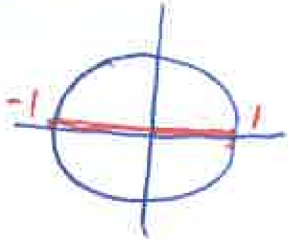
Calculus Lect 11
A. Ram

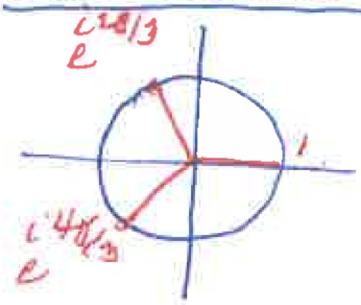
Square roots of 1



$$\{z \in \mathbb{C} \mid z^2 = 1\} = \{1, -1\} = \{e^{i0}, e^{i\pi}\}$$

$$z^2 - 1 = (z-1)(z-(-1))$$

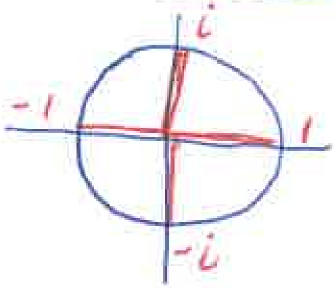
Cube roots of 1



$$\{z \in \mathbb{C} \mid z^3 = 1\} = \{1, e^{i\frac{2\pi}{3}}, e^{i\frac{4\pi}{3}}\}$$

$$z^3 - 1 = (z-1)(z - e^{i\frac{2\pi}{3}})(z - e^{i\frac{4\pi}{3}})$$

Fourth roots of 1



$$\{z \in \mathbb{C} \mid z^4 = 1\} = \{1, e^{i\frac{\pi}{2}}, e^{i\pi}, e^{i\frac{3\pi}{2}}\}$$

$$z^4 - 1 = (z-1)(z-i)(z-(-1))(z-(-i))$$

Geometric series

Since $(z-1)(z+1) = z^2 + z - z - 1 = z^2 - 1$ then

$$\frac{z^2-1}{z-1} = z+1 = \frac{\cancel{(z-1)}(z+1)}{\cancel{(z-1)}} = (z+1)$$

Since $(z-1)(z^2+z+1) = z^3 + z^2 + z - z^2 - z - 1 = z^3 - 1$ then

$$z^2 + z + 1 = \frac{z^3-1}{z-1} = \frac{\cancel{(z-1)}(z - e^{i\frac{2\pi}{3}})(z - e^{i\frac{4\pi}{3}})}{\cancel{(z-1)}} = (z - e^{i2\pi/3})(z - e^{i\frac{4\pi}{3}})$$

Since

$$(z-1)(z^3+z^2+z+1) = z^4+z^3+z^2+z - z^3 - z^2 - z - 1 = z^4 - 1$$

then

$$z^4+z^3+z^2+z = \frac{z^4-1}{z-1} = \frac{\cancel{(z-1)}(z-i)(z+1)(z+i)}{\cancel{(z-1)}} = (z-i)(z+i)\cdot(z+1)$$

Fundamental theorem of algebra

$$\mathbb{C}[z] = \left\{ \begin{array}{c} \text{polynomials in } z \text{ with} \\ \text{complex coefficients} \end{array} \right\}$$

$$= \left\{ a_0 + a_1 z + a_2 z^2 + \cdots + a_\ell z^\ell \;\middle|\; \begin{array}{l} \ell \in \mathbb{Z}_{\geq 0} \text{ and} \\ a_i \in \mathbb{C} \end{array} \right\}$$

Examples 2.55 and 2.54

$$z^3 - 3iz^2 - 2z = z(z^2 - 3iz - 2)$$

$$= z(z-2i)(z-i) = (z-0)(z-2i)(z-i).$$

Theorem Every polynomial in $\mathbb{C}[z]$ can be completely factored in $\mathbb{C}[z]$ with factors $(z-a)$.

