The Math Book:
Algebra, Graphing, Numbers, Sets and Proof

Arun Ram   email: aram@unimelb.edu.au

Abstract

These are notes about algebra, graphing, numbers, sets and proof, with a few fun lectures added at the end.

Key words— algebra, graphing, sets, numbers, proof

Contents

AMS Subject Classifications: Primary 05E05; Secondary 20G99.

1 Introduction

In 2024 I decided to start my third act.

First, I read a biography of Pauline Viardot Garcia, a great opera star, a great teacher, an inspiring person, of the 1800's. Her husband Louis Viardot translated the classic Spanish novel: Don Quijote de La Mancha, by Miguel de Cervantes Saavedra. I decided I should read Viardot's translation and compare it to the original Spanish to see for myself whether Louis' translation is any good. I rarely trust the critics without making an assessment for myself. Of course, my French is better than my Spanish, so it will take me some analysis and thought to produce a critical analysis of the quality of L's translation.

I was discussing my plans with my friend Persi Diaconis and he told me that his concept of Don Quixote is coloured by a Broadway performance of Man of La Mancha from the late 60's. In this time that we live in, in 2024, I soon found a video of the Broadway production on YouTube. This book was compiled while watching this wonderful video.

This is the Math Book. To dream the impossible dream, I hope you enjoy this book and fall in love with the beautiful mathematics that it is based on. Explanations of things in this book that might not seem immediately obvious can be found in the sequel to this book, which is entitled 'The Math Book – the proof'. Perhaps the sequel will justify the initial escapade.

If, by chance, this author does not complete the sequel, I hope that others will. It is the proof that completes the quest.

A 'fun' part (at least for the author) of this text is Section 7, entitled 'Some fun lectures'. I hope that at least some readers will read Section 7.6 first. I hope that all readers will feel free to skip around to any part of the book that looks interesting or useful at any given moment. This text is **not** designed to be read like a novel by starting at page 1 and then reading page 2 and then reading page 3 and then page 4 and then page 5 and then page 6 and then page 7 and then page 8 and then page 9 and then page 10 and then page 11 and then page 12 and then page 13 and then page 14 and then page 15 and then page 16 and then page 17 and then page 18 and then page 19 and then page 20 and then page 21 and then page 22 and then page 23 and then page 24 and then page 25 and then page 26 and then page 27 and then page 28 and then page 29 and then page 30 and then page 31 and then page 32 and then page 33 and then page 34 and then page 35 and then page 36 and then page 37 and then page 38 and then page 39 and then page 40 and then page 41 and then page 42 and then page 43 and then page 44 and then page 45 and then page 46 and then page 47 and then page 48 and then page 49 and then page 50 and then page 51 and then page 52 and then page 53 and then page 54 and then page 55 and then page 56 and then page 57 and then page 58 and then page 59 and then page 60 and then page 61 and then page 62 and then page 63 and then page 64 and then page 65 and then page 66 and then page 67 and then page 68 and then page 69 and then page 70 and then page 71 and then page 72 and then page 73 and then page 74 and then page 75 and then page 76 and then page 77 and then page 78 and then page 79 and then page 80 and then page 81 and then page 82 and then page 83 and then page 84 and then page 85 and then page 86 and then page 87 and then page 88 and then page 89 and then page 90. I think it's more fun, and more instructive, to be more original than that.

2 exponentials, derivatives and integrals

2.1 The number system $\mathbb{Q}[[x]]$, the exponential and the logarithm

The number system $\mathbb{Q}[x]$ is the collection of polynomials in a variable x with coefficients that are rational numbers. Addition, multiplication and scalar multiplication are operations with polynomials.

If $r \in \mathbb{Z}_{>0}$ then $(1-x)(1+x+x^2+\cdots+x^{r-1}) = 1-x^r$ and

$$\frac{1-x^r}{1-x} = 1+x+x^2+\cdots+x^{r-1}, \qquad \text{in the number system } \mathbb{Q}[x].$$

The number system $\mathbb{Q}[[x]]$ is the collection of, possibly infinite, polynomials.
Favorite elements of $\mathbb{Q}[[x]]$ are

$$\frac{1}{1-x} = 1+x+x^2+x^3+x^4+\cdots, \qquad \log(1-x) = -(x+\tfrac{1}{2}x^2+\tfrac{1}{3}x^3+\cdots),$$
$$\frac{1}{1+x} = 1-x+x^2-x^3+x^4-\cdots, \qquad \log(1+x) = x-\tfrac{1}{2}x^2+\tfrac{1}{3}x^3-\tfrac{1}{4}x^4+\cdots,$$

$$e^x = 1+x+\tfrac{1}{2!}x^2+\tfrac{1}{3!}x^3+\cdots,$$

The number e^x in the number system $\mathbb{Q}[[x]]$ is the most important number in mathematics:

e^x is the most important number in mathematics.

(Yes, this sentence is intentionally repeated and highlighted; it is a very important sentence.)

The *derivative with respect to* x is the function $\frac{d}{dx}\colon \mathbb{Q}[[x]] \to \mathbb{Q}[[x]]$ determined by

$$\frac{dx}{dx} = 1, \qquad \frac{d(c_1f+c_2g)}{dx} = c_1\frac{df}{dx}+c_2\frac{dg}{dx}, \qquad \frac{d(fg)}{dx} = f\frac{dg}{dx}+\frac{df}{dx}g,$$

for $c_1, c_2 \in \mathbb{Q}$ and $f, g \in \mathbb{Q}[[x]]$.

HW: Prove, by induction on r, that if $r \in \mathbb{Z}_{\geq 0}$ then $\dfrac{dx^r}{dx} = rx^{r-1}$.

HW: Take derivatives with respect to x and check that

$$\frac{de^x}{dx} = e^x, \qquad \frac{d\log(1-x)}{dx} = \frac{1}{1-x}, \qquad \frac{d\log(1+x)}{dx} = \frac{1}{1+x}.$$

HW: Show that

$$\text{if } xy = yx \quad \text{then} \quad e^{x+y} = e^x e^y.$$

HW: Show that

$$e^0 = 1, \qquad e^{-x} = \frac{1}{e^x} \qquad \text{and} \qquad \frac{de^x}{dx} = e^x.$$

HW: Show that e^x is characterized by the conditions $\frac{de^x}{dx} = e^x$ and $e^0 = 1$.
HW: Show that if $f(x) \in \mathbb{Q}[[x]]$ satisfies $f(x+y) = f(x)+f(y)$ then

$$\text{there exists } a \in \mathbb{Q} \text{ such that } f(x) = e^{ax}.$$

HW: Show that $e^{\log(1+x)} = 1+x$.
HW: Show that $\log(e^x) = x$.

2.2 Hyperbolic functions

The *hyperbolic functions* $\sinh x$ and $\cosh x$ are given by

$$\sinh(x) = \tfrac{1}{2}(e^x - e^{-x}) \qquad \text{and} \qquad \cosh(x) = \tfrac{1}{2}(e^x + e^{-x}).$$

Define

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)}, \qquad \coth(x) = \frac{1}{\tanh(x)}, \qquad \mathrm{sech}(x) = \frac{1}{\cosh(x)}, \qquad \mathrm{csch}(x) = \frac{1}{\sinh(x)}.$$

HW: Show that

$$\sinh(x) = x + \tfrac{1}{3!}x^3 + \tfrac{1}{5!}x^5 + \cdots \qquad \text{and} \qquad \cosh(x) = 1 + \tfrac{1}{2!}x^2 + \tfrac{1}{4!}x^4 + \cdots .$$

HW: Show that $\quad e^x = \cosh(x) + \sinh(x) \quad$ and $\quad e^{-x} = \cosh(x) - \sinh(x).$

2.3 Circular functions

Let i be such that $i^2 = -1$. The *circular functions* $\sin x$ and $\cos x$ are given by

$$\cos(x) = \tfrac{1}{2}(e^{ix} + e^{-ix}) \qquad \text{and} \qquad \sin(x) = (-i)\tfrac{1}{2}(e^{ix} - e^{-ix}).$$

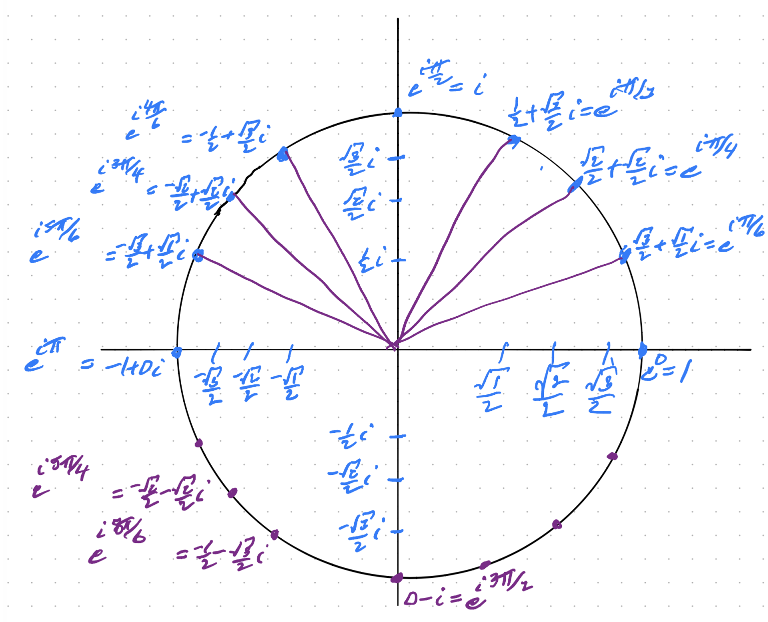
Define

$$\tan(x) = \frac{\sin(x)}{\cos(x)}, \qquad \cot(x) = \frac{\cos(x)}{\sin(x)}, \qquad \sec(x) = \frac{1}{\cos(x)}, \qquad \csc(x) = \frac{1}{\sin(x)}.$$

HW: Show that

$$\cos(x) = 1 - \tfrac{1}{2!}x^2 + \tfrac{1}{4!}x^4 - \cdots \qquad \text{and} \qquad \sin(x) = x - \tfrac{1}{3!}x^3 + \tfrac{1}{5!}x^5 - \cdots .$$

HW:. Show that $\quad e^{ix} = \cos(x) + i\sin(x) \quad$ and $\quad e^{-ix} = \cos(x) - i\sin(x).$



sines and cosines of the favorite angles

For $a, \theta \in \mathbb{R}$, let $r = e^a$ and $x = r\cos(\theta)$ and $y = r\sin(\theta)$. Then

$$z = e^{a+i\theta} = e^a e^{i\theta} = re^{i\theta} = r(\cos\theta + i\sin\theta) = (r\cos\theta) + i(r\sin\theta) = x + iy.$$

2.4 Inverse "functions"

This is a single page to exposit inverse "functions". The most important point is "undoes" and this is repeated several time on this page for emphasis. The other most important point is that an inverse "function" is often not a function.

