A. Ram 28.04.2025
Calculus

<u>4.19</u> A stone is dropped off a 40m high cliff.

Position: $p(t) = 40 - 4.9t^2$

Velocity is change of position as time changes

$$v(t) = \frac{dp}{dt} = 0 - 2\cdot(4.9)t = -9.8t$$

Acceleration is change of velocity as time changes

$$a(t) = \frac{dv}{dt} = -9.8$$

The stone hits the ground when

$0 = p(t) = 40 - 4.9t^2$. So $t^2 = \frac{-40}{-4.9} = \frac{400}{49} = \frac{20^2}{7^2}$.

The velocity when $t = \frac{20}{7}$ is

$$v\left(\frac{20}{7}\right) = -9.8 \cdot \frac{20}{7} = -2 \cdot 4.9 \cdot \frac{20}{7} = -2 \cdot \frac{49}{10} \cdot \frac{20}{7}$$

$$= -2 \cdot 7 \cdot 2 = -28.$$

i.e. 28 m/s downward.

2.48 4^{th} roots of 1 are

$$\{z \in \mathbb{C} \mid z^4 = 1\} = \{1, i, -1, i\} = \{e^0, e^{i\pi/2}, e^{i\pi}, e^{i3\pi/2}\}$$

$$= \{e^0, e^{i6\pi/12}, e^{i12\pi/12}, e^{i18\pi/12}\}$$

The number $2-2i\sqrt{3}$ has

length $|2-2i\sqrt{3}| = \sqrt{2^2 + (-2\sqrt{3})^2} = \sqrt{4+12} = 4.$

and

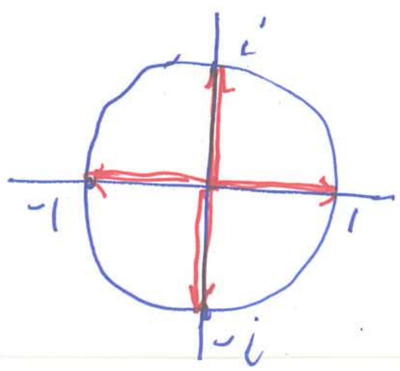
$$2-2i\sqrt{3} = 4\left(\frac{2}{4} - \frac{2\sqrt{3}}{4} i\right) = 4\left(\frac{1}{2} - \frac{\sqrt{3}}{2} i\right) = 4e^{-i\pi/3}$$

so $4^{1/4}(e^{-i\pi/3})^{1/4} = \sqrt{2}\, e^{-i\pi/12}$ is a 4^{th} root of $2-2i\sqrt{3}$

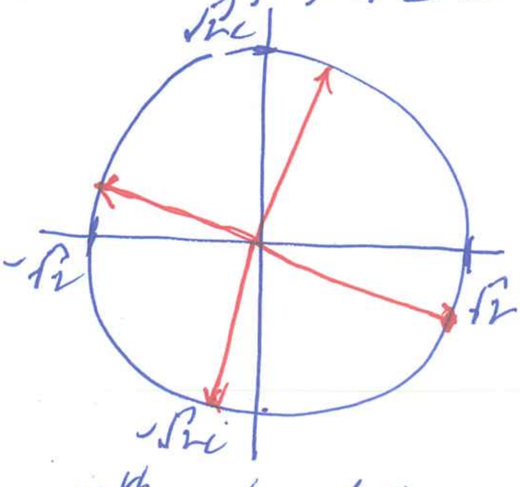
The 4^{th} roots of $2-2i\sqrt{3}$ are

$$\sqrt{2}\, e^{-i\pi/12} \cdot 1^{1/4} = \left\{ \begin{array}{l} \sqrt{2}\, e^{-i\pi/12} \cdot e^0, \ \sqrt{2}\, e^{-i\pi/12} \cdot e^{i6\pi/12}, \\ \sqrt{2}\, e^{-i\pi/12} e^{i12\pi/12}, \ \sqrt{2}\, e^{-i\pi/12} e^{i18\pi/12} \end{array} \right\}$$

$$= \{\sqrt{2}\, e^{-i\pi/12}, \sqrt{2}\, e^{i5\pi/12}, \sqrt{2}\, e^{i11\pi/12}, \sqrt{2}\, e^{i17\pi/12}\}$$



4^{th} roots of 1



4^{th} roots of $2-2i\sqrt{3}$.

