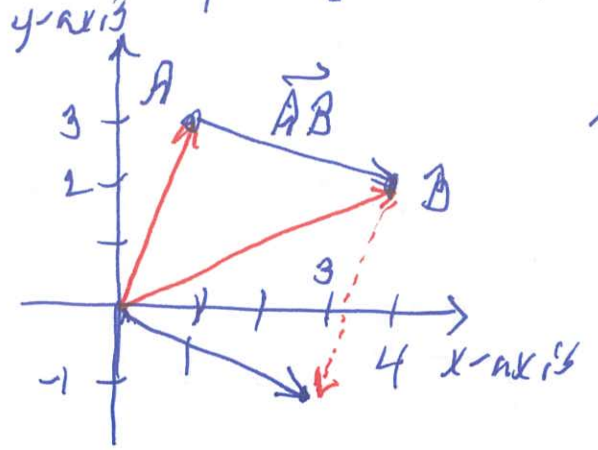
Then

$$d(P, Q) = \| \overrightarrow{PQ} \| = \| Q - P \| = \| (3, -1, 1) - (-2, 1, 0) \|$$
$$= \| (3 - (-2)), (-1 - 1), (1 - 0) \| = \| (5, -2, 1) \|$$
$$= \sqrt{(3 - (-2))^2 + (-1-1)^2 + (1-0)^2} = \sqrt{5^2 + (-2)^2 + 1^2}$$
$$= \sqrt{30}.$$

7.4 $\|(1,2)\| = \sqrt{1^2+2^2} = \sqrt{5}$ and

$\|(-1,3,2)\| = \sqrt{(-1)^2+3^2+2^2} = \sqrt{1+9+4} = \sqrt{14}$.

7.5 Graphing $A=(1,3)$ and $B=(4,2)$



$\vec{AB} = B - A = (4,2)-(1,3)$
$= (4-1, 2-3) = (3,-1)$

A unit vector is a vector u of length 1.

7.14 Let v be the vector from $A=(2,0,-1)$ to $B=(1,2,-3)$.

$v = \vec{AB} = B - A = (1,2,-3)-(2,0,-1) = (-1,2,-2)$

$\|v\| = \|(-1,2,-2)\| = \sqrt{(-1)^2+2^2+(-2)^2} = \sqrt{9} = 3$.

$\frac{1}{\|v\|}\cdot v = \frac{1}{3}(-1,2,-2) = \left(\frac{-1}{3}, \frac{2}{3}, \frac{-2}{3}\right)$ is a unit vector

since $\left\|\left(\frac{-1}{3}, \frac{2}{3}, \frac{-2}{3}\right)\right\| = \sqrt{\left(\frac{-1}{3}\right)^2+\left(\frac{2}{3}\right)^2+\left(\frac{-2}{3}\right)^2} = \sqrt{\frac{1}{9}+\frac{4}{9}+\frac{4}{9}}$

$= \sqrt{\frac{9}{9}} = \sqrt{1} = 1$.

$\left(\frac{-1}{3}, \frac{2}{3}, \frac{-2}{3}\right)$ and $\left(\frac{1}{3}, \frac{-2}{3}, \frac{2}{3}\right)$ are unit vectors parallel to v.

The <u>standard basis of $\mathbb{R}^2$</u> is

$\{\hat{\imath}, \hat{\jmath}\}$ where $\hat{\imath} = (1,0)$ and $\hat{\jmath} = (0,1)$.

The <u>standard basis of $\mathbb{R}^3$</u> is

$\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ where $\hat{\imath} = (1,0,0)$, $\hat{\jmath} = (0,1,0)$, $\hat{k} = (0,0,1)$.

<u>7.15</u> $v = (1,2) = (1,0) + (0,2) = (1,0) + 2(0,1)$

$= \hat{\imath} + 2\hat{\jmath}$,

$u = (-1,3) = -\hat{\imath} + 3\hat{\jmath}$.

<u>7.16</u> $v = (-3,1,2) = -3\hat{\imath} + \hat{\jmath} + 2\hat{k}$ and

$u = (-1,1,-2) = -\hat{\imath} + \hat{\jmath} - 2\hat{k}$.

The <u>angle between u and v</u> is the function

$\theta : \mathbb{R}^n \times \mathbb{R}^n \to [-\pi, \pi)$ given by

$$\theta((a_1, \dots, a_n), (b_1, \dots, b_n)) = \arccos\left(\frac{\langle u, v\rangle}{\|u\| \cdot \|v\|}\right)$$

where $u = (a_1, \dots, a_n)$ and $v = (b_1, \dots, b_n)$.

<u>7.22</u> $\theta((-2,1,2),(1,-1,0))$

$$= \arccos\left(\frac{(-2,1,2)\cdot(1,-1,0)}{\|(-2,1,2)\| \cdot \|(1,-1,0)\|}\right)$$

$$= \arccos\left(\frac{-2+(-1)+0}{\sqrt{2^2+1^2+2^2}\sqrt{1^2+(-1)^2+0^2}}\right)$$

$$= \arccos\left(\frac{-3}{\sqrt{9}\sqrt{2}}\right) = \arccos\left(\frac{-1}{\sqrt{2}}\right) = \frac{3\pi}{4}.$$