If $p(z) = a_n z^n + \cdots + a_1 z + a_0 \in \mathbb{C}[z]$ with $a_n = 1$ then there exist $r_1, \ldots, r_n \in \mathbb{C}$ such that

$$z^n + \cdots + a_1 z + a_0 = (z-r_1)(z-r_2)\cdots(z-r_n).$$

28.03.2015
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<u>2.61</u> If $z^4+z^2-12=0$ then

$$z^2 = \frac{-1\pm\sqrt{1^2-4\cdot1\cdot(-12)}}{2} = \frac{-1\pm\sqrt{1+48}}{2} = \frac{-1\pm\sqrt{49}}{2}$$

$$= \frac{-1\pm7}{2} = \left\{\frac{6}{2}, \frac{-8}{2}\right\} = \{3,-4\}.$$

So

$$z^4+z^2-12 = (z^2-3)(z^2-(-4))$$

$$= (z-\sqrt{3})(z+\sqrt{3})\cdot(z-2i)(z+2i)$$

<u>2.53</u> $$\sin(x)^4 = \left(\frac{1}{2i}(e^{ix}-e^{-ix})\right)^4 = \frac{1}{2^4 i^4}(e^{ix}-e^{-ix})^4$$

$$= \frac{1}{16}\cdot\frac{1}{1}\left(e^{ix}+(-e^{-ix})\right)^4$$

$$= \frac{1}{16}\left((e^{ix})^4+4(e^{ix})^3(-e^{-ix})+6(e^{ix})^2(-e^{-ix})^2 + 4e^{ix}(-e^{-ix})^3+(-e^{-ix})^4\right)$$

$$= \frac{1}{16}\left(e^{i4x}-4e^{i(3x-x)}+6e^{i(2x-2x)} - 4e^{i(x-3x)}+e^{-i4x}\right)$$

$$= \frac{1}{16}\left(e^{i4x}+e^{-i4x}-4(e^{i2x}+e^{-i2x})+6\right)$$

$$= \frac{1}{16}\left(2\tfrac{1}{2}(e^{i4x}+e^{-i4x})-4\cdot2\tfrac{1}{2}(e^{i2x}+e^{-i2x})+6\right)$$

$$= \frac{1}{16}\left(2\cos(4x)-8\cos(2x)+6\right)$$

$$= \frac{1}{8}\cos(4x)-\frac{1}{2}\cos(2x)+3/8.$$

Not fundamental theorem of algebra

$\mathbb{R}[z] = \left\{ \begin{array}{c} \text{polynomials in } z \text{ with} \\ \text{real coefficients} \end{array} \right\}$

$= \left\{ a_0 + a_1 z + \cdots + a_\ell z^\ell \;\middle|\; \begin{array}{c} \ell \in \mathbb{Z}_{\geq 0} \text{ and} \\ a_i \in \mathbb{R} \end{array} \right\}$

Theorem Every polynomial in $\mathbb{R}[z]$ can be factored completely in $\mathbb{R}[z]$ with factors $(z-r)$ is a false statement.

Example: $z^2+1 \in \mathbb{R}[z]$ cannot be factored in $\mathbb{R}[z]$.

$z^2+1 \in \mathbb{C}[z]$ can be factored in $\mathbb{C}[z]$:

$$z^2+1 = (z-i)(z+i).$$

Theorem Every polynomial in $\mathbb{R}[z]$ can be factored completely in $\mathbb{R}[z]$ with factors

$$z-r, \text{ with } r \in \mathbb{R}$$

and z^2+bz+c, with $b, c \in \mathbb{R}$ such that $b^2-4ac < 0$.

Example $z^4+z^2-12 \in \mathbb{R}[z]$ and

$$z^4+z^2-12 = (z^2-3)(z^2+4) = (z-\sqrt{3})(z+\sqrt{3})(z^2+4)$$

in $\mathbb{R}[z]$ (all coefficients in $\mathbb{R}$).