31.03.2025

Calculus Lect. 12 (1)

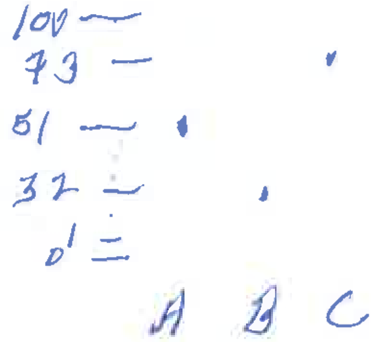
A. Ram

Functions

Example Let $S = \{A, B, C\}$ and $M = \{0, 1, \ldots, 100\}$.

$$f : \{A, B, C\} \to \{0, 1, \ldots, 100\}$$

$$A \longmapsto 51$$
$$B \longmapsto 32$$
$$C \longmapsto 73$$



has

$$\Gamma_f = \{(A, 51), (B, 32), (C, 73)\} \subseteq S \times M.$$

Definition A function $f : S \to M$ is a subset

$$\Gamma_f \subseteq S \times M \quad \text{such that}$$

(a) Every student gets a mark

if $s \in S$ then there exists $m \in M$ such that $(s, m) \in \Gamma_f$

(b) Every student gets a unique mark

if $(s, m_1) \in \Gamma_f$ and $(s, m_2) \in \Gamma_f$ then $m_1 = m_2$.

S is the source, or domain, of $f : S \to M$

M is the target, or codomain, of $f : S \to M$

Γ_f is the graph of $f : S \to M$

$$\Gamma_f = \{(s, f(s)) \mid s \in S\}$$

Examples 3.2, 3.3, 3.4

$f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \dfrac{x+1}{x^2+4}$ is a function.

$g : \mathbb{C} \to \mathbb{C}$ given by $g(x) = \dfrac{x+1}{x^2+4}$ is <u>not</u> a function

since $g(2i)$ is not defined
(i.e. the student $2i$ does not get a mark)

$h : \mathbb{C} \setminus \{2i, -2i\} \to \mathbb{C}$ given by $g(x) = \dfrac{x+1}{x^2+4}$ is a function

$\alpha : \left\{ \begin{matrix} \text{words in the} \\ \text{dictionary} \end{matrix} \right\} \to \{a, b, \dots, z\}$ given by

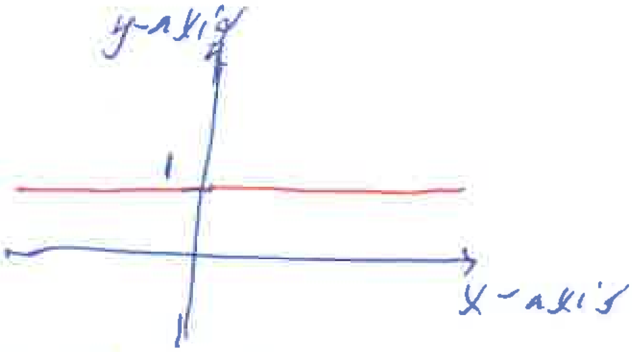
$\alpha(w) =$ first letter of w

is a function.

Example 3.5

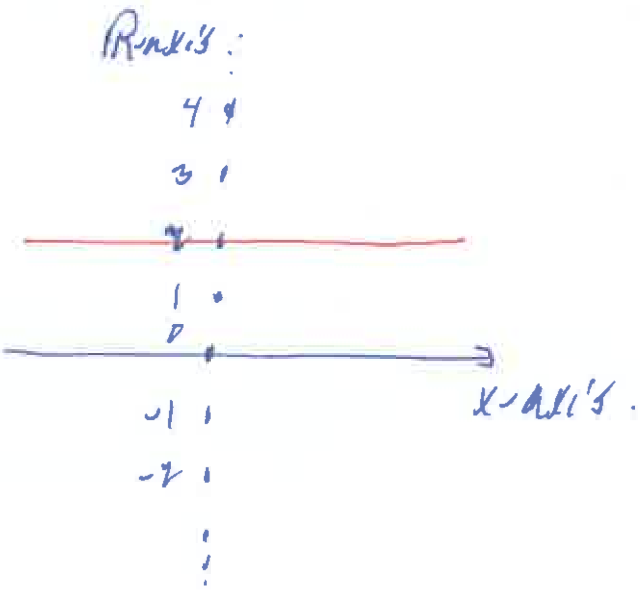
The graph of $f : \mathbb{R} \to \mathbb{R}$
$x \longmapsto 1$

is $\Gamma_f = \{(x, 1) \mid x \in \mathbb{R}\}$



The graph of $g : \mathbb{R} \to \mathbb{Z}$
$x \longmapsto 2$

is $\Gamma_g = \{(x, 2) \mid x \in \mathbb{R}\}$



The graph of $h: \mathbb{Z} \to \mathbb{Z}$, $x \mapsto 3$ is

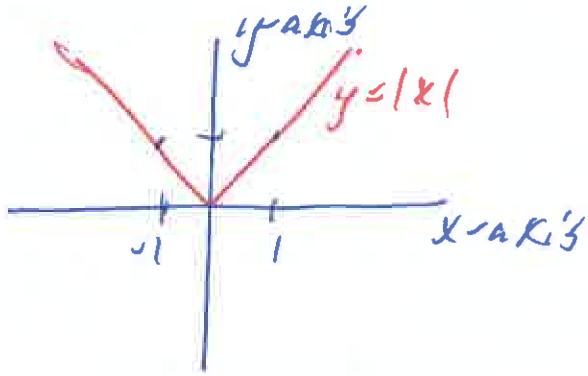
$$\Gamma_h = \{(x,3) \mid x \in \mathbb{Z}\}$$

page 10 the **absolute value function**

$\mathbb{R} \to \mathbb{R}$, $x \mapsto |x|$ given by $|x| = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x < 0 \end{cases}$