$\sqrt{x}$ is the "function" that undoes x^2. This means that

$$\sqrt{x^2} = x \qquad \text{and} \qquad (\sqrt{x})^2 = x.$$

$\log x$ is the "function" that undoes e^x. This means that

$$\log(e^x) = x \qquad \text{and} \qquad e^{\log x} = x.$$

$\int \quad dx$ is the "function" that undoes $\frac{d}{dx}$. This means that

$$\int \frac{df}{dx} dx = f \qquad \text{and} \qquad \frac{d}{dx}\Big(\int f dx\Big) = f.$$

arcsin$x$ is the "function" that undoes $\sin x$. This means that

$$\text{arcsin}(\sin x) = x \qquad \text{and} \qquad \sin(\text{arcsin}x) = x.$$

arccos$x$ is the "function" that undoes $\cos x$. This means that

$$\text{arccos}(\cos x) = x \qquad \text{and} \qquad \cos(\text{arccos}x) = x.$$

arctan$x$ is the "function" that undoes $\tan x$. This means that

$$\text{arctan}(\tan x) = x \qquad \text{and} \qquad \tan(\text{arctan}x) = x.$$

arccot$x$ is the "function" that undoes $\cot x$. This means that

$$\text{arccot}(\cot x) = x \qquad \text{and} \qquad \cot(\text{arccot}x) = x.$$

arcsec$x$ is the "function" that undoes $\sec x$. This means that

$$\text{arcsec}(\sec x) = x \qquad \text{and} \qquad \sec(\text{arcsec}x) = x.$$

arccsc$x$ is the "function" that undoes $\csc x$. This means that

$$\text{arccsc}(\csc x) = x \qquad \text{and} \qquad \csc(\text{arccsc}x) = x.$$

$\log_a x$ is the "function" that undoes a^x. This means that

$$\log_a(a^{\sqrt{7}\pi i \sin 32}) = \sqrt{7}\pi i \sin 32 \qquad \text{and} \qquad a^{\log_a(\sqrt{7}\pi i \sin 32)} = \sqrt{7}\pi i \sin 32.$$

WARNING: In spite of the name, an inverse "function" is rarely a function. The output of an inverse function is usually a set of values, as opposed to a single value. For example

$$\sqrt{9} = \{3, -3\} \qquad \text{since } 3^2 = 9 \text{ and } (-3)^2 = 9.$$

Similarly,

$$\log(1) = \{0 + k2i\pi \mid k \in \mathbb{Z}\}, \qquad \text{since} \quad e^{0+k2i\pi} = (e^{i2\pi})^k = 1^k = 1.$$

and

$$\int 2x dx = \{x^2 + c \mid c \text{ is a constant}\}, \qquad \text{since} \quad \frac{d(x^2+c)}{dx} = \frac{dx^2}{dx} + \frac{dc}{dx} = 2x + 0 = 2x$$

when c is a constant.

2.5 Derivatives and integrals

This is a single page to cover derivatives and integrals. The reader may treat the Theorems as HW questions. The notation "Theorem" indicates that this HW question is often useful for doing other HW questions.

2.5.1 Derivatives

Let $\mathbb{C}((x)) = \{\frac{a(x)}{b(x)} \mid a(x), b(x) \in \mathbb{C}[[x]] \text{ and } b(x) \neq 0\}$ with $\frac{a}{b} = \frac{c}{d}$ if $ad = bc$. The *derivative with respect to* x on $\mathbb{C}((x))$ is the function $\frac{d}{dx} \colon \mathbb{C}((x)) \to \mathbb{C}((x))$ such that

$$\frac{d(f+g)}{dx} = \frac{df}{dx} + \frac{dg}{dx}, \qquad \frac{d(cf)}{dx} = c\frac{df}{dx}, \qquad \frac{d(fg)}{dx} = f\frac{dg}{dx} + \frac{df}{dx}g \qquad \text{and} \qquad \frac{dx}{dx} = 1,$$

for $f, g \in \mathbb{C}((x))$ and $c \in \mathbb{C}$ and where x denotes the identity function $\mathrm{id} \colon \mathbb{R} \to \mathbb{R}$.

Theorem 2.1. *(Chain rule and power formula)*

$$\frac{d(f \circ g)}{dx} = \frac{df}{dg}\frac{dg}{dx} \qquad \text{and} \qquad \frac{d(f^g)}{dx} = f^g\Big(\frac{g}{f}\frac{df}{dx} + \log f\frac{dg}{dx}\Big).$$

Theorem 2.2. *If* $n \in \mathbb{Z}_{\geq 0}$ *then*

$$\frac{dx^n}{dx} = nx^{n-1} \qquad \text{and} \qquad \frac{d\ e^x}{dx} = e^x.$$

Theorem 2.3. *If* $a \in \mathbb{C}$ *then*

$$\frac{d\ x^a}{dx} = ax^{a-1}, \qquad \frac{d\log x}{dx} = \frac{1}{x}, \qquad \frac{d\sin x}{dx} = \cos x, \qquad \frac{d\ \cos x}{dx} = -\sin x.$$

2.5.2 Integrals

The *integral* is backwards of the derivative, so that

$$\int \frac{df}{dx}\,dx = f. \tag{intdef}$$

The `product rule` for derivatives gives the formula for *integration by parts*:

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx \qquad \Big(\text{The } \texttt{product rule} \text{ is } \quad \frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}.\Big) \tag{IPR}$$

The `chain rule` for derivatives gives the formula for *substitution*:

$$\int u dv = \int u\frac{dv}{dx}dx \qquad \Big(\text{The } \texttt{chain rule} \text{ is } \quad \frac{d(f\circ v)}{dx} = \frac{df}{dv}\frac{dv}{dx}.\Big) \tag{ICR}$$

(get this by replacing $u = \frac{df}{dv}$ in the left side and the right side of $\int \frac{df}{dv}dv = f = \int \frac{df}{dx}dx = \int \frac{df}{dv}\frac{dv}{dx}dx$).

Remark 2.4. A more accurate expression for the integral is that the output of the integral is a set,

$$\int \frac{df}{dx}\,dx = \Big\{f + c \mid \frac{dc}{dx} = 0\Big\}, \qquad \text{since} \quad \frac{d(f+c)}{dx} = \frac{df}{dx} + \frac{dc}{dx} = \frac{df}{dx} + 0 = \frac{df}{dx}$$

when $\frac{dc}{dx} = 0$. Alternatively, write

$$\int \frac{df}{dx}\,dx = f + C, \qquad \text{where} \quad C = \Big\{c \mid \frac{dc}{dx} = 0\Big\}$$

(the set C is the *kernel* of $\frac{d}{dx}$). □

3 Graphing

3d-space is $\mathbb{R}^3 = \{|u_1, u_2, u_3\rangle \mid u_1, u_2, u_3 \in \mathbb{R}\}$ and *1d-space* is $\mathbb{R}^1 = \mathbb{R} = \{|u_1\rangle \mid u_1 \in \mathbb{R}\}$. (This notation follows the historical framework of Descartes and Dirac.)

3.1 Parallelipipeds

Let $u_1, u_2, u_3, v_1, v_2, v_3 \in \mathbb{R}$. Define

$$\det(u_1) = u_1, \qquad \det\begin{pmatrix} v_1 & v_2 \\ w_1 & w_2 \end{pmatrix} = v_1w_2 - w_1v_2$$

and

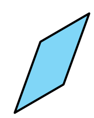
$$\det\begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix} = u_1(v_2w_3 - v_3w_2) - u_2(v_1w_3 - v_3w_1) + u_3(v_1w_2 - v_2w_1).$$

Proposition 3.1. *(Lengths of segments in $\mathbb{R}$) Let P be the segment with vertices $|0\rangle$ and $|u_1\rangle$. Then*

$$\text{(Length of segment } P) = \left|\det\left(u_1\right)\right|. \qquad \text{(lengthdetB)}$$

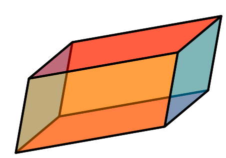
(Areas of parallelograms in $\mathbb{R}^2$) Let P be the parallelogram with vertices $|0, 0\rangle$, $|v_1, v_2\rangle$, $|w_1, w_2\rangle$ and $|v_1 + w_1, v_2 + w_2\rangle$. Then

$$(\text{Area of } P) = \left|\det\begin{pmatrix} v_1 & v_2 \\ w_1 & w_2 \end{pmatrix}\right|. \qquad \text{(areadetB)}$$



(Volumes of parallelipipeds $\mathbb{R}^3$) Let P be the parallelipiped with vertices $|0, 0, 0\rangle$, $(u_1, u_2, u_3\rangle$, $|v_1, v_2, v_3\rangle$, $|w_1, w_2, w_3\rangle$, $|u_1 + v_1, u_2 + v_2, u_3 + v_3\rangle$, $|u_1 + w_1, u_2 + w_2, u_3 + w_3\rangle$, $|v_1 + w_1, v_2 + w_2, v_3 + w_3\rangle$ and $|u_1 + v_1 + w_1, u_2 + v_2 + w_2, u_3 + v_3 + w_3\rangle$. Then

$$(\text{Volume of parallelipiped } P) = \left|\det\begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix}\right|. \qquad \text{(volumedetB)}$$



Let $u = |u_1, u_2, u_3\rangle$, $v = |v_1, v_2, v_3\rangle$, $w = |w_1, w_2, w_3\rangle$ in $\mathbb{R}^3$. The *cross product of v and w* is

$$v \times w = |\ v_2w_3 - v_3w_2, -(v_1w_3 - v_3w_1), v_1w_2 - v_2w_1\ \rangle \qquad \text{in } \mathbb{R}^3.$$

The *inner product of u and v* is

$$\langle u \mid v\rangle = u_1v_1 + u_2v_2 + u_3v_3 \qquad \text{and} \qquad \|u\| = \sqrt{u_1^2 + u_2^2 + u_3^2}$$

is the *length of u* and the *angle between v and w* is $\theta(v, w) \in \mathbb{R}_{[0,\pi]}$ given by

$$\cos(\theta(v, w)) = \frac{\langle v \mid w\rangle}{\|v\| \cdot \|w\|}.$$

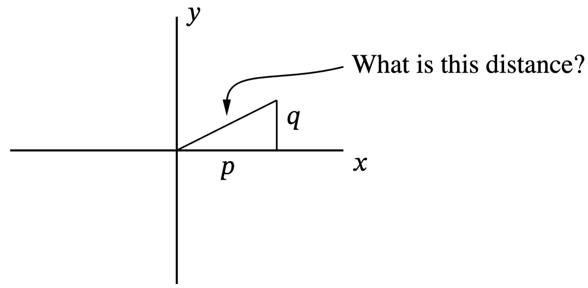
Proposition 3.2. *Let $u = |u_1, u_2, u_3\rangle$, $v = |v_1, v_2, v_3\rangle$, $w = |w_1, w_2, w_3\rangle$ in $\mathbb{R}^3$ and let $\theta(v, w)$ be the angle between v and w. Then*

$$\langle u|v \times w\rangle = \det\begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix}, \qquad \|v \times w\| = \|v\| \cdot \|w\| \sin(\theta(v, w)), \qquad \text{(crosslengthB)}$$

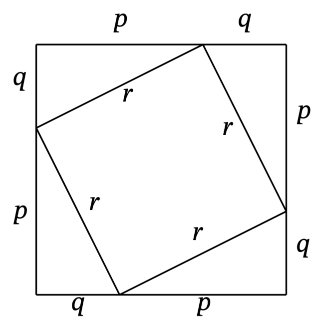
$$\text{and} \quad \hat{n} = \frac{1}{\|v \times w\|} v \times w \quad \text{is a unit vector orthogonal to } v \text{ and } w.$$

3.2 Circles

3.2.1 distances, the Pythagorean theorem and the equation of the basic circle



For the moment call it r.



Then

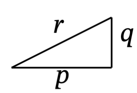
$$\begin{array}{ccccc} \text{Area of the outer square} & = & \text{Area of the inner square} & + & \text{Area of 4 triangles} \\ (p+q)^2 & = & r^2 & + & 4\cdot\frac{1}{2}pq. \end{array}$$

Solve for r.

$$\begin{aligned} r^2 &= (p+q)^2 - 4\cdot\tfrac{1}{2}pq \\ &= p^2+2pq+q^2-2pq = p^2+q^2. \end{aligned}$$

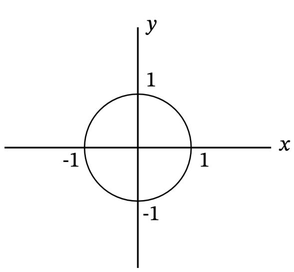
So

$$r^2 = p^2+q^2 \qquad \text{and} \qquad r=\sqrt{p^2+q^2}$$



This is the heavily used Pythagorean theorem. So what is the equation of the basic circle?

