5 exponentials, derivatives and integrals

5.1 The number system $\mathbb{Q}[[x]]$, the exponential and the logarithm

The number system $\mathbb{Q}[x]$ is the collection of polynomials in a variable x with coefficients that are rational numbers. Addition, multiplication and scalar multiplication are operations with polynomials.

If $r \in \mathbb{Z}_{>0}$ then $(1-x)(1+x+x^2+\cdots+x^{r-1}) = 1-x^r$ and

$$\frac{1-x^r}{1-x} = 1+x+x^2+\cdots+x^{r-1}, \qquad \text{in the number system } \mathbb{Q}[x].$$

The number system $\mathbb{Q}[[x]]$ is the collection of, possibly infinite, polynomials.
Favorite elements of $\mathbb{Q}[[x]]$ are

$$\frac{1}{1-x} = 1+x+x^2+x^3+x^4+\cdots, \qquad \log(1-x) = -(x+\tfrac{1}{2}x^2+\tfrac{1}{3}x^3+\cdots),$$
$$\frac{1}{1+x} = 1-x+x^2-x^3+x^4-\cdots, \qquad \log(1+x) = x-\tfrac{1}{2}x^2+\tfrac{1}{3}x^3-\tfrac{1}{4}x^4+\cdots,$$

$$e^x = 1+x+\tfrac{1}{2!}x^2+\tfrac{1}{3!}x^3+\cdots,$$

The number e^x in the number system $\mathbb{Q}[[x]]$ is the most important number in mathematics:

$$e^x \text{ is the most important number in mathematics.}$$

The *derivative with respect to* x is the function $\frac{d}{dx}\colon \mathbb{Q}[[x]] \to \mathbb{Q}[[x]]$ determined by

$$\frac{dx}{dx} = 1, \qquad \frac{d(c_1f+c_2g)}{dx} = c_1\frac{df}{dx}+c_2\frac{dg}{dx}, \qquad \frac{d(fg)}{dx} = f\frac{dg}{dx}+\frac{df}{dx}g,$$

for $c_1, c_2 \in \mathbb{Q}$ and $f, g \in \mathbb{Q}[[x]]$.

HW: Prove, by induction on r, that if $r \in \mathbb{Z}_{\geq 0}$ then $\dfrac{dx^r}{dx} = rx^{r-1}$.

HW: Take derivatives with respect to x and check that

$$\frac{de^x}{dx} = e^x, \qquad \frac{d\log(1-x)}{dx} = \frac{1}{1-x}, \qquad \frac{d\log(1+x)}{dx} = \frac{1}{1+x}.$$

HW: Show that

$$\text{if } xy = yx \quad \text{then} \quad e^{x+y} = e^xe^y.$$

HW: Show that

$$e^0 = 1, \qquad e^{-x} = \frac{1}{e^x} \qquad \text{and} \qquad \frac{de^x}{dx} = e^x.$$

HW: Show that e^x is characterized by the conditions $\frac{de^x}{dx} = e^x$ and $e^0 = 1$.

HW: Show that if $f(x) \in \mathbb{Q}[[x]]$ satisfies $f(x+y) = f(x)+f(y)$ then

$$\text{there exists } a \in \mathbb{Q} \text{ such that } f(x) = e^{ax}.$$

HW: Show that $e^{\log(1+x)} = 1+x$.

HW: Show that $\log(e^x) = x$.

5.2 Hyperbolic functions

The *hyperbolic functions* $\sinh x$ and $\cosh x$ are given by

$$\sinh(x) = \tfrac{1}{2}(e^x - e^{-x}) \qquad \text{and} \qquad \cosh(x) = \tfrac{1}{2}(e^x + e^{-x}).$$

Define

$$\tanh x = \frac{\sinh x}{\cosh x}, \qquad \coth x = \frac{1}{\tanh x}, \qquad \text{sech } x = \frac{1}{\cosh x}, \qquad \text{csch } x = \frac{1}{\sinh x}.$$

HW: Show that

$$\sinh(x) = x + \tfrac{1}{3!}x^3 + \tfrac{1}{5!}x^5 + \cdots \qquad \text{and} \qquad \cosh(x) = 1 + \tfrac{1}{2!}x^2 + \tfrac{1}{4!}x^4 + \cdots .$$

HW: Show that $\quad e^x = \cosh x + \sinh x \quad$ and $\quad e^{-x} = \cosh x - \sinh x.$

5.3 Circular functions

Let $i \in \mathbb{C}$ be such that $i^2 = -1$. The *circular functions* $\sin x$ and $\cos x$ are given by

$$\cos(x) = \tfrac{1}{2}(e^{ix} + e^{-ix}) \qquad \text{and} \qquad \sin(x) = (-i)\tfrac{1}{2}(e^{ix} - e^{-ix}).$$

