24.03.2025 ①

Calculus Lect. 9

A. Ram

Algebra

$(x+y)^2 = (x+y)(x+y) = xx + xy + yx + y^2$

$= x^2 + 2xy + y^2$

$(x+y)^3 = (x+y)(x+y)^2 = (x+y)(x^2+2xy+y^2)$

$= x^3 + yx^2 + 2x^2y + 2xy^2 + xy^2 + y^3$

$= x^3 + 3x^2y + 3xy^2 + y^3$

$(x+y)^4 = (x+y)(x+y)^3 = (x+y)(x^3+3x^2y+3xy^2+y^3)$

$= x^4 + 3yx^3 + 3x^3y + 3yx^2y + 3x^2y^2 + 3yxy^2 + xy^3 + y^4$

$= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

$(x+y)^0$	1	1
$(x+y)^1$	$x+y$	1 1
$(x+y)^2$	$x^2+2xy+y^2$	1 2 1
$(x+y)^3$	$x^3+3x^2y+3xy^2+y^3$	1 3 3 1
$(x+y)^4$	$x^4+4x^3y+6x^2y^2+4xy^3+y^4$	1 4 6 4 1

So 2.52

$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$ and

$(a-b)^4 = (a+(-b))^4 = a^4 + 4a^3(-b) + 6a^2(-b)^2 + 4a(-b)^3 + (-b)^5$

$= a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4.$

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Calculus Lect 4.
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Trigonometry

From tutorial 1:

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$$
$$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y).$$

Let
$$p(x) = \cos(x) + i\sin(x)$$
Then 2.30
$$p(x)p(y) = (\cos(x) + i\sin(x))(\cos(y) + i\sin(y))$$
$$= \cos(x)\cos(y) + i\sin(x)\cos(y)$$
$$+ i\cos(x)\sin(y) + i^2\sin(x)\sin(y)$$
$$= \cos(x)\cos(y) - \sin(x)\sin(y)$$
$$+ i(\sin(x)\cos(y) + \cos(x)\sin(y))$$
$$= \cos(x+y) + i\sin(x+y)$$
$$= p(x+y).$$

So p converts multiplication to addition!!

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<u>Converting multiplication to addition</u> Calculus Lect. 4

A. Ram

Find $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$

such that $p(x+y) = p(x)p(y)$.

$$
\begin{aligned}
p(x)p(y) &= (a_0 + a_1 x + a_2 x^2 + \cdots)(a_0 + a_1 y + a_2 y^2 + \cdots) \\
&= a_0^2 + a_0 a_1 y + a_0 a_2 y^2 + \cdots \\
&\quad + a_1 a_0 x + a_1^2 xy + a_1 a_2 x y^2 + \cdots \\
&\quad + a_2 a_0 x + a_2 a_1 x^2 y + a_2 x^2 y^2 + \cdots
\end{aligned}
$$

and

$$
\begin{aligned}
p(x+y) = {} & a_0 \\
& + a_1 x + a_1 y \\
& + a_2 x^2 + 2a_2 xy + a_2 y^2 \\
& + a_3 x^3 + 3a_3 x^2 y + 3a_3 x y^2 + a_3 y^3 + \cdots
\end{aligned}
$$

So $a_1 x = a_1 a_0 x$, $a_1^2 xy = 2a_2 xy$, $a_1 a_2 x y^2 = 3a_3 x y^2, \ldots$

So $a_0 = 1$, $a_2 = \frac{1}{2} a_1^2$, $a_3 = \frac{1}{3} a_1 a_2 = \frac{1}{3 \cdot 2} a_1 a_1^2 = \frac{1}{3!} a_1^3$

So

$$
\begin{aligned}
p(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \\
&= 1 + a_1 x + \frac{1}{2} a_1^2 x^2 + \frac{1}{3!} a_1^3 x^3 + \cdots \\
&= e^{a_1 x}.
\end{aligned}
$$

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So there exists $a_1 \in \mathbb{C}$ such that

$$e^{a_1 x} = \cos(x) + i\sin(x)$$

Then $\frac{d}{dx} e^{a_1 x} = a_1 e^{a_1 x}$ and

$$\frac{d}{dx}\left(\cos(x) + i\sin(x)\right) = -\sin(x) + i\cos(x)$$
$$= i\left(\cos(x) + i\sin(x)\right)$$

So $a_1 = i$ and

$$e^{ix} = \cos(x) + i\sin(x).$$

Then

$$|e^{ix}| = \sqrt{\cos(x)^2 + \sin(x)^2} = \sqrt{1} = 1. \quad \text{and}$$

<u>2.28</u>
<u>If</u> $z \in \mathbb{C}$ then $\frac{1}{z} = \frac{1}{|z|^2}\bar{z}$, so

$$e^{-ix} = \frac{1}{e^{ix}} = \frac{1}{1^2}\left(\overline{e^{ix}}\right) = 1\cdot\left(\cos(x) - i\sin(x)\right)$$

So

$$e^{-ix} = \cos(x) - i\sin(x)$$

and <u>2.49</u>, <u>2.50</u>

$$\cos(x) = \frac{1}{2}\left(e^{ix} + e^{-ix}\right) \quad \text{and}$$

$$\sin(x) = (-i)\frac{1}{2}\left(e^{ix} - e^{-ix}\right).$$