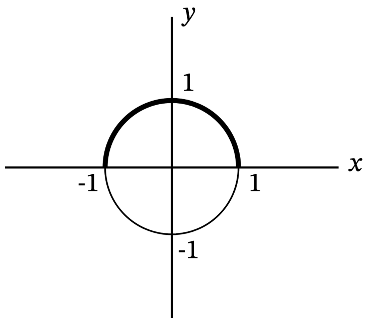
$$\sqrt{x^2+y^2}=1 \qquad \text{is} \quad \text{all points } (x,y) \text{ that are distance 1 from the origin.}$$



All points that are distance 1 from the origin

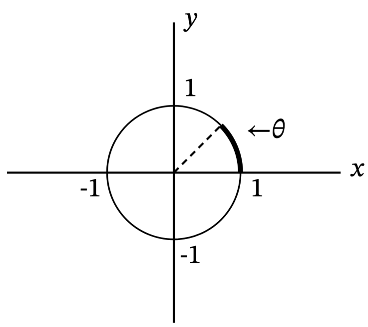
The basic circle $x^2+y^2=1$

3.2.2 angles, arc lengths, cosine, sine and the circumference of a circle



π is the distance halfway around a circle of radius 1

Measure angles according to the distance traveled on a circle of radius 1.



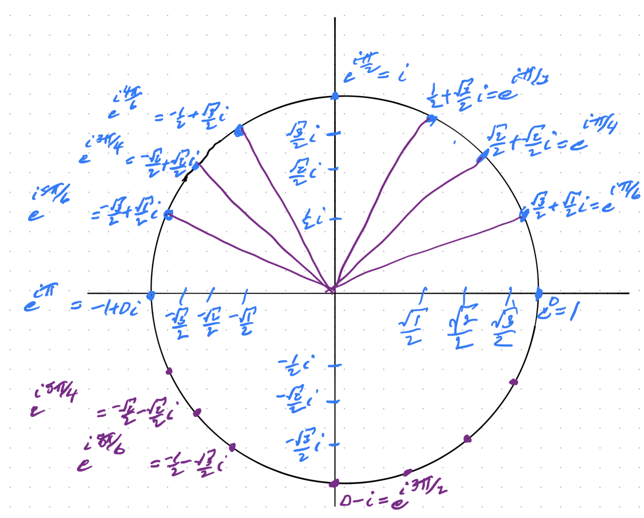
the angle θ is measured by traveling
a distance θ on a circle of radius 1

The *cosine* and *sine* are the x and y coordinates of the point at angle θ on a circle of radius 1,

$$\cos(\theta) = (x\text{-coordinate of point at angle } \theta \text{ on a circle of radius 1}),$$
$$\sin(\theta) = (y\text{-coordinate of point at angle } \theta \text{ on a circle of radius 1}).$$

HW: A priori, the functions $\cos\colon \mathbb{R} \to \mathbb{R}_{[-1,1]}$ and $\sin\colon \mathbb{R} \to \mathbb{R}_{[-1,1]}$ have nothing to do with the expressions $\cos(x)$ and $\sin(x)$ in $\mathbb{Q}[[x]]$ defined in Section 2.3. Make a precise connection and justify it carefully and thoroughly.

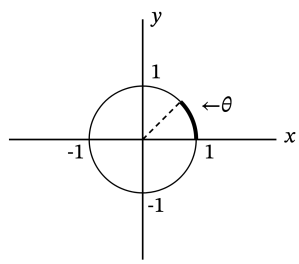
3.2.3 Favorite angles



sines and cosines of the favorite angles

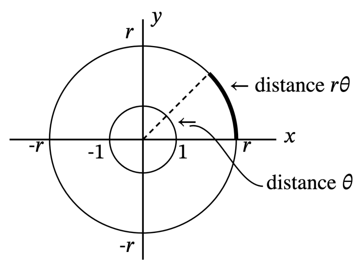
3.2.4 Circumference of a circle

Start with a circle of radius 1.



the angle θ is measured by traveling a distance θ on a circle of radius 1

Stretch both x-axis and the y-axis to get a circle of radius r.



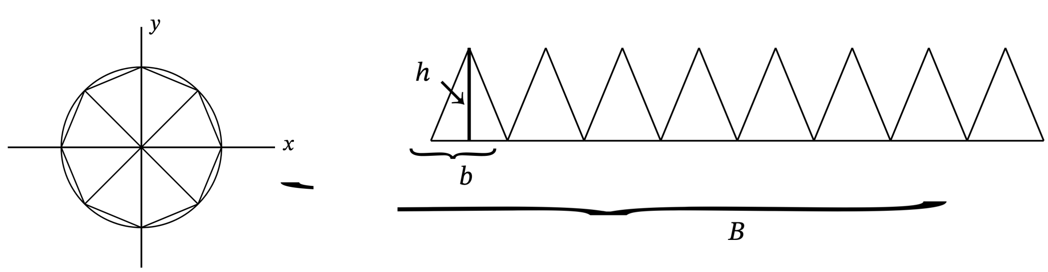
The distance θ stretches to $r\theta$

The distance 2π around a circle of radius 1 stretches to $2\pi r$ around a circle of radius r.

So the circumference of a circle of radius r is $2\pi r$.

3.2.5 Area of a circle

To find the area of a circle first approximate with a polygon inscribed in the circle. The eight triangles form an octagon P_8 in the circle. The area of the octagon is almost the same as the area of the circle. Unwrap the octagon.



The area of the octagon is the area of the 8 triangles. The area of each tringle is $\frac{1}{2}bh$. So the area of the octagon is $\frac{1}{2}Bh$, where $B = 8b$.

Teke the limit as the number of triangles in the interior polygon gets larger and larger (the polygon gets closer and closer to being the circle). If the base of each triangle is b and $B = nb$ then

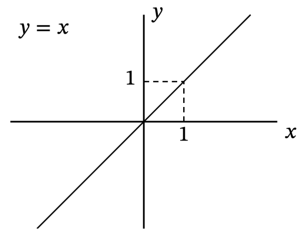
$$\begin{aligned}\text{Area of the circle} &= \lim_{n\to\infty}(\text{area of the } n\text{-sided polygon } P_n) = \lim_{n\to\infty}\left(n\tfrac{1}{2}bh\right) \\ &= \lim_{n\to\infty}\left(\tfrac{1}{2}Bh\right) = \tfrac{1}{2}\Big(\lim_{n\to\infty} B\Big)\cdot h = \tfrac{1}{2}\cdot 2\pi r\cdot r = \pi r^2,\end{aligned}$$

where B is the total base, h is the height of the triangle and $2\pi r$ is the length of an unwrapped circle, and r is the radius of the circle.

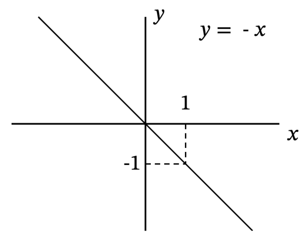
So the area of a circle of radius r is πr^2.

3.3 basic graphs

3.3.1 The basic line $y = x$

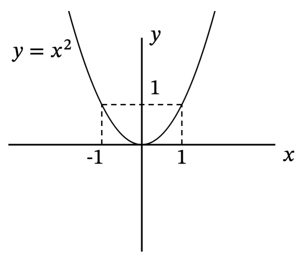


The line $y = x$

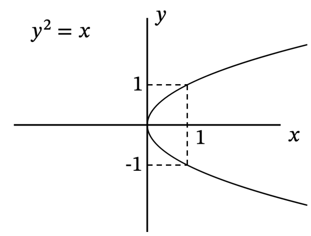


the line $y = -x$

3.3.2 The basic parabola $y = x^2$

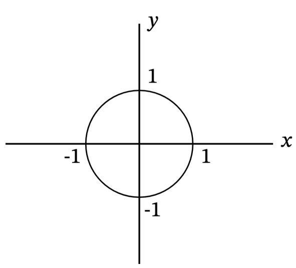


The parabola $y = x^2$



the parabola $y^2 = x$

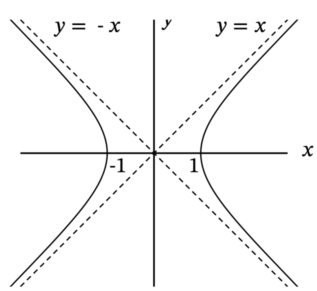
3.3.3 The basic circle $x^2 + y^2 = 1$



The circle $x^2 + y^2 = 1$

All points in $\mathbb{R}^2$ that are distance 1 from the origin.

3.3.4 The basic hyperbola $x^2 - y^2 = 1$



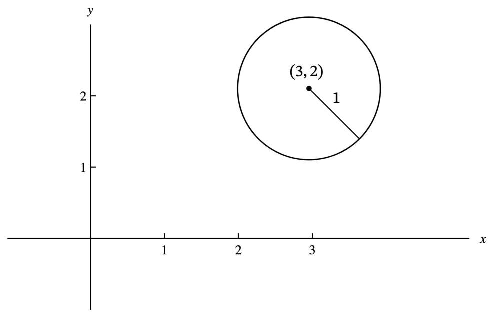
The hyperbola $x^2 - y^2 = 1$

See Section 3.5 for notes on how to derive the graph of the basic hyperbola $x^2 - y^2 = 1$.

3.4 Graphing: Shifting, scaling and flipping

3.4.1 Shifting

Example:. Graph $\{(x,y) \in \mathbb{R}^2 \mid (x-3)^2 + (y-2)^2 = 1\}$.

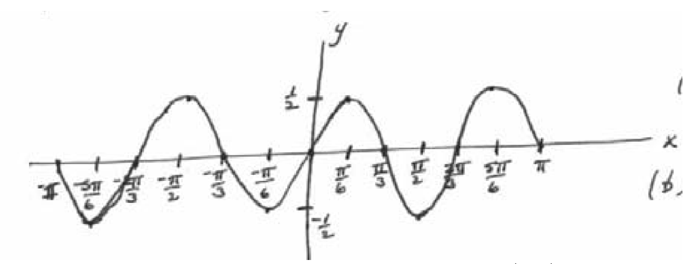


A circle of radius 1 and center $(3,2)$

To graph $(x-3)^2 + (y-2)^2 = 1$:
(a) $x^2 + y^2 = 1$ is a basic circle of radius 1.
(b) The center is shifted by
3 to the right in the x-direction,
2 upwards in the y-diection.

3.4.2 Scaling

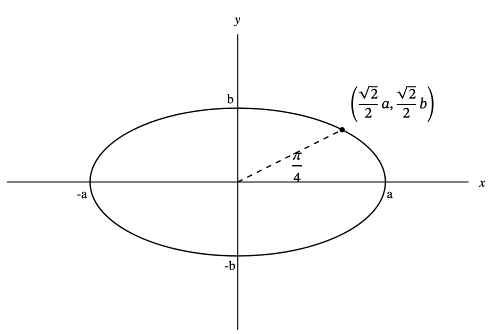
Example:. Graph $\{(x,y) \in \mathbb{R}^2 \mid 2y = \sin(3x)\}$.



Real solutions of $2y = \sin(3x)$

To graph $2y = \sin(3x)$:
(a) $y = \sin x$ is the basic graph.
(b) The x-axis is scaled (squished) by 3.
(c) The y-axis is scaled by 2.

