13.4 Example proofs

The following example proofs have been chosen because they are results that are often assumed, are needed for many topics in algebra and analysis and topology and are rarely proved carefully in an undergraduate curriculum; facts like, if $a \neq 0$ then $a^2 > 0$. These often seem "obvious", until you meet that first example, like a field witih 5 elements, where $2 \neq 0$ and $2^2 = -1$. After getting over the initial shock, then one begins to wonder why such a fact might ever be true, and how it might be proved when it is. It is proved in Proposition 13.4(b), below.

13.4.1 An inverse function to f exists if and only if f is bijective.

Theorem 13.1. *Let $f\colon S \to T$ be a function. The inverse function to f exists if and only if f is bijective.*

Proof.

$\Rightarrow$: Assume $f\colon S \to T$ has an inverse function $f^{-1}\colon T \to S$.
To show: (a) f is injective.
(b) f is surjective.

(a) Assume $s_1, s_2 \in S$ and $f(s_1) = f(s_2)$.
To show: $s_1 = s_2$.

$$s_1 = f^{-1}f(s_1)) = f^{-1}f(s_2)) = s_2.$$

So f is injective.

(b) Let $t \in T$.
To show: There exists $s \in S$ such that $f(s) = t$.
Let $s = f^{-1}(t)$.
Then

$$f(s) = f(f^{-1}(t)) = t.$$

So f is surjective.

So f is bijective.

$\Leftarrow$: Assume $f\colon S \to T$ is bijective.
To show: f has an inverse function.
We need to define a function $\varphi\colon T \to S$.
Let $t \in T$.
Since f is surjective there eists $s \in S$ such that $f(s) = t$.
Define $\varphi(t) = s$.
To show: (a) φ is well defined.
(b) φ is an inverse function to f.

(a) To show: (aa) If $t \in T$ then $\varphi(t) \in S$.
(ab) If $t_1, t_2 \in T$ and $t_1 = t_2$ then $\varphi(t_1) = \varphi(t_2)$.

(aa) This follows from the definition of φ.
(ab) Assume $t_1, t_2 \in T$ and $t_1 = t_2$.
Let $s_1, s_2 \in S$ such that $f(s_1) = t_1$ and $f(s_2) = t_2$.

Since $t_1 = t_2$ then $f(s_1) = f(s_2)$.
Since f is injective this implies that $s_1 = s_2$.
So $\varphi(t_1) = s_1 = s_2 = \varphi(t_2)$.

So φ is well defined.

(b) To show: (ba) If $s \in S$ then $\varphi(f(s)) = s$.
(bb) If $t \in T$ then $f(\varphi(t)) = t$.

(ba) This follows from the definition of φ.
(bb) Assume $t \in T$.
Let $s \in S$ be such that $f(s) = t$.
Then

$$f(\varphi(t)) = f(s) = t.$$

So $\varphi \circ f$ and $f \circ \varphi$ are the identity functions on S and T, respectively.

So φ is an inverse function to f.

□

13.4.2 An equivalence relation on S and a partition of S are the same data.

Let S be a set.

- A *relation* $\sim$ *on* S is a subset $R_\sim$ of $S \times S$. Write $s_1 \sim s_2$ if the pair (s_1, s_2) is in the subset $R_\sim$ so that
$$R_\sim = \{(s_1, s_2) \in S \times S \mid s_1 \sim s_2\}.$$
- An *equivalence relation* on S is a relation $\sim$ on S such that
  (a) if $s \in S$ then $s \sim s$,
  (b) if $s_1, s_2 \in S$ and $s_1 \sim s_2$ then $s_2 \sim s_1$,
  (c) if $s_1, s_2, s_3 \in S$ and $s_1 \sim s_2$ and $s_2 \sim s_3$ then $s_1 \sim s_3$.

Let $\sim$ be an equivalence relation on a set S and let $s \in S$. The *equivalence class of* s is the set

$$[s] = \{t \in S \mid t \sim s\}.$$

A *partition of a set* S is a collection $\mathcal{P}$ of subsets of S such that

(a) If $s \in S$ then there exists $P \in \mathcal{P}$ such that $s \in P$, and
(b) If $P_1, P_2 \in \mathcal{P}$ and $P_1 \cap P_2 \neq \emptyset$ then $P_1 = P_2$.

Theorem 13.2.

(a) If S is a set and let $\sim$ be an equivalence relation on S then

$$\textit{the set of equivalence classes of } \sim \qquad \textit{is a partition of } S.$$

(b) If S is a set and $\mathcal{P}$ is a partition of S then

$$\textit{the relation defined by} \qquad s \sim t \quad \textit{if } s \textit{ and } t \textit{ are in the same } P \in \mathcal{P}$$

is an equivalence relation on S.

Proof.
(a) To show: (aa) If $s \in S$ then s is in some equivalence class.