(Fa) If $a, b, c \in \mathbb{F}$ then $(a+b)+c = a+(b+c)$,
(Fb) If $a, b \in \mathbb{F}$ then $a+b = b+a$,
(Fc) There exists $0 \in \mathbb{F}$ such that

$$\text{if } a \in \mathbb{F} \quad \text{then} \quad 0+a = a \text{ and } a+0 = a,$$

(Fd) If $a \in \mathbb{F}$ then there exists $-a \in \mathbb{F}$ such that $a+(-a) = 0$ and $(-a)+a = 0$,
(Fe) If $a, b, c \in \mathbb{F}$ then $(ab)c = a(bc)$,
(Ff) If $a, b, c \in \mathbb{F}$ then

$$(a+b)c = ac+bc \qquad \text{and} \qquad c(a+b) = ca+cb,$$

(Fg) There exists $1 \in \mathbb{F}$ such that

$$\text{if } a \in \mathbb{F} \quad \text{then} \quad 1 \cdot a = a \text{ and } a \cdot 1 = a,$$

(Fh) If $a \in \mathbb{F}$ and $a \neq 0$ then there exists $a^{-1} \in \mathbb{F}$ such that $aa^{-1} = 1$ and $a^{-1}a = 1$,
(Fi) If $a, b \in \mathbb{F}$ then $ab = ba$.

Proposition 13.3. *Let $\mathbb{F}$ be a field.*

(a) If $a \in \mathbb{F}$ then $a \cdot 0 = 0$.

(b) If $a \in \mathbb{F}$ then $-(-a) = a$.

(c) If $a \in \mathbb{F}$ and $a \neq 0$ then $(a^{-1})^{-1} = a$.

(d) If $a \in \mathbb{F}$ then $a(-1) = -a$.

(e) If $a, b \in \mathbb{F}$ then $(-a)b = -ab$.

(f) If $a, b \in \mathbb{F}$ then $(-a)(-b) = ab$.

Proof.

(a) Assume $a \in \mathbb{F}$.

$$\begin{aligned} a \cdot 0 &= a \cdot (0+0), \quad \text{by (Fc)}, \\ &= a \cdot 0 + a \cdot 0, \quad \text{by (Ff)}. \end{aligned}$$

Add $-a \cdot 0$ to each side and use (Fd) to get $0 = a \cdot 0$.

(b) Assume $a \in \mathbb{F}$.
By (Fd),

$$-(-a)+(-a) = 0 = a+(-a).$$

Add $-a$ to each side and use (Fd) to get $-(-a) = a$.

(c) Assume $a \in \mathbb{F}$ and $a \neq 0$.
By (Fh),

$$(a^{-1})^{-1} \cdot a^{-1} = 1 = a \cdot a^{-1}.$$

Multiply each side by a and use (Fh) and (Fg) to get $(a^{-1})^{-1} = a$.

(d) Assume $a \in \mathbb{F}$.
By (Ff),

$$a(-1) + a \cdot 1 = a(-1+1) = a \cdot 0 = 0,$$

where the last equality follows from part (a).

So, by (Fg), $a(-1) + a = 0$.
Add $-a$ to each side and use (Fd) and (Fc) to get $a(-1) = -a$.

(e) Assume $a, b \in \mathbb{F}$.

$$\begin{aligned}(-a)b + ab &= (-a + a)b, \quad \text{by (Ff)},\\ &= 0 \cdot b, \quad \text{by (Fd)},\\ &= 0, \quad \text{by part (a)}.\end{aligned}$$

Add $-ab$ to each side and use (Fd) and (Fc) to get $(-a)b = -ab$.

(f) Assume $a, b \in \mathbb{F}$.

$$\begin{aligned}(-a)(-b) &= -(a(-b)), \quad \text{by (e)},\\ &= -(-ab), \quad \text{by (e)},\\ &= ab, \quad \text{by part (b)}.\end{aligned}$$

□

13.4.4 Identities in an ordered field

An *ordered field* is a field $\mathbb{F}$ with a total order $\le$ such that

(OFa) If $a, b, c \in \mathbb{F}$ and $a \le b$ then $a + c \le b + c$,

(OFb) If $a, b \in \mathbb{F}$ and $a \ge 0$ and $b \ge 0$ then $ab \ge 0$.

Proposition 13.4. *Let $\mathbb{F}$ be an ordered field.*

(a) If $a \in \mathbb{F}$ and $a > 0$ then $-a < 0$.