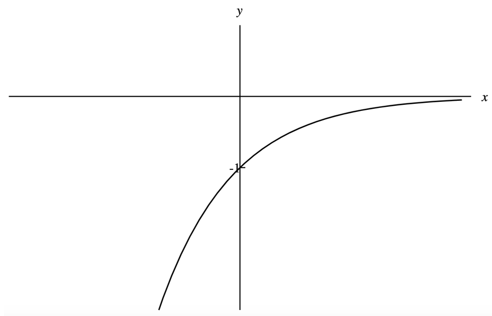
Example: Let $a, b \in \mathbb{R}_{>0}$. Graph $\{(x,y) \in \mathbb{R}^2 \mid \frac{1}{a^2}x^2 + \frac{1}{b^2}y^2 = 1\}$.



An ellipse with width $2a$ and height $2b$

3.4.3 Flipping

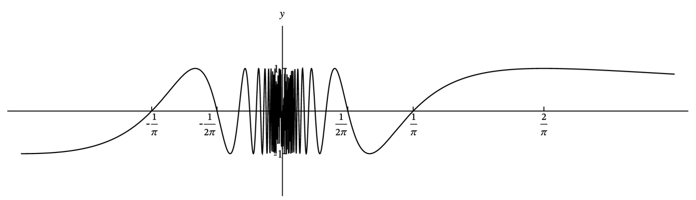
Example:. Graph $\{(x,y) \in \mathbb{R}^2 \mid y = -e^{-x}\}$.



Real solutions of $y = -e^{-x}$

To graph solutions of $y = -e^{-x}$:
(a) $y = e^x$ is the basic graph.
(b) $y = -e^{-x}$ is the same as $-y = e^{-x}$.
(c) The x-axis is flipped (around $x = 0$).
(d) The y-axis is flipped (around $y = 0$).

Example:. Graph $\{(x,y) \in \mathbb{R}^2 \mid y = \sin\left(\frac{1}{x}\right)\}$.

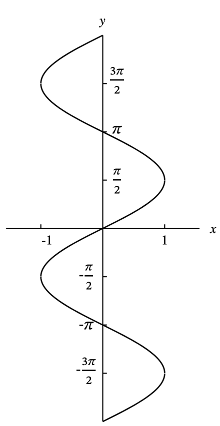


Real solutions of $y = \sin\left(\frac{1}{x}\right)$

To graph solutions of $y = \sin\left(\frac{1}{x}\right)$:

(a) $y = \sin x$ is the basic graph.
(b) The positive x axis is flipped (around $x = 1$).
(c) The negative x axis is flipped (around $x = -1$).
(d) As $x \to \infty$ then $\sin\left(\frac{1}{x}\right)$ is positive and gets close to 0.
(e) As $x \to -\infty$ then $\sin\left(\frac{1}{x}\right)$ is negative and gets close to 0.
(f) If x is positive and gets close to 0 then $\sin\left(\frac{1}{x}\right)$ oscillates between $+1$ and -1.

Example:. Graph $\{(x,y) \in \mathbb{R}^2 \mid y = \arcsin(x)\}$.



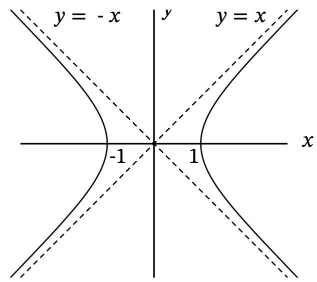
Real solutions of $y = \arcsin(x)$

To graph solutions $y = \arcsin(x)$:
(a) The graph of solutions of $y = \sin(x)$ is the basic graph.
(b) $y = \arcsin(x)$ is the same as $\sin(y) = x$.
So the x and y axis are switched from $y = \sin(x)$.
So the graph for $y = \sin(x)$ is flipped across the line $x = y$.

3.5 Asymptotes

A **asymptote** of a graph $y = f(x)$ as $x \to a$ is another graph $y = g(x)$ that the original graph $y = f(x)$ gets closer and closer to as x gets closer and closer to a.

Example: Graph the basic hyperbola $x^2 - y^2 = 1$.



The hyperbola $x^2 - y^2 = 1$

Graphing notes:

(a) If $y = 0$ then $x^2 = 1$. So $x = \pm 1$.

(b) If $x = 0$ then $-y^2 = 1$ which is impossible for $y \in \mathbb{R}$.

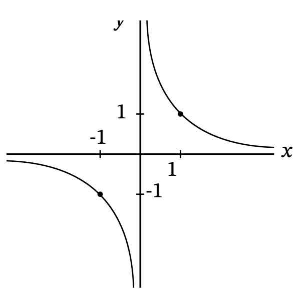
(c) The equation is $1 - \left(\frac{y}{x}\right)^2 = \left(\frac{1}{x}\right)^2$.

If x gets very big then $\frac{1}{x}$ gets closer and closer to 0 and the equation gets closer and closer to $1 - \left(\frac{y}{x}\right)^2 = 0$. This is the same as $\left(\frac{y}{x}\right)^2 = 1$, which is the same as $\frac{y}{x} = \pm 1$, i.e. $y = \pm x$. So, as x gets very large the equation gets closer and closer to $y = x$ and $y = -x$. As x gets very negative the basic hyperbola gets closer and closer to $y = x$ and $y = -x$.

Asymptotes:

$y = x$ is an asymptote of the basic hyperbola as $x \to +\infty$;
$y = -x$ is an asymptote of the basic hyperbola as $x \to +\infty$;
$y = x$ is an asymptote of the basic hyperbola as $x \to -\infty$;
$y = -x$ is an asymptote of the basic hyperbola as $x \to -\infty$.

Example: Graph $\{(x, y) \in \mathbb{R}^2 \mid y = \frac{1}{x}\}$.



Real solutions of $y = \frac{1}{x}$

(a) As x gets large $\frac{1}{x}$ gets closer and closeer to 0.
(b) As x gets closer to 0 (from the positive side) then $\frac{1}{x}$ gets larger and larger.
(c) As x gets closer to 0 (from the negative side) then $\frac{1}{x}$ gets more and more negative.
(d) As x gets more and more negative $\frac{1}{x}$ gets closer and closer to 0.
(e) If $x = 1$ then $y = 1$.
(f) If $x = -1$ then $y = -1$.

Asymptotes:

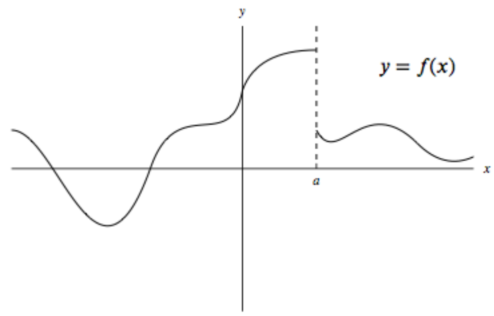
$y = 0$ (the x axis) is an asymptote to $y = \frac{1}{x}$ as $x \to +\infty$;
$y = 0$ (the x axis) is an asymptote to $y = \frac{1}{x}$ as $x \to -\infty$;
$x = 0$ (the y axis) is an asymptote to $y = \frac{1}{x}$ as $x \to 0^+$;
$x = 0$ (the y axis) is an asymptote to $y = \frac{1}{x}$ as $x \to 0^-$.

3.6 continuity and differentiablility – jumps and slopes

Let $a \in \mathbb{R}$. A function $f: \mathbb{R} \to \mathbb{R}$ is *continuous at a* if

$$\lim_{x \to a} f(x) = f(a).$$

In other words, a function $f: \mathbb{R} \to \mathbb{R}$ is continuous at a if the graph of f doesn't jump at $x = a$.

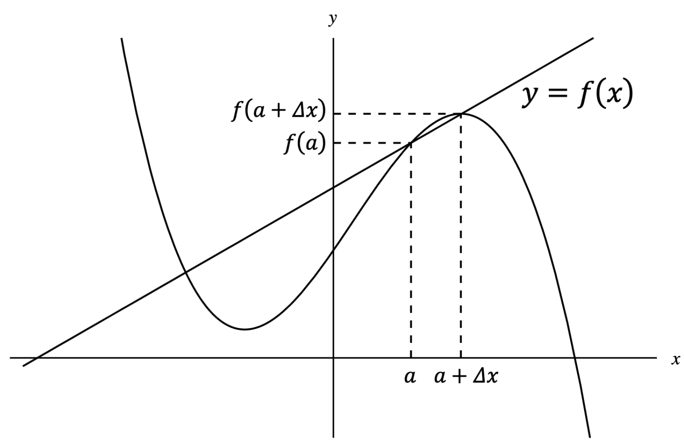


$f: \mathbb{R} \to \mathbb{R}$ is not continuous at a

Think about

$$\left.\frac{df}{dx}\right]_{x=a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

in terms of the graph



The slope of $f(x)$ at $x = a$

$$\begin{aligned}\frac{f(a+\Delta x) - f(a)}{\Delta x} &= \frac{\text{change in } f}{\text{change in } x} \\ &= \frac{\text{rise}}{\text{run}} \\ &= \text{slope of line connecting } (a, f(a)) \text{ and } (a + \Delta x, f(a + \Delta x)).\end{aligned}$$

This gives that

$$\lim_{\Delta x \to 0} \frac{f(a+\Delta x) - f(a)}{\Delta x} = (\text{slope of } f \text{ at the point } x = a).$$

A function is *differentiable at $x = a$* if the slope of graph of $f(x)$ at $x = a$ exists.

3.7 increasing, decreasing, concave up and concave down

A function $f(x)$ is **increasing** at $x = a$ if it is going up at $x = a$,

i.e., a function $f(x)$ is increasing at $x = a$ if $f(a + \Delta x) > f(a)$ for all small $\Delta x > 0$,

i.e., a function $f(x)$ is increasing at $x = a$ if the slope of $f(x)$ at $x = a$ is positive,

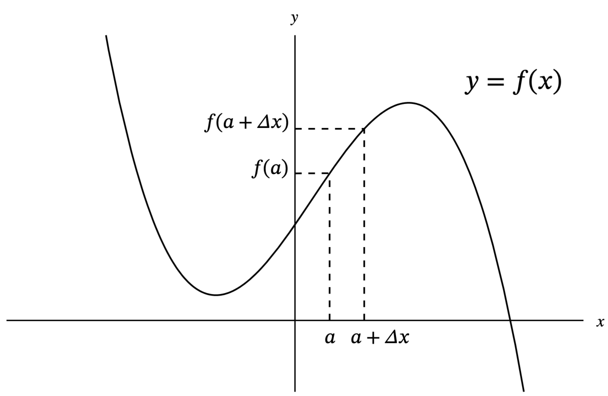
i.e., a function $f(x)$ is increasing at $x = a$ if $\frac{df}{dx}\Big]_{x=a} > 0$.

A function $f(x)$ is **decreasing** at $x = a$ if it is going up at $x = a$,

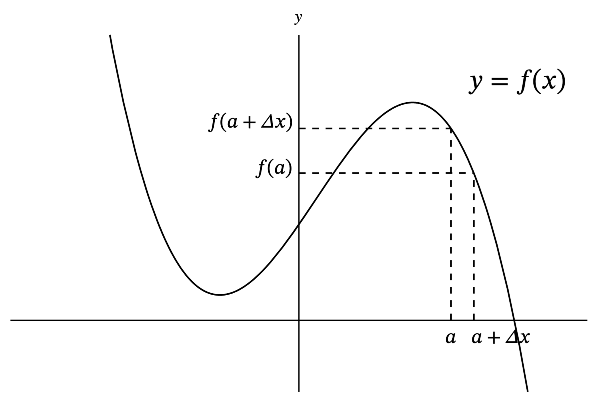
i.e., a function $f(x)$ is *decreasing* at $x = a$ if $f(a + \Delta x) < f(x)$ for all small $\Delta x < 0$,

i.e., a function $f(x)$ is *decreasing* at $x = a$ if the slope of $f(x)$ at $x = a$ is negative,

i.e., a function $f(x)$ is *decreasing* at $x = a$ if $\frac{df}{dx}\Big]_{x=a} < 0$.