Indefinite integrals (inverse derivatives)

$$\int \frac{df}{dx}\, dx = f.$$

5.7 $\int x^2 dx = \frac{1}{3}x^3 + 38$

since $\frac{d}{dx}\left(\frac{1}{3}x^3 + 38\right) = \frac{1}{3}3x^2 + 0 = x^2$

To write all correct answers at once,

$\int x^2 dx = \frac{1}{3}x^3 + c$, where c is a constant.

5.10 $\int\left(5e^{2x} - \frac{3}{x^2}\right)dx = \int\left(\frac{5}{2}\,2e^{2x} - 3x^{-2}\right)dx$

$= \int\left(\frac{5}{2}\,2e^{2x} + 3(-1)x^{-2}\right)dx$

$= \frac{5}{2}e^{2x} + 3x^{-1} + c$, where c is a constant

since

$\frac{d}{dx}\left(\frac{5}{2}e^{2x} + 3x^{-1} + c\right) = \frac{5}{2}\,2e^{2x} + 3(-1)x^{-2} + 0$

$= 5e^{2x} - \frac{3}{x^2}.$

Linearity for integrals

If c_1, c_2 are constants then

$$\frac{d}{dx}(c_1 f + c_2 g) = c_1 \frac{df}{dx} + c_2 \frac{dg}{dx} \quad \text{gives}$$

$$\int (c_1 f + c_2 g)\, dx = c_1 \int f\, dx + c_2 \int g\, dx.$$

Chain rule for integrals

$$\frac{df}{dx} = \frac{df}{du}\frac{du}{dx} \quad \text{gives} \quad \int f \frac{du}{dx}\, dx = \int f\, du$$

Product rule for integrals

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + \frac{du}{dx} v \quad \text{gives}$$

$$\int u \frac{dv}{dx}\, dx = \left(-\int v \frac{du}{dx}\right) + uv$$

or $$\int u \frac{dv}{dx}\, dx = uv - \int v \frac{du}{dx}.$$

5.21 $\int \frac{1}{2x-3}\,dx = \int \frac{1}{2}\left(\frac{1}{x-\frac{3}{2}}\right)dx$

$= \frac{1}{2}\log\left(x-\frac{3}{2}\right)+c$, where c is a constant,

since $\frac{d}{dx}\left(\frac{1}{2}\log\left(x-\frac{3}{2}\right)+c\right) = \frac{1}{2}\frac{1}{\left(x-\frac{3}{2}\right)}+0 = \frac{1}{2x-3}$.

5.13 $\int \cos(3x)\sqrt{\sin(3x)+4}\,dx$.

Let $u = \sin(3x)+4$. Then $\frac{du}{dx} = 3\cos(3x)$

$\int \frac{1}{3}\,3\cos(3x)(\sin(3x)+4)^{\frac{1}{2}}\,dx$

$= \int \frac{1}{3}\frac{du}{dx}u^{\frac{1}{2}}\,dx = \int \frac{1}{3}u^{\frac{1}{2}}\frac{du}{dx}\,dx = \int \frac{1}{3}u^{\frac{1}{2}}\,du$

$= \frac{1}{3}\cdot\frac{2}{3}u^{3/2}+c = \frac{2}{9}u^{3/2}+c$

$= \frac{2}{9}(\sin(3x)+4)^{3/2}+c$, where c is a constant.

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<u>5.14</u> $\int \frac{4x^2}{\sqrt{1-2x}}\,dx.$

Let $u = 1-2x$.
Then $x = \frac{1-u}{2}$ and
$\frac{dx}{du} = -\frac{1}{2}$.

So

$$\int \frac{4x^2}{\sqrt{1-2x}}\,dx = \int \frac{4\left(\frac{1-u}{2}\right)^2}{\sqrt{u}}\,dx = \int (1-u)^2 u^{-\frac{1}{2}} \frac{dx}{du}\,du$$

$$= \int (1-2u+u^2)u^{-\frac{1}{2}}\left(\frac{-1}{2}\right)du$$

$$= \frac{-1}{2}\int \left(u^{-\frac{1}{2}} - 2u^{\frac{1}{2}} + u^{3/2}\right)du$$

$$= \frac{-1}{2}\left(\frac{2}{1}u^{\frac{1}{2}} - 2\,\frac{2}{3}u^{3/2} + \frac{2}{5}u^{5/2}\right) + c$$

$$= -u^{\frac{1}{2}} + \frac{2}{3}u^{3/2} - \frac{1}{5}u^{5/2} + c$$

$$= -(1-2x)^{\frac{1}{2}} + \frac{2}{3}(1-2x)^{3/2} - \frac{1}{5}(1-2x)^{5/2} + c,$$

where c is a constant.