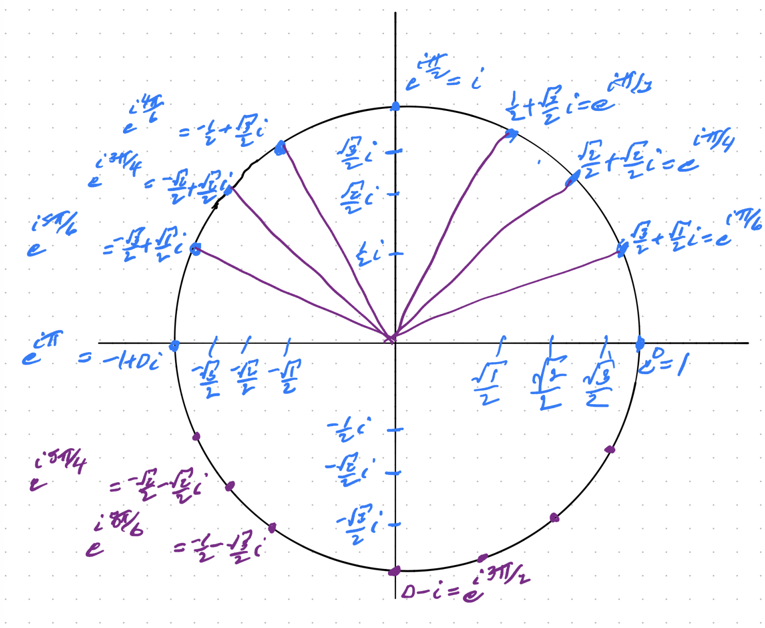
Define

$$\tan(x) = \frac{\sin(x)}{\cos(x)}, \qquad \cot(x) = \frac{\cos(x)}{\sin(x)}, \qquad \sec(x) = \frac{1}{\cos(x)}, \qquad \csc(x) = \frac{1}{\sin(x)}.$$

HW: Show that

$$\cos(x) = 1 - \tfrac{1}{2!}x^2 + \tfrac{1}{4!}x^4 - \cdots \qquad \text{and} \qquad \sin(x) = x - \tfrac{1}{3!}x^3 + \tfrac{1}{5!}x^5 - \cdots .$$

HW:. Show that $\quad e^{ix} = \cos(x) + i\sin(x) \quad$ and $\quad e^{-ix} = \cos(x) - i\sin(x).$



sines and cosines of the favorite angles

For $a, \theta \in \mathbb{R}$, let $r = e^a$ and $x = r\cos(\theta)$ and $y = r\sin(\theta)$. Then

$$z = e^{a+i\theta} = e^a e^{i\theta} = re^{i\theta} = r(\cos\theta + i\sin\theta) = (r\cos\theta) + i(r\sin\theta) = x + iy.$$

5.4 Inverse "functions"

$\sqrt{x}$ is the "function" that undoes x^2. This means that

$$\sqrt{x^2} = x \qquad \text{and} \qquad (\sqrt{x})^2 = x.$$

$\log x$ is the "function" that undoes e^x. This means that

$$\log(e^x) = x \qquad \text{and} \qquad e^{\log x} = x.$$

$\int \quad dx$ is the "function" that undoes $\frac{d}{dx}$. This means that

$$\int \frac{df}{dx} dx = f \qquad \text{and} \qquad \frac{d}{dx}\Big(\int f dx \Big) = f.$$

arcsin$x$ is the "function" that undoes $\sin x$. This means that

$$\arcsin(\sin x) = x \qquad \text{and} \qquad \sin(\text{arcsin}x) = x.$$

arccos$x$ is the "function" that undoes $\cos x$. This means that

$$\arccos(\cos x) = x \qquad \text{and} \qquad \cos(\text{arccos}x) = x.$$

arctan$x$ is the "function" that undoes $\tan x$. This means that

$$\arctan(\tan x) = x \qquad \text{and} \qquad \tan(\text{arctan}x) = x.$$

arccot$x$ is the "function" that undoes $\cot x$. This means that

$$\text{arccot}(\cot x) = x \qquad \text{and} \qquad \cot(\text{arccot}x) = x.$$

arcsec$x$ is the "function" that undoes $\sec x$. This means that

$$\text{arcsec}(\sec x) = x \qquad \text{and} \qquad \sec(\text{arcsec}x) = x.$$

arccsc$x$ is the "function" that undoes $\csc x$. This means that

$$\text{arccsc}(\csc x) = x \qquad \text{and} \qquad \csc(\text{arccsc}x) = x.$$

$\log_a x$ is the "function" that undoes a^x. This means that

$$\log_a(a^{\sqrt{7}\pi i \sin 32}) = \sqrt{7}\pi i \sin 32 \qquad \text{and} \qquad a^{\log_a(\sqrt{7}\pi i \sin 32)} = \sqrt{7}\pi i \sin 32.$$

WARNING: In spite of the name, an inverse "function" is rarely a function. The output of an inverse function is usually a set of values, as opposed to a single value. For example

$$\sqrt{9} = \{3, -3\} \qquad \text{since } 3^2 = 9 \text{ and } (-3)^2 = 9.$$

Similarly,

$$\log(1) = \{0 + k2i\pi \mid k \in \mathbb{Z}\}, \qquad \text{since} \quad e^{0+k2i\pi} = (e^{i2\pi})^k = 1^k = 1.$$

and

$$\int 2x dx = \{x^2 + c \mid c \text{ is a constant}\}, \qquad \text{since} \quad \frac{d(x^2+c)}{dx} = \frac{dx^2}{dx} + \frac{dc}{dx} = 2x + 0 = 2x$$

when c is a constant.