Increasing at $x = a$



Decreasing at $x = a$

A function $f(x)$ is **concave up** at $x = a$ if it is right side up bowl shaped at $x = a$,

i.e., a function $f(x)$ is **concave up** at $x = a$ if the slope of f is getting larger at $x = a$,

i.e., a function $f(x)$ is **concave up** at $x = a$ if $\frac{df}{dx}$ is increasing at $x = a$,

i.e., a function $f(x)$ is **concave up** at $x = a$ if $\frac{d^2f}{dx^2}\Big]_{x=a} > 0$

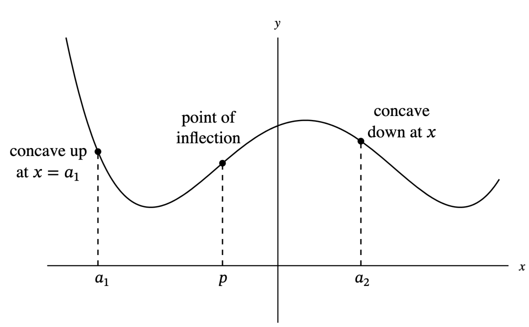
A function $f(x)$ is **concave down** at $x = a$ if it is up side down bowl shaped at $x = a$,

i.e., a function $f(x)$ is **concave down** at $x = a$ if the slope of f is getting smaller at $x = a$,

i.e., a function $f(x)$ is **concave down** at $x = a$ if $\frac{df}{dx}$ is decreasing at $x = a$,

i.e., a function $f(x)$ is **concave down** at $x = a$ if $\frac{d^2f}{dx^2}\Big]_{x=a} < 0$

A **point of inflection** is a point where f changes from concave up to concave down, or from concave down to concave up.

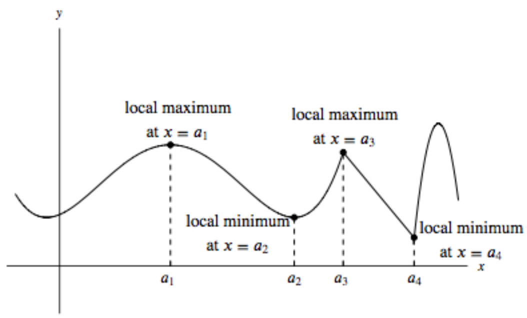


Concave up, concave down and points of inflection

3.8 local maxima and minima

A **local maximum** is a point $x = a$ where $f(a)$ is bigger than the $f(x)$ around it.

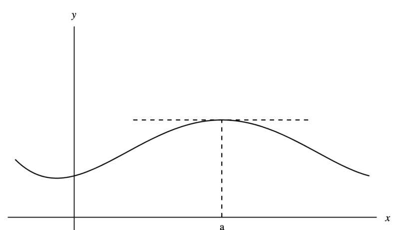
A **local minimum** is a point $x = a$ where $f(a)$ is smaller than the $f(x)$ around it i.e., $f(a) < f(a + \Delta x)$ for small Δx.



Local maximums and mniimums

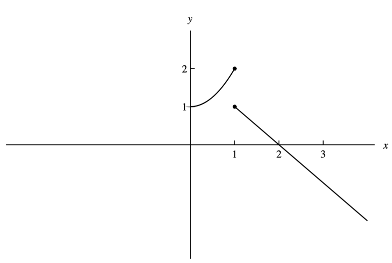
Where can a maximum or minimum occur?

(a) A point $x = a$ where $f(x)$ is differentiable and $\frac{df}{dx}\big]_{x=a} = 0$.



A $\frac{df}{dx}\big]_{x=a} = 0$ critical point

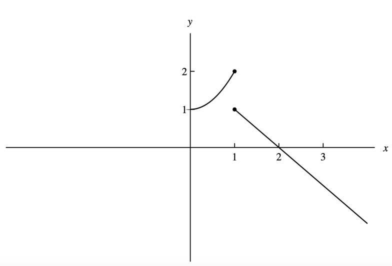
(b) A point $x = a$ where $f(x)$ is not continuous.



$$f(x) = \begin{cases} x^2 + 1, & \text{if } x \in \mathbb{R}_{[0,1]}, \\ 2 - x, & \text{if } x \in \mathbb{R}_{>1}. \end{cases}$$

A discontinuity critical point $\quad x = 1$ is a minimum in this example.

(c) A point $x = a$ on the boundary of where $f(x)$ is defined.



$$f(x) = \begin{cases} x^2 + 1, & \text{if } x \in \mathbb{R}_{[0,1]}, \\ 2 - x, & \text{if } x \in \mathbb{R}_{>1}. \end{cases}$$

A boundary critical point $\quad x = 0$ is a minimum in this example.

3.9 Relating limits and derivatives and relating limits and integrals

If
$$\frac{df}{dx} = g$$
then define
$$\frac{df}{dx}\Big]_{x=a} = g(a) \qquad \text{and} \qquad \Big(\int g\,dx\Big)\Big]_{x=a}^{x=b} = f(b) - f(a).$$

The fundamental theorem of change.

$$\frac{df}{dx}\Big]_{x=a} = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}.$$

Fundamental theorem of calculus.

$$\Big(\int g\,dx\Big)\Big]_{x=a}^{x=b} = \lim_{N\to\infty} \Big(g(a)\tfrac{1}{N} + g(a+\tfrac{1}{N})\tfrac{1}{N} + \cdots + g(b-\tfrac{1}{N})\tfrac{1}{N}\Big).$$

3.10 The fundamental theorem of change

Think about
$$\frac{df}{dx}\Big]_{x=a} = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}.$$
The core of what makes this equality true is that the weird limit on the right hand side satisfies the product rule: Assume that
$$D_f(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \qquad \text{and} \qquad D_g(a) = \lim_{h\to 0} \frac{g(a+h)-g(a)}{h}$$
both exist. Then
$$\begin{aligned}
D_{fg}\big]_{x=a} &= D_{fg}(a) = \lim_{h\to 0} \frac{(fg)(a+h)-(fg)(a)}{h} = \lim_{h\to 0} \frac{f(a+h)g(a+h)-f(a)g(a)}{h} \\
&= \lim_{h\to 0} \frac{(f(a+h)-f(a))(g(a+h)-g(a)+f(a+h)g(a)+f(a)g(a+h)-2f(a)g(a)}{h} \\
&= \lim_{h\to 0} \Big(h\frac{f(a+h)-f(a)}{h}\frac{g(a+h)-g(a)}{h} + \frac{(f(a+h)-f(a))g(a)}{h} + \frac{f(a)(g(a+h)-g(a)}{h} \\
&= \Big(\lim_{h\to 0} h\Big)\Big(\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}\Big)\Big(\lim_{h\to 0} \frac{g(a+h)-g(a)}{h}\Big) \\
&\qquad + \Big(\lim_{h\to 0} \frac{(f(a+h)-f(a))}{h} g(a)\Big) + \Big(\lim_{h\to 0} f(a)\frac{(g(a+h)-g(a)}{h}\Big) \\
&= 0\cdot D_f(a)D_f(b) + D_f(a)g(a) + f(a)D_g(a) \\
&= 0 + D_f(a)g(a) + f(a)D_g(a) \\
&= D_f(a)g(a) + f(a)D_g(a) \\
&= \big(D_f g + f D_g\big)\big]_{x=a} \qquad \Big(\text{The } \texttt{product rule} \text{ is } \quad \frac{d(fg)}{dx} = \frac{df}{dx}g + f\frac{dg}{dx}.\Big)
\end{aligned}$$

3.11 The fundamental theorem of calculus: interpreting the limit via areas

If

$$\frac{df}{dx} = g$$

then define

$$\frac{df}{dx}\Big]_{x=a} = g(a) \qquad \text{and} \qquad \left(\int g\,dx\right)\Big]_{x=a}^{x=b} = f(b) - f(a).$$

Fundamental theorem of change.

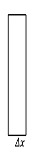
$$\frac{df}{dx}\Big]_{x=a} = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}.$$

Fundamental theorem of calculus.

$$\left(\int g\,dx\right)\Big]_{x=a}^{x=b} = \lim_{N\to\infty} \left(g(a)\tfrac{1}{N} + g(a+\tfrac{1}{N})\tfrac{1}{N} + \cdots + g(b-\tfrac{1}{N})\tfrac{1}{N}\right).$$

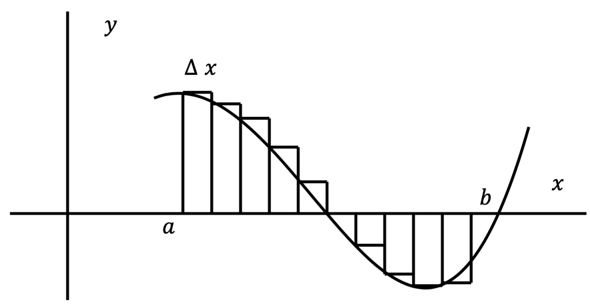
The right hand side

$$\lim_{N\to\infty} \left(g(a)\tfrac{1}{N} + g(a+\tfrac{1}{N})\tfrac{1}{N} + \cdots + g(b-\tfrac{1}{N})\tfrac{1}{N}\right)$$
$$= \lim_{N\to\infty} (\text{add up the areas of the little boxes of width } \Delta x = \tfrac{1}{N} \text{ and height } g(a+k\tfrac{1}{N}))$$



$g(a+\Delta x)$

little box



How little boxes are used to calculate an integral

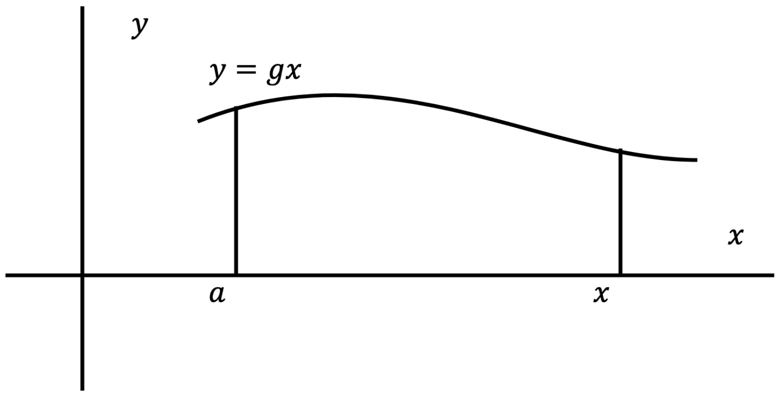
The leftmost box has area $g(a)\Delta x = g(a)\frac{1}{N}$.
The second box has area $g(a+\Delta x)\Delta x = g(a+\frac{1}{N})\frac{1}{N}$.
Continue this process.
So think of $\lim_{N\to\infty}\left(g(a)\frac{1}{N} + g(a+\frac{1}{N})\frac{1}{N} + \cdots + g(b-\frac{1}{N})\frac{1}{N}\right)$ as adding up areas from a to b of infinitesimally small boxes with area $g(x)\Delta x$.

