<u>2.14</u> Let $z, w \in \mathbb{C}$. Show that

$$\overline{z+w} = \overline{z} + \overline{w} \quad \text{and} \quad \overline{zw} = \overline{z}\,\overline{w}.$$

<u>Proof</u>: Assume $z, w \in \mathbb{C}$.

To show: (a) $\overline{z+w} = \overline{z} + \overline{w}$

(b) $\overline{zw} = \overline{z} \cdot \overline{w}$

Since $z, w \in \mathbb{C}$ then there exist $x, y, a, b \in \mathbb{R}$ such that

$$z = x + iy \quad \text{and} \quad w = a + ib.$$

By definition of conjugate

$$\overline{z} = x - iy \quad \text{and} \quad \overline{w} = a - ib.$$

(a) $\overline{z+w} = \overline{x+iy+a+ib} = \overline{(x+a)+i(y+b)}$

$= (x+a) - i(y+b)$

and $\overline{z} + \overline{w} = x - iy + a - ib = (x+a) - i(y+b).$

So $\overline{z+w} = \overline{z} + \overline{w}.$

(b) $\overline{zw} = \overline{(x+iy)(a+ib)} = \overline{xa + ixb + iya + i^2 yb}$

$= \overline{xa - yb + i(xb+ya)} = (xa-yb) - i(xb+ya)$

and

$\overline{z} \cdot \overline{w} = (x-iy)\cdot(a-ib) = xa - ixb - iya + i^2 yb$

$= xa - yb - i(xb+ya).$ So $\overline{zw} = \overline{z} \cdot \overline{w}.$ //

2.20 Let $z, w \in \mathbb{C}$. Show that

$$|zw| = |z|\cdot|w| \quad \text{and} \quad \mathrm{Arg}(zw) = \mathrm{Arg}(z) + \mathrm{Arg}(w).$$

Proof: Assume $z, w \in \mathbb{C}$. Then there exist $r, s \in \mathbb{R}_{>0}$ and $\theta, \varphi \in \mathbb{R}$ such that

$$z = re^{i\theta} \quad \text{and} \quad w = se^{i\varphi}.$$

To show: (a) $|zw| = |z|\cdot|w|$

(b) $\mathrm{Arg}(zw) = \mathrm{Arg}(z) + \mathrm{Arg}(w)$.

By definition of modulus and argument

$$|z| = r, \quad \mathrm{Arg}(z) = \theta \quad \text{and} \quad |w| = s, \quad \mathrm{Arg}(w) = \varphi.$$

(a) $$|zw| = |re^{i\theta} se^{i\varphi}| = |rse^{i\theta + i\varphi}|$$
$$= |rse^{i(\theta+\varphi)}| = rs$$

and $|z|\cdot|w| = r\cdot s$. So $|zw| = |z|\cdot|w|$.

(b) $$\mathrm{Arg}(zw) = \mathrm{Arg}(re^{i\theta} se^{i\varphi}) = \mathrm{Arg}(rse^{i\theta+i\varphi})$$
$$= \mathrm{Arg}(rse^{i(\theta+\varphi)}) = \theta + \varphi.$$

and $\mathrm{Arg}(z) + \mathrm{Arg}(w) = \theta + \varphi$.

So $\mathrm{Arg}(zw) = \mathrm{Arg}(z) + \mathrm{Arg}(w)$. //

Looking at $\frac{1}{z}$ geometrically

Let $z = x + iy$. Then $\bar{z} = x - iy$.

$$|z| = \sqrt{x^2+y^2} \quad \text{and} \quad |\bar{z}| = \sqrt{x^2+(-y)^2} = \sqrt{x^2+y^2} = |z|.$$

Then

$$\frac{1}{z} = \frac{1}{(x+iy)} \cdot \frac{(x-iy)}{(x-iy)} = \frac{(x-iy)}{x^2+y^2} = \frac{1}{x^2+y^2}(x-iy)$$