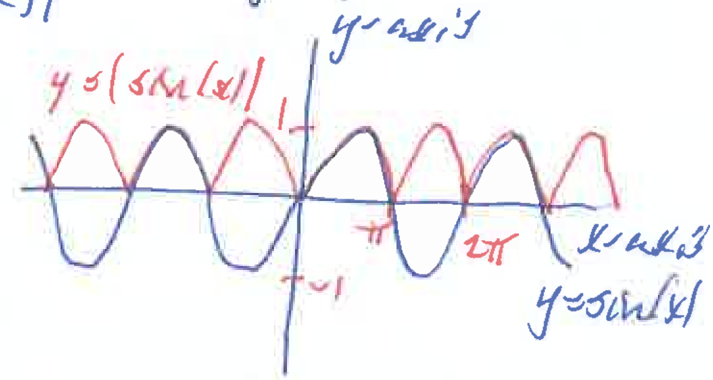
has graph

$$\Gamma_f = \{(x,x) \mid x \in \mathbb{R}_{\geq 0}\} \cup \{(-x,x) \mid x \in \mathbb{R}_{\geq 0}\}$$



Example 3.6 $f: \mathbb{R} \to \mathbb{R}_{\geq 0}$, $x \mapsto |\sin(x)|$ has graph

$$\Gamma_f = \{(x, |\sin(x)|) \mid x \in \mathbb{R}\}$$



An **injective function** is a function $f: S \to M$ such that every student gets a different mark:

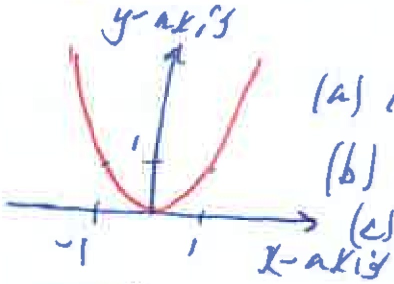
if $s_1, s_2 \in S$ and $s_1 \neq s_2$ then $f(s_1) \neq f(s_2)$

A <u>surjective function</u> is a function $f: S \to M$ such that all possible marks are given:
if $m \in M$ then there exists $s \in S$ such that $f(s) = m$.

A <u>bijective function</u> is a function $f: S \to M$ that is both injective and surjective.

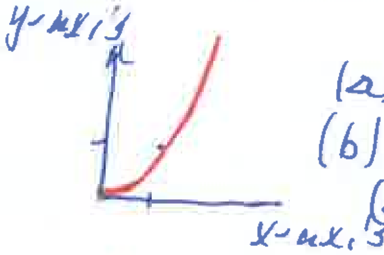
<u>Example 3.15</u>

$f: \mathbb{R} \to \mathbb{R}_{\geq 0}$
$x \longmapsto x^2$



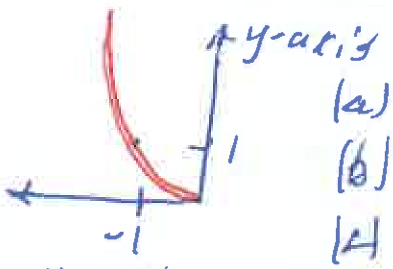
(a) is <u>not</u> injective,
(b) is surjective,
(c) is not bijective

$g: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$
$x \longmapsto x^2$



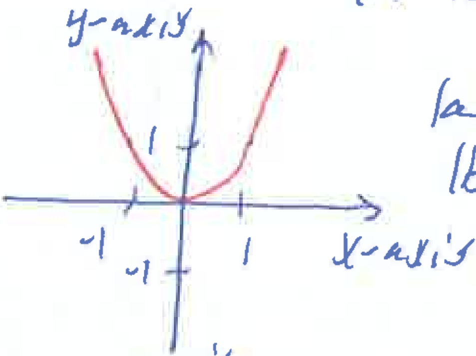
(a) is injective
(b) is surjective
(c) is bijective

$h: \mathbb{R}_{\leq 0} \to \mathbb{R}_{\geq 0}$
$x \longmapsto x^2$



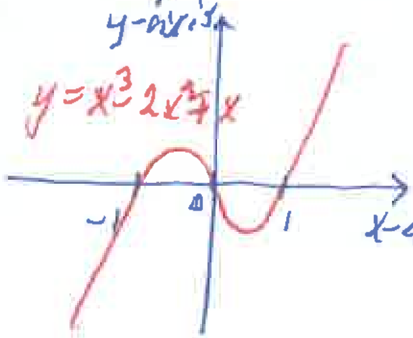
(a) is injective
(b) is surjective
(c) is bijective

$\sigma: \mathbb{R} \to \mathbb{R}$
$x \longmapsto x^2$



(a) is <u>not</u> injective
(b) is <u>not</u> surjective
(c) is not bijective

$\tau: \mathbb{R} \to \mathbb{R}$
$x \mapsto x(x-1)(x+1)$



(a) is <u>not</u> injective
(b) is surjective
(c) is <u>not</u> bijective.