3.12 Why the fundamental theorem of calculus works

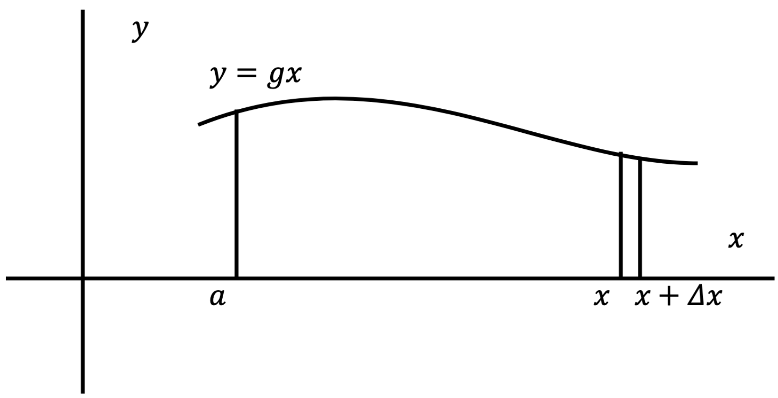
The fundamental theorem of calculus says that

$$(\text{Area under } g(x) \text{ from } a \text{ to } b) = A(b) - A(a), \qquad \text{where} \quad \int g(x)dx = A(x) + c.$$

Why does this work?
Let $A(x) =$ (area under $g(x)$ from a to x).



Area under $g(x)$ from a to b



Difference in area is the last little box

Then

$$\begin{aligned}\frac{dA}{dx} &= \lim_{\Delta x \to 0} \frac{A(x) - \Delta x - A(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{\text{area of last little box}}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{g(x)\Delta x}{\Delta x} \\ &= g(x).\end{aligned}$$

So

$$\begin{aligned}A(b) - A(a) &= (\text{area under } g(x) \text{ from } a \text{ to } b) - (\text{area under } g(x) \text{ from } a \text{ to } a) \\ &= (\text{area under } g(x) \text{ from } a \text{ to } b).\end{aligned}$$

4 Sets and functions

4.1 Sets

A *set* is a collection of objects which are called *elements*.
Write

$$s \in S \text{ if } s \text{ is an element of the set } S.$$

- The *empty set* $\emptyset$ is the set with no elements.
- A *subset* T of a set S is a set T such that if $t \in T$ then $t \in S$.

Write

$$T \subseteq S \text{ if } T \text{ is a subset of } S \text{, and}$$
$$T = S \text{ if the set } T \text{ is equal to the set } S.$$

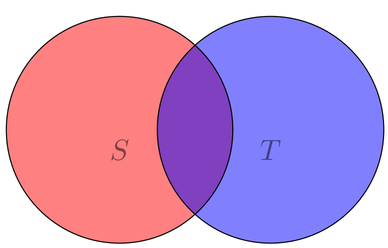
More precisely, $T = S$ if $T \subseteq S$ and $S \subseteq T$.

Let S and T be sets.

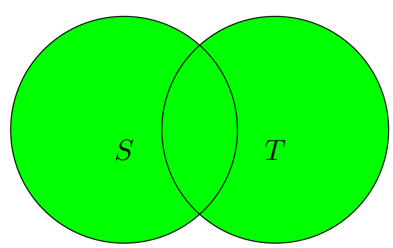
- The *union of* S *and* T is the set $S \cup T$ of all u such that $u \in S$ or $u \in T$,
$$S \cup T = \{u \mid u \in S \text{ or } u \in T\}.$$
- The *intersection of* S *and* T is the set $S \cup T$ of all u such that $u \in S$ and $u \in T$,
$$S \cap T = \{u \mid u \in S \text{ and } u \in T\}.$$
- The *product* S *and* T is the set $S \times T$ of all ordered pairs (s, t) where $s \in S$ and $t \in T$,
$$S \times T = \{(s, t) \mid s \in S \text{ and } t \in T\}.$$

The sets S and T are *disjoint* if $S \cap T = \emptyset$.
The set S is a *proper subset* of T if $S \subseteq T$ and $S \neq T$. Write

$$S \subsetneq T \qquad \text{if } S \text{ is a proper subset of } T.$$



The red (and purple) region is S
The blue (and purple) region is T
the purple region is $S \cap T$



the green region is $S \cup T$

4.2 Functions

Functions are for comparing sets.

Let S and T be sets. A *function from S to T* is a subset $\Gamma_f \subseteq S \times T$ such that

$$\text{if } s \in S \quad \text{then} \quad \text{there exists a unique } t \in T \text{ such that } (s,t) \in \Gamma_f.$$

Write

$$\Gamma_f = \{(s, f(s)) \mid s \in S\}$$

so that the function Γ_f can be expressed as

$$\text{an ``assignment''} \qquad \begin{array}{rccc} f\colon & S & \to & T \\ & s & \mapsto & f(s) \end{array}$$

which must satisfy

(a) If $s \in S$ then $f(s) \in T$, and

(b) If $s_1, s_2 \in S$ and $s_1 = s_2$ then $f(s_1) = f(s_2)$.

Let S and T be sets.

- Two functions $f\colon S \to T$ and $g\colon S \to T$ are *equal* if they satisfy

$$\text{if } s \in S \quad \text{then} \quad f(s) = g(s).$$

- A function $f\colon S \to T$ is *injective* if f satisfies the condition

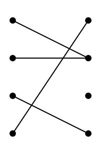
$$\text{if } s_1, s_2 \in S \text{ and } f(s_1) = f(s_2) \quad \text{then} \quad s_1 = s_2.$$

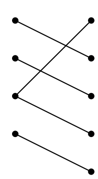
- A function $f\colon S \to T$ is *surjective* if f satisfies the condition

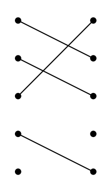
$$\text{if } t \in T \quad \text{then} \quad \text{there exists } s \in S \text{ such that } f(s) = t.$$

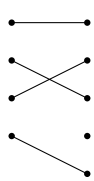
- A function $f\colon S \to T$ is *bijective* if f is both injective and surjective.

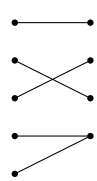
Examples. It is useful to visualize a function $f\colon S \to T$ as a graph with edges $(s, f(s))$ connecting elements $s \in S$ and $f(s) \in T$. With this in mind the following are examples:

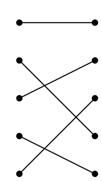

(a) a function


(b) not a function


(c) not a function


(d) an injective function


(e) a surjective function


(f) a bijective function

In these pictures the elements of the left column are the elements of the set S and the elements of the right column are the elements of the set T. In order to be a function the graph must have exactly one edge adjacent to each point in S. The function is injective if there is at most one edge adjacent to each point in T. The function is surjective if there is at least one edge adjacent to each point in T.

4.3 Composition of functions

Let $f\colon S \to T$ and $g\colon T \to U$ be functions. The *composition* of f and g is the function

$$g \circ f \quad \text{given by} \quad \begin{array}{rccc} g\circ f\colon & S & \to & U \\ & s & \mapsto & g(f(s)) \end{array}$$

Let S be a set. The *identity map on* S is the function given by

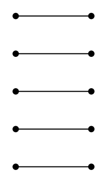
$$\begin{array}{rccc} \mathrm{id}_S\colon & S & \to & S \\ & s & \mapsto & s \end{array}$$

Let $f\colon S \to T$ be a function. The *inverse function to* f is a function

$$f^{-1}\colon T \to S \qquad \text{such that} \qquad f\circ f^{-1} = \mathrm{id}_T \quad \text{and} \quad f^{-1}\circ f = \mathrm{id}_S.$$

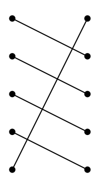
Theorem 4.1. *Let $f\colon S \to T$ be a function. An inverse function to f exists if and only if f is bijective.*

Representing functions as graphs, the identity function id_S looks like

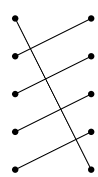


(a) the identity function id_S

In the pictures below, if the left graph is a pictorial representation of a function $f\colon S \to T$ then the inverse function to f, $f^{-1}\colon T \to S$, is represented by the graph on the right; the graph for f^{-1} is the mirror-image of the graph for f.

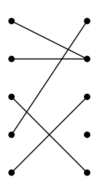


(b) the function f

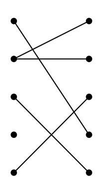


(c) the function f^{-1}

Graph (d) below, represents a function $g\colon S \to T$ which is not bijective. The inverse function to g does not exist in this case: the graph (e) of a possible candidate, is not the graph of a function.



(d) the function g



(e) not a function

4.4 Cardinality

Let S and T be sets. The sets *S and T are isomorphic*, or *have the same cardinality*

$$\text{if there is a bijective function} \quad \varphi\colon S \to T.$$

Write

$$\mathrm{Card}(S) = \mathrm{Card}(T) \qquad \text{if } S \text{ and } T \text{ have the same cardinality.}$$

Notation: Let S be a set. Write

$$\mathrm{Card}(S) = \begin{cases} 0, & \text{if } S = \emptyset, \\ n, & \text{if } \mathrm{Card}(S) = \mathrm{Card}(\{1, 2, \ldots, n\}), \\ \infty, & \text{otherwise.} \end{cases}$$

Note that even in the cases where $\mathrm{Card}(S) = \infty$ and $\mathrm{Card}(T) = \infty$ it may be that $\mathrm{Card}(S) \neq \mathrm{Card}(T)$.

Let S be a set.

- The set S is *finite* if there exists $n \in \mathbb{Z}_{\geq 0}$ such that $\mathrm{Card}(S) = \mathrm{Card}(\{1, \ldots, n\})$.
- The set S is *infinite* if $\mathrm{Card}(S)$ is not finite.
- The set S is *countable* if $\mathrm{Card}(S) = \mathrm{Card}(\mathbb{Z}_{>0})$.
- The set S is *uncountable* if S is not countable.

Some authors define: 'The set S is *countable* if $\mathrm{Card}(S) = \mathrm{Card}(\mathbb{Z}_{>0})$ or S is finite.' The choice in this book is that finite sets are finite and countable sets are not finite.

4.5 Images and fibers

Let $f\colon S \to T$ be a function. Let $A \subseteq S$ and let $B \subseteq T$. The *image of A* is

$$f(A) = \{f(a) \mid a \in A\} \qquad \text{and} \qquad f^{-1}(B) = \{s \in S \mid f(s) \in B\},$$

is the *fiber over B*. Let $t \in T$. The *fiber over t* is

$$f^{-1}(t) = f^{-1}(\{t\}) = \{s \in S \mid f(s) = t\} \qquad \text{and} \qquad \mathrm{im}(f) = f(S) = \{f(s) \mid s \in S\}$$

is the *image of f*.

Let $S/(f)$ be the set of fibers of f,

$$S(f) = \{f^{-1}(t) \mid t \in T\}.$$

The elements of $S(f)$ are, themselves, sets. Then

$$\begin{array}{rrcl} \hat{f}\colon & S_f & \to & \mathrm{im}(f) \\ & f^{-1}(t) & \mapsto & t, \end{array} \qquad \begin{array}{rrcl} p\colon & S & \to & S_f \\ & s & \to & f^{-1}(f(s)), \end{array} \qquad \begin{array}{rrcl} \iota\colon & \mathrm{im}(f) & \to & T \\ & f(s) & \to & f(s) \end{array}$$

define functions such that

(a) $\hat{f}$ is bijective,
(b) p is surjective, and $f = \iota \circ \hat{f} \circ p$.
(c) ι is injective,

4.6 Relations, equivalence relations and partitions

Let S be a set.

- A *relation* $\sim$ *on* S is a subset $R_\sim$ of $S \times S$. Write $s_1 \sim s_2$ if the pair (s_1, s_2) is in the subset $R_\sim$ so that
$$R_\sim = \{(s_1, s_2) \in S \times S \mid s_1 \sim s_2\}.$$
- An *equivalence relation* on S is a relation $\sim$ on S such that
  (a) if $s \in S$ then $s \sim s$,
  (b) if $s_1, s_2 \in S$ and $s_1 \sim s_2$ then $s_2 \sim s_1$,
  (c) if $s_1, s_2, s_3 \in S$ and $s_1 \sim s_2$ and $s_2 \sim s_3$ then $s_1 \sim s_3$.