(b) If $a \in \mathbb{F}$ and $a \ne 0$ then $a^2 > 0$.

(c) $1 \ge 0$.

(d) If $a \in \mathbb{F}$ and $a > 0$ then $a^{-1} > 0$.

(e) If $a, b \in \mathbb{F}$ and $a \ge 0$ and $b \ge 0$ then $a + b \ge 0$.

(f) If $a, b \in \mathbb{F}$ and $0 < a < b$ then $b^{-1} < a^{-1}$.

Proof.

(a) Assume $a \in \mathbb{F}$ and $a > 0$.
Then $a + (-a) > 0 + (-a)$, by (OFb).
So $0 > -a$, by (Fd) and (Fc).

(b) Assume $a \in \mathbb{F}$ and $a \ne 0$.
Case 1: $a > 0$.
Then $a \cdot a > a \cdot 0$, by (OFb).
So $a^2 > 0$, by part (a).
Case 2: $a < 0$.
Then $-a > 0$, by part (a).
Then $(-a)^2 > 0$, by Case 1.
So $a^2 > 0$, by Proposition 13.3 (f).

(c) To show: $1 \geq 0$.
$1 = 1^2 \geq 0$, by part (b).

(d) Assume $a \in \mathbb{F}$ and $a > 0$.
By part (b), $a^{-2} = (a^{-1})^2 > 0$.
So $a(a^{-1})^2 > a \cdot 0$, by (OFb).
So $a^{-1} > 0$, by (Fh) and Proposition 13.3 (a).

(e) Assume $a, b \in \mathbb{F}$ and $a \geq 0$ and $b \geq 0$.

$$\begin{aligned} a + b &\geq 0 + b, \quad \text{by (OFa)}, \\ &\geq 0 + 0, \quad \text{by (OFa)}, \\ &= 0, \quad \text{by (Fc)}. \end{aligned}$$

(f) Assume $a, b \in \mathbb{F}$ and $0 < a < b$.
So $a > 0$ and $b > 0$.
Then, by part (d), $a^{-1} > 0$ and $b^{-1} > 0$.
Thus, by (OFb), $a^{-1}b^{-1} > 0$.
Since $a < b$, then $b - a > 0$, by (OFa).
So, by (OFb), $a^{-1}b^{-1}(b - a) > 0$.
So, by (Fh), $a^{-1} - b^{-1} > 0$.
So, by (OFa), $a^{-1} > y^{-1}$.

□

Proposition 13.5. *Let $\mathbb{F}$ be an ordered field and let $x, y \in \mathbb{F}$ with $x \geq 0$ and $y \geq 0$. Then*

$$x \leq y \qquad \text{if and only if} \qquad x^2 \leq y^2.$$

Proof. Assume $x, y \in S$ and $x \geq 0$ and $y \geq 0$.
To show: (a) If $x \leq y$ then $x^2 \leq y^2$.
(b) If $x^2 \leq y^2$ then $x \leq y$.

(b) Assume $x^2 \leq y^2$.
Adding $(-x^2)$ to each side and using (OFa) gives $y^2 + (-x^2) \geq x^2 + (-x^2) = 0$.
So $y^2 - x^2 \geq 0$.
Using Proposition 13.3(e) and axioms (Ff) and (Fi),

$$\begin{aligned} (y - x)(y + x) &= yy + (-x)y + yx + (-x)x = y^2 + (-xy) + xy + (-xx) \\ &= y^2 + 0 - x^2 = y^2 - x^2. \end{aligned}$$

So $(y - x)(y + x) \geq 0$.
By Proposition 13.4(e) and Proposition 13.4 (d),
since $x \geq 0$ and $y \geq 0$ then $x + y \geq 0$ and $(x + y)^{-1} > 0$ (or $x = 0$ and $y = 0$).
So, by (OFb), $(y - x)(y + x)(x + y)^{-1} \geq 0$.
Using (Fg), then $y - x \geq 0$.
Adding x to both sides and using (OFa) gives $y \geq x$.

(a) Assume $y \geq x$.
Then $y - x \geq 0$.

Since $y \geq 0$ and $x \geq 0$ then, by (OFa), $(y + x) \geq y + 0 = y \geq 0$.
So, by (OFb), $(y - x)(y + x) \geq 0$.
So $y^2 - x^2 \geq 0$.
So $y^2 \geq x^2$.

□