3.2 Functions

Functions are for comparing sets.

Let S and T be sets. A *function from S to T* is a subset $\Gamma_f \subseteq S \times T$ such that

$$\text{if } s \in S \quad \text{then} \quad \text{there exists a unique } t \in T \text{ such that } (s,t) \in \Gamma_f.$$

Write

$$\Gamma_f = \{(s, f(s)) \mid s \in S\}$$

so that the function Γ_f can be expressed as

$$\text{an “assignment”} \qquad \begin{array}{rccc} f\colon & S & \to & T \\ & s & \mapsto & f(s) \end{array}$$

which must satisfy

(a) If $s \in S$ then $f(s) \in T$, and

(b) If $s_1, s_2 \in S$ and $s_1 = s_2$ then $f(s_1) = f(s_2)$.

Let S and T be sets.

- Two functions $f\colon S \to T$ and $g\colon S \to T$ are *equal* if they satisfy

$$\text{if } s \in S \quad \text{then} \quad f(s) = g(s).$$

- A function $f\colon S \to T$ is *injective* if f satisfies the condition

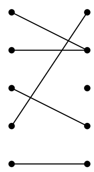
$$\text{if } s_1, s_2 \in S \text{ and } f(s_1) = f(s_2) \quad \text{then} \quad s_1 = s_2.$$

- A function $f\colon S \to T$ is *surjective* if f satisfies the condition

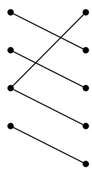
$$\text{if } t \in T \quad \text{then} \quad \text{there exists } s \in S \text{ such that } f(s) = t.$$

- A function $f\colon S \to T$ is *bijective* if f is both injective and surjective.

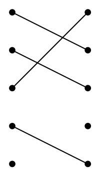
Examples. It is useful to visualize a function $f\colon S \to T$ as a graph with edges $(s, f(s))$ connecting elements $s \in S$ and $f(s) \in T$. With this in mind the following are examples:



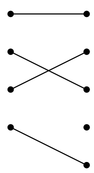
(a) a function



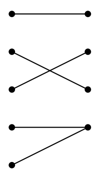
(b) not a function



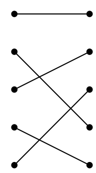
(c) not a function



(d) an injective function



(e) a surjective function



(f) a bijective function

In these pictures the elements of the left column are the elements of the set S and the elements of the right column are the elements of the set T. In order to be a function the graph must have exactly one edge adjacent to each point in S. The function is injective if there is at most one edge adjacent to each point in T. The function is surjective if there is at least one edge adjacent to each point in T.

3.3 Composition of functions

Let $f\colon S \to T$ and $g\colon T \to U$ be functions. The *composition* of f and g is the function

$$g \circ f \quad \text{given by} \quad \begin{array}{rccc} g\circ f\colon & S & \to & U \\ & s & \mapsto & g(f(s)) \end{array}$$

Let S be a set. The *identity map on S* is the function given by

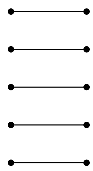
$$\begin{array}{rccc} \mathrm{id}_S\colon & S & \to & S \\ & s & \mapsto & s \end{array}$$

Let $f\colon S \to T$ be a function. The *inverse function to f* is a function

$$f^{-1}\colon T \to S \qquad \text{such that} \qquad f \circ f^{-1} = \mathrm{id}_T \quad \text{and} \quad f^{-1} \circ f = \mathrm{id}_S.$$

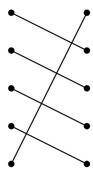
Theorem 3.1. *Let $f\colon S \to T$ be a function. An inverse function to f exists if and only if f is bijective.*

Representing functions as graphs, the identity function id_S looks like

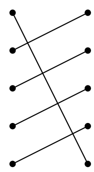


(a) the identity function id_S

In the pictures below, if the left graph is a pictorial representation of a function $f\colon S \to T$ then the inverse function to f, $f^{-1}\colon T \to S$, is represented by the graph on the right; the graph for f^{-1} is the mirror-image of the graph for f.

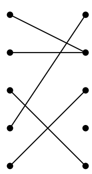


(b) the function f

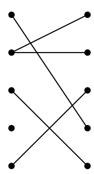


(c) the function f^{-1}

Graph (d) below, represents a function $g\colon S \to T$ which is not bijective. The inverse function to g does not exist in this case: the graph (e) of a possible candidate, is not the graph of a function.



(d) the function g



(e) not a function

3.4 Cardinality

Let S and T be sets. The sets *S and T are isomorphic*, or *have the same cardinality*

$$\text{if there is a bijective function} \quad \varphi\colon S \to T.$$

Write

$$\mathrm{Card}(S) = \mathrm{Card}(T) \qquad \text{if } S \text{ and } T \text{ have the same cardinality.}$$

Notation: Let S be a set. Write

$$\mathrm{Card}(S) = \begin{cases} 0, & \text{if } S = \emptyset, \\ n, & \text{if } \mathrm{Card}(S) = \mathrm{Card}(\{1, 2, \ldots, n\}), \\ \infty, & \text{otherwise.} \end{cases}$$

Note that even in the cases where $\mathrm{Card}(S) = \infty$ and $\mathrm{Card}(T) = \infty$ it may be that $\mathrm{Card}(S) \neq \mathrm{Card}(T)$.

Let S be a set.

- The set S is *finite* if there exists $n \in \mathbb{Z}_{\geq 0}$ such that $\mathrm{Card}(S) = \mathrm{Card}(\{1, \ldots, n\})$.
- The set S is *infinite* if $\mathrm{Card}(S)$ is not finite.
- The set S is *countable* if $\mathrm{Card}(S) = \mathrm{Card}(\mathbb{Z}_{>0})$ or S is finite.
- The set S is *countably infinite* if S is countable and infinite.
- The set S is *uncountable* if S is not countable.

3.5 Images and fibers

Let $f\colon S \to T$ be a function. Let $A \subseteq S$ and let $B \subseteq T$. The *image of A* is

$$f(A) = \{f(a) \mid a \in A\} \qquad \text{and} \qquad f^{-1}(B) = \{s \in S \mid f(s) \in B\},$$

is the *fiber over B*. Let $t \in T$. The *fiber over t* is

$$f^{-1}(t) = f^{-1}(\{t\}) = \{s \in S \mid f(s) = t\} \qquad \text{and} \qquad \mathrm{im}(f) = f(S) = \{f(s) \mid s \in S\}$$

is the *image of f*.

Let $S/(f)$ be the set of fibers of f,

$$S(f) = \{f^{-1}(t) \mid t \in T\}.$$

The elements of $S(f)$ are, themselves, sets. Then

$$\begin{array}{rccc} \hat{f}\colon & S_f & \to & \mathrm{im}(f) \\ & f^{-1}(t) & \mapsto & t, \end{array} \qquad \begin{array}{rccc} p\colon & S & \to & S_f \\ & s & \to & f^{-1}(f(s)), \end{array} \qquad \begin{array}{rccc} \iota\colon & \mathrm{im}(f) & \to & T \\ & f(s) & \to & f(s) \end{array}$$

define functions such that

$$\begin{array}{l} \text{(a) } \hat{f} \text{ is bijective,} \\ \text{(b) } p \text{ is surjective,} \\ \text{(c) } \iota \text{ is injective,} \end{array} \qquad \text{and} \qquad f = \iota \circ \hat{f} \circ p.$$