Let $\sim$ be an equivalence relation on a set S and let $s \in S$. The *equivalence class of* s is the set
$$[s] = \{t \in S \mid t \sim s\}.$$

A *partition of a set* S is a collection $\mathcal{P}$ of subsets of S such that

(a) If $s \in S$ then there exists $P \in \mathcal{P}$ such that $s \in P$, and

(b) If $P_1, P_2 \in \mathcal{P}$ and $P_1 \cap P_2 \neq \emptyset$ then $P_1 = P_2$.

Theorem 4.2.

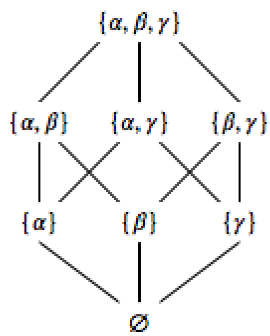
(a) If S is a set and let $\sim$ be an equivalence relation on S then
$$\textit{the set of equivalence classes of} \quad \sim \quad \textit{is a partition of } S.$$

(b) If S is a set and $\mathcal{P}$ is a partition of S then
$$\textit{the relation defined by} \qquad s \sim t \quad \textit{if } s \textit{ and } t \textit{ are in the same } P \in \mathcal{P}$$
is an equivalence relation on S.

4.7 Partially ordered sets

Let S be a set.

- A *partial order* on S is a relation $\leq$ on S such that
  (a) If $x \in A$ then $x \leq x$,
  (b) If $x, y, z \in S$ and $x \leq y$ and $y \leq z$ then $x \leq z$, and
  (c) If $x, y \in S$ and $x \leq y$ and $y \leq x$ then $x = y$.
- A *total order* on S is a partial order $\leq$ such that
  (d) If $x, y \in S$ then $x \leq y$ or $y \leq x$.
- A *partially ordered set*, or *poset*, is a set S with a partial order $\leq$ on S.
- A *totally ordered set* is a set S with a total order $\leq$ on S.



The poset of subsets of $\{\alpha, \beta, \gamma\}$ with inclusion as $\le$

Let S be a poset. Write

$$x < y \quad \text{if} \quad x \le y \text{ and } x \ne y.$$

- The *Hasse diagram* of S is the graph with vertices S and directed edges given by

$$x \to y \qquad \text{if } x \le y.$$

- A *lower order ideal of* S is a subset E of S such that

$$\text{if } y \in E \text{ and } x \in S \text{ and } x \le y \quad \text{then} \quad x \in E.$$

- The *intervals in* S are the sets

$$\begin{array}{ll} S_{[a,b]} = \{x \in S \mid a \le x \le b\} & S_{(a,b)} = \{x \in S \mid a < x < b\} \\ S_{[a,b)} = \{x \in S \mid a \le x < b\} & S_{(a,b]} = \{x \in S \mid a < x \le b\} \\ S_{(-\infty,b]} = \{x \in S \mid x \le b\} & S_{[a,\infty)} = \{x \in S \mid a \le x\} \\ S_{(-\infty,b)} = \{x \in S \mid x < b\} & S_{(a,\infty)} = \{x \in S \mid a < x\} \end{array}$$

  for $a, b \in S$.

4.8 Upper and lower bounds, sup and inf

Let S be a poset and let E be a subset of S.

- An *upper bound of* E *in* S is an element $b \in S$ such that if $y \in E$ then $y \le b$.
- A *lower bound of* E *in* S is an element $l \in S$ such that if $y \in E$ then $l \le y$.
- A *greatest lower bound of* E *in* S is an element $\inf(E) \in S$ such that
  (a) $\inf(E)$ is a lower bound of E in S, and
  (b) If $l \in S$ is a lower bound of E in S then $l \le \inf(E)$.
- A *least upper bound of* E *in* S is an element $\sup(E) \in S$ such that
  (a) $\sup(E)$ is a upper bound of E in S, and
  (b) If $b \in S$ is a upper bound of E in S then $\sup(E) \le b$.
- The set E is *bounded in* S if E has both an upper bound and a lower bound in S.

Proposition 4.3. *Let S be a poset and let E be a subset of S. If $\sup(E)$ exists then $\sup(E)$ is unique.*

5 The "number systems" $\mathbb{Z}_{>0}$, $\mathbb{Z}_{\geq 0}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{R}^2$, $\mathbb{R}^n$

5.1 Numbers and intervals

The positive integers: $\mathbb{Z}_{>0} = \{1, 2, 3, \ldots\}$.

The nonnegative integers: $\mathbb{Z}_{\geq 0} = \{0, 1, 2, 3, \ldots\}$.

The integers: $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.

The rational numbers: $\mathbb{Q} = \left\{ \frac{a}{b} \mid a \in \mathbb{Z},\ b \in \mathbb{Z}_{\neq 0} \text{ and } \frac{a}{b} = \frac{c}{d} \text{ if } ad = bc \right\}$.

The real numbers:

$$\mathbb{R} = \{\pm a_\ell a_{\ell-1} \ldots a_1 a_0 . a_{-1} a_{-2} \ldots \mid \ell \in \mathbb{Z}_{\geq 0},\ a_i \in \{0, \ldots, 9\},\ a_\ell \neq 0 \text{ if } \ell > 0\}.$$

with a requirement that if $a_k \neq 9$ then $\pm a_\ell ... a_{k+1} a_k 9999... = \pm a_\ell ... a_{k+1}(a_k + 1)000...$ so that, for example, $0.9999... = 1.0000....$

The complex numbers:

$$\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\} \qquad \text{with } i^2 = -1.$$

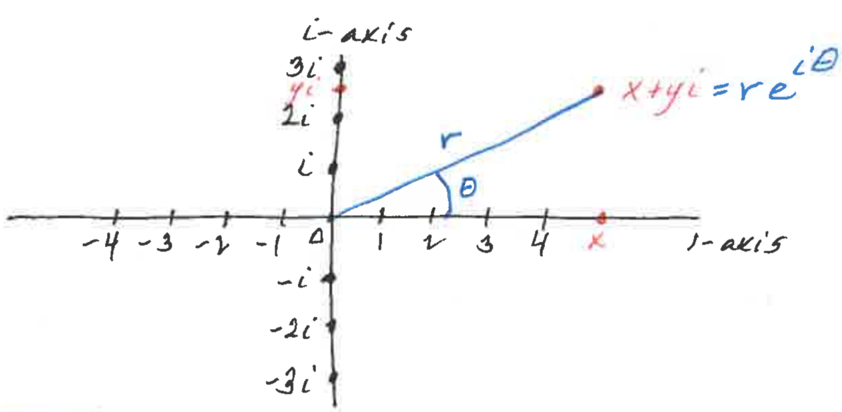
Let $a, b \in \mathbb{R}$ with $a < b$. Define

$$\mathbb{R}_{(a,b)} = \{x \in \mathbb{R} \mid a < x < b\}, \qquad \mathbb{R}_{[a,b)} = \{x \in \mathbb{R} \mid a \leq x < b\}$$

$$\mathbb{R}_{(a,b]} = \{x \in \mathbb{R} \mid a < x \leq b\}, \qquad \mathbb{R}_{[a,b]} = \{x \in \mathbb{R} \mid a \leq x \leq b\}$$

$$\mathbb{R}_{(a,\infty)} = \{x \in \mathbb{R} \mid a < x\}, \qquad \mathbb{R}_{[a,\infty)} = \{x \in \mathbb{R} \mid a \leq x\}$$

$$\mathbb{R}_{(\infty,a)} = \{x \in \mathbb{R} \mid x < a\}, \qquad \mathbb{R}_{(\infty,a]} = \{x \in \mathbb{R} \mid x \leq a\}.$$



Picture of $\mathbb{Z} \subseteq \mathbb{R} \subseteq \mathbb{C}$

What does $\frac{1}{a}$ really mean?

$$\frac{1}{a} \text{ is the number that when multiplied by } a \text{ gives } 1.$$

5.2 The number systems $\mathbb{R}$, $\mathbb{Q}_p$ and $\mathbb{R}((t))$

5.2.1 The real numbers

The real numbers $\mathbb{R}$ is the set of decimal expansions.

The *real numbers* $\mathbb{R}$ contain the *integers* $\mathbb{Z}$.

$$\begin{aligned}
\mathbb{R} &= \left\{ \pm(a_{-\ell}\left(\tfrac{1}{10}\right)^{-\ell} + a_{-\ell+1}\left(\tfrac{1}{10}\right)^{-\ell+1} + a_{-\ell+2}\left(\tfrac{1}{10}\right)^{-\ell+2} + \cdots) \mid \ell \in \mathbb{Z},\ a_j \in \frac{\mathbb{Z}}{10\mathbb{Z}} \right\} \\
&\cup| \\
\mathbb{Z} &= \left\{ \pm(a_{-\ell}\left(\tfrac{1}{10}\right)^{-\ell} + a_{-\ell+1}\left(\tfrac{1}{10}\right)^{-\ell+1} + \cdots + a_{-1}\left(\tfrac{1}{10}\right) + a_0) \mid \ell \in \mathbb{Z}_{\geq 0},\ a_j \in \frac{\mathbb{Z}}{10\mathbb{Z}} \right\}
\end{aligned}$$

where $\frac{\mathbb{Z}}{10\mathbb{Z}} = \{0,1,2,3,4,5,6,7,8,9\}$ and the addition and multiplication in $\mathbb{R}$ are compatible with the addition and multiplication in $\mathbb{Z}$.

5.2.2 The p-adic numbers

Let $p \in \mathbb{Z}_{>0}$. The *p-adic numbers* $\mathbb{Q}_p$ contain the *p-adic integers* $\mathbb{Z}_p$ and the *nonnegative integers* $\mathbb{Z}_{\geq 0}$.

$$\begin{aligned}
\mathbb{Q}_p &= \left\{ a_{-\ell}p^{-\ell} + a_{-\ell+1}p^{-\ell+1} + a_{-\ell+2}p^{-\ell+2} + \cdots \;\middle|\; \ell \in \mathbb{Z},\ a_j \in \frac{\mathbb{Z}}{p\mathbb{Z}} \right\} \\
&\cup| \\
\mathbb{Z}_p &= \left\{ a_0p^0 + a_1p^1 + a_2p^2 + \cdots \;\middle|\; a_j \in \frac{\mathbb{Z}}{p\mathbb{Z}} \right\} \\
&\cup| \\
\mathbb{Z}_{\geq 0} &= \left\{ a_0p^0 + a_1p^1 + a_2p^2 + \cdots \;\middle|\; a_j \in \frac{\mathbb{Z}}{p\mathbb{Z}} \text{ and all but a finite number of the } a_j \text{ are } 0 \right\},
\end{aligned}$$

where $\frac{\mathbb{Z}}{p\mathbb{Z}} = \{0,1,2,\ldots,p-2,p-1\}$ and the addition and multiplication in $\mathbb{Q}_p$ and $\mathbb{Z}_p$ are compatible with the addition and multiplication in $\mathbb{Z}$.

5.2.3 Extended polynomials

Let t be a variable.

The *rational functions* $\mathbb{R}((t))$ contain the *formal power series* $\mathbb{R}[[t]]$ and the *polynomials* $\mathbb{R}[t]$.

$$\begin{aligned}
\mathbb{R}((t)) &= \left\{ a_{-\ell}t^{-\ell} + a_{-\ell+1}t^{-\ell+1} + a_{-\ell+2}t^{-\ell+2} + \cdots \mid \ell \in \mathbb{Z},\ a_j \in \mathbb{R} \right\} \\
&\cup| \\
\mathbb{R}[[t]] &= \left\{ a_0t^0 + a_1t^1 + a_2t^2 + \cdots \mid a_j \in \mathbb{R} \right\} \\
&\cup| \\
\mathbb{R}[t] &= \left\{ a_0t^0 + a_1t^1 + a_2t^2 + \cdots \mid a_j \in \mathbb{R} \text{ and all but a finite number of the } a_j \text{ are } 0 \right\},
\end{aligned}$$

where $\mathbb{R}$ is the real numbers and the addition and multiplication in $\mathbb{R}((t))$ and $\mathbb{R}[[t]]$ are compatible with the addition and multiplication in $\mathbb{R}$.