So

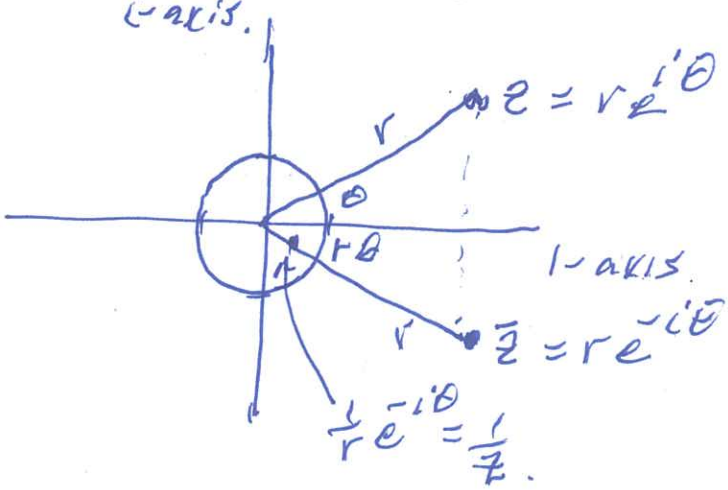
$$\frac{1}{z} = \frac{1}{z} \cdot \frac{\bar{z}}{\bar{z}} = \frac{1}{z \cdot \bar{z}} \bar{z} = \frac{1}{|z|^2} \bar{z}.$$

Let $z = re^{i\theta}$. Then $\bar{z} = re^{-i\theta}$.

$$|z| = r \quad \text{and} \quad |\bar{z}| = |re^{-i\theta}| = r = |z|.$$

Then

$$\frac{1}{z} = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta} = \frac{1}{r^2} re^{-i\theta} = \frac{1}{|z|^2}\bar{z}.$$



Using the tattoo

$1+\sqrt{3}i$ has length $\sqrt{1^2+(\sqrt{3})^2} = \sqrt{1+3} = \sqrt{4} = 2$

and $1+\sqrt{3}i = 2\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = 2e^{i\pi/3}$.

$1+i$ has length $\sqrt{1^2+1^2} = \sqrt{1+1} = \sqrt{2}$

and $1+i = \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) = \sqrt{2}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) = \sqrt{2}e^{i\pi/4}$.

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Calculus Lect 8
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2.42 $(1+\sqrt{3}i)^6 = (2e^{i\pi/3})^6 = 2^6(e^{i\pi/3})^6$

$= 2^6 e^{i\frac{6\pi}{3}} = 2^6 e^{i2\pi} = 2^6 e^{i0} = 2^6 \cdot 1 = 2^6$

2.44 $(1+\sqrt{3}i)^8 = (2e^{i\pi/3})^8 = 2^8 e^{i8\pi/3} = 2^8 e^{i\frac{6\pi}{3}} e^{i\frac{2\pi}{3}}$

$= 2^8 \cdot 1 \cdot \left(-\frac{\sqrt{1}}{2} + \frac{\sqrt{3}}{2}i\right) = -2^7 + 2^7\sqrt{3}i.$

2.45 $\left(\frac{2}{1+i}\right)^4 = \left(\frac{2}{\sqrt{2}e^{i\pi/4}}\right)^4 = \left(\sqrt{2}e^{-i\pi/4}\right)^4 = \left(2^{\frac{1}{2}}e^{-i\pi/4}\right)^4$

$= 2^{4/2} e^{-i\frac{4\pi}{4}} = 2^2 e^{-i\pi} = 4\cdot(-1) = -4.$

2.47a $\{\text{cube roots of } 1\} = \{z \in \mathbb{C} \mid z^3 = 1\}$

$= \{re^{i\theta} \mid (re^{i\theta})^3 = 1\}$

$= \{re^{i\theta} \mid r^3 e^{i3\theta} = 1e^{i0}\}$

$= \{e^{i\theta} \mid e^{i3\theta} = 1\} = \{e^{i2\pi/3}, e^{i\frac{4\pi}{3}}, e^{i\frac{6\pi}{3}}\}$

2.48a $\{4^{th} \text{ roots of } 1\} = \{(e^{i0})^{\frac{1}{4}}, (e^{i2\pi})^{\frac{1}{4}}, (e^{i4\pi})^{\frac{1}{4}}, (e^{i6\pi})^{\frac{1}{4}}\}$

$= \{e^{i\frac{0\pi}{4}}, e^{i\frac{2\pi}{4}}, e^{i\frac{4\pi}{4}}, e^{i\frac{6\pi}{4}}\}$

$= \{1, i, -1, -i\}.$