5.2.4 Some examples to check.

In $\mathbb{R}$,

$$\begin{aligned}
\tfrac{1}{2} &= .5000000\ldots = 5\cdot 10^{-1} + 0\cdot 10^{-2} + 0\cdot 10^{-3} + \cdots, \\
-1 &= -(1\cdot 10^{0} + 0\cdot 10^{-1} + 0\cdot 10^{-2} + \cdots), \\
\pi &= 3.1415926\ldots = 3\cdot 10^{0} + 1\cdot 10^{-1} + 4\cdot 10^{-2} + \cdots, \\
1 &= 1.00000.... = 1\cdot 10^{0} + 0\cdot 10^{-1} + 0\cdot 10^{-2} + \cdots \\
&= 0.999999 = 9\cdot 10^{-1} + 9\cdot 10^{-2} + 9\cdot 10^{-3} + 9\cdot 10^{-4} + \cdots.
\end{aligned}$$

In $\mathbb{Q}_7$,

$$\begin{aligned}
888 &= 6 + 0\cdot 7 + 4\cdot 7^2 + 1\cdot 7^3 + 0\cdot 7^4 + 0\cdot 7^5 + 0\cdot 7^6 + \cdots, \\
-\tfrac{1}{6} &= \frac{1}{1-7} = 1 + 1\cdot 7 + 1\cdot 7^2 + 1\cdot 7^3 + 1\cdot 7^4 + \cdots, \\
-1 &= 6\cdot\left(-\tfrac{1}{6}\right) = 6 + 6\cdot 7 + 6\cdot 7^2 + 6\cdot 7^3 + 6\cdot 7^4 + \cdots, \\
\tfrac{1}{2} &= 1 + 3\cdot\left(-\tfrac{1}{6}\right) = 4 + 3\cdot 7 + 3\cdot 7^2 + 3\cdot 7^3 + 3\cdot 7^4 + \cdots, \\
-6 &= 1 + 7\cdot(-1) = 1 + 6\cdot 7 + 6\cdot 7^2 + 6\cdot 7^3 + 6\cdot 7^4 + \cdots,
\end{aligned}$$

In $\mathbb{R}((t))$,

$$\begin{aligned}
\frac{1}{1-t} &= 1 + t + t^2 + t^3 + t^4 + \cdots, \\
e^t &= 1 + t + \frac{1}{2!}t^2 + \frac{1}{3!}t^3 + \frac{1}{4!}t^4 + \cdots, \\
\sin t &= t - \frac{1}{3!}t^3 + \frac{1}{5!}t^5 - \frac{1}{7!}t^7 + \cdots, \\
\frac{1}{t^3(1-t)} &= t^{-3} + t^{-2} + t^{-1} + t + t^2 + \cdots.
\end{aligned}$$

5.2.5 $\mathbb{R}$ and $\mathbb{Q}_p$ and $\mathbb{R}((t))$ are metric spaces

Fix a number $e \in \mathbb{R}_{>0}$.
If $x, y \in \mathbb{R}$ the *distance* between x and y is

$$d(x,y) = e^{-\mathrm{val}_{1/10}(y-x)}, \quad \text{where}$$

$$\mathrm{val}_{1/10}\big(\pm(a_\ell\left(\tfrac{1}{10}\right)^{\ell} + a_{\ell-1}\left(\tfrac{1}{10}\right)^{\ell+1} + a_{l-2}\left(\tfrac{1}{10}\right)^{\ell+2} + \cdots)\big) = \ell$$

if $\ell \in \mathbb{Z}$ is minimal such that $a_\ell \neq 0$.
If $x, y \in \mathbb{Q}_p$ then the *distance* between x and y is

$$d(x,y) = e^{-\mathrm{val}_p(y-x)}, \quad \text{where} \quad \mathrm{val}_p\big(a_\ell p^\ell + a_{\ell+1}p^{\ell+1} + a_{\ell+2}p^{\ell+2} + \cdots\big) = \ell$$

if $\ell \in \mathbb{Z}$ is minimal such that $a_\ell \neq 0$.
If $x, y \in \mathbb{R}((t))$ then the *distance* between x and y is

$$d(x,y) = e^{-\mathrm{val}_t(y-x)} \quad \text{where} \quad \mathrm{val}_t\big(a_\ell t^\ell + a_{\ell+1}t^{\ell+1} + a_{\ell+2}t^{\ell+2} + \cdots\big) = \ell$$

if $\ell \in \mathbb{Z}$ is minimal such that $a_\ell \neq 0$.

5.3 The complex numbers $\mathbb{C}$

The *complex numbers* is the $\mathbb{R}$-algebra

$$\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R}\} \qquad \text{with} \quad i^2 = -1,$$

so that if $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ then

$$\begin{aligned} z_1 + z_2 &= (x_1 + iy_1) + (x_2 + iy_2) \\ &= (x_1 + x_2) + i(y_1 + y_2) \end{aligned} \qquad \text{and} \qquad \begin{aligned} z_1 z_2 &= (x_1 + iy_1)(x_2 + iy_2) \\ &= x_1x_2 + i(x_1y_2 + x_2y_1) + i^2 y_1y_2 \\ &= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1). \end{aligned}$$

The *complex conjugation*, or *Galois automorphism*, is the function

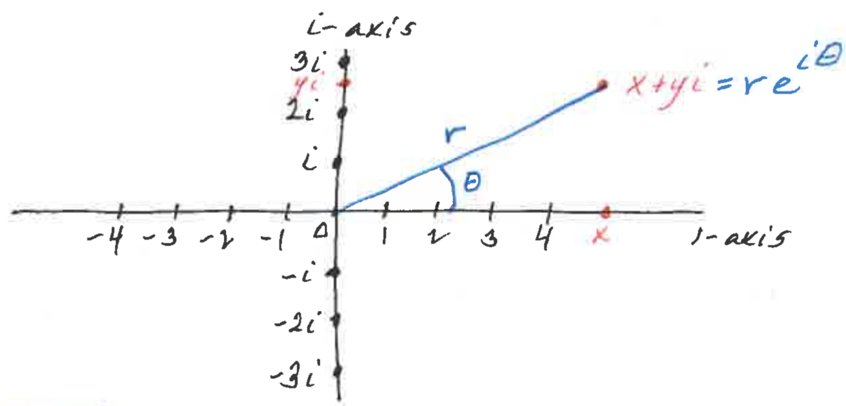
$$\overline{} \colon \mathbb{C} \to \mathbb{C} \qquad \text{given by} \quad \overline{x + iy} = x - iy.$$

The *norm*, or *length function*, on $\mathbb{C}$ is the function

$$|\ |\colon \mathbb{C} \to \mathbb{R}_{\ge 0} \qquad \text{given by} \quad |x + iy| = \sqrt{x^2 + y^2}.$$

The *Hermitian form*, or *inner product*, on $\mathbb{C}$ is

$$\langle , \rangle \colon \mathbb{C} \times \mathbb{C} \to \mathbb{C} \qquad \text{given by} \quad \langle z_1, z_2 \rangle = z_1 \overline{z_2}.$$



Graphing complex numbers

If $r \in \mathbb{R}_{\ge 0}$ and $\theta \in \mathbb{R}$ then

$$re^{i\theta} = r\cos\theta + ir\sin\theta.$$

If $z \in \mathbb{C}$ and $z \neq 0$ then

$$z^{-1} = \frac{1}{|z|^2}\overline{z},$$

since if $z = x + iy$ then

$$\frac{1}{z} = \frac{1}{x + iy} = \frac{1}{(x + iy)}\frac{(x - iy)}{(x - iy)} = \frac{x - iy}{x^2 + y^2} = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2}.$$

HW: Show that if $z \in \mathbb{C}$ then $|z|^2 = z\overline{z}$.
HW: Show that if $z_1, z_2 \in \mathbb{C}$ then $|z_1 z_2| = |z_1|\,|z_2|$.

5.4 The 3d-space-time $\mathbb{D}$

The 3*d-space-time*, or the *Hamiltonians*, or the *quaternions*, is the vector space

$$\mathbb{D} = \mathbb{R}\text{-span}\{1, i, j, k\} = \{t + xi + yj + zk \mid t, x, y, z \in \mathbb{R}\}$$

with product determined by

$$i^2 = j^2 = k^2 = -1, \qquad ij = -ji = k, \qquad jk = -kj = i, \qquad ki = -ik = j,$$

and the distributive laws. The 3d-space-time $\mathbb{D}$ is a generalization of 1d-space-time $\mathbb{C}$, where

$$\mathbb{C} = \{xi + t \mid x, t \in \mathbb{R}\} \qquad \text{with product determined by} \quad i^2 = -1 \quad \text{and the distributive laws.}$$

The 3*d-space* $\mathbb{R}^3$ is a subspace of 3d-space-time $\mathbb{D}$,

$$\begin{array}{ll} \mathbb{D} & = \{t + xi + yj + zk \mid t, x, y, z \in \mathbb{R}\} \\ \cup| & \\ \mathbb{R}^3 & = \{xi + yj + zk \mid x, y, z \in \mathbb{R}\} \end{array}$$

For $v_1 = x_1 i + y_1 j + z_1 k$ and $v_2 = x_2 i + y_2 j + z_2 k$ in $\mathbb{R}^3$ define

$$\begin{aligned} \langle v_1 \,|\, v_2 \rangle &= x_1 y_1 + y_1 y_2 + z_1 z_2 \qquad \text{and} \\ v_1 \times v_2 &= (y_1 z_2 - z_1 y_2) i - (x_1 z_2 - z_1 x_2) j + (x_1 y_2 - y_1 x_2) k. \end{aligned}$$

Then

$$\begin{aligned} v_1 v_2 &= (x_1 i + y_1 j + z_1 k)(x_2 i + y_2 j + z_2 k) \\ &= -(x_1 x_2 + y_1 y_2 + z_1 z_2) + (x_1 y_2 ij + y_1 x_2 ji) + (x_1 z_2 ik + z_1 x_2 ki) + (y_1 z_2 jk + z_1 y_2 kj) \\ &= -\langle v_1, v_2 \rangle + (x_1 y_2 - y_1 x_2) k - (x_1 z_2 - z_1 x_2) j + (y_1 z_2 - z_1 y_2) i \\ &= -\langle v_1 \,|\, v_2 \rangle + v_1 \times v_2. \end{aligned}$$

This computation shows how the standard inner product and the cross product arise from the multiplication in 3d-space-time.

The *Galois automorphism*, or *conjugation map*, is the function $\overline{} \colon \mathbb{D} \to \mathbb{D}$ given by

$$\overline{t + xi + yj + zk} = t - xi - yj - zk, \qquad \text{for } t, x, y, z \in \mathbb{R}.$$

The *norm on* $\mathbb{D}$ is the function $\|\ \| \colon \mathbb{D} \to \mathbb{R}_{\geq 0}$ given by

$$\|d\| = \sqrt{d\,\overline{d}}.$$

HW:. Show that if $d_1, d_2 \in \mathbb{D}$ then $\overline{d_1 d_2} = \overline{d_1} \cdot \overline{d_2}$.

HW:. Show that if $t, x, y, z \in \mathbb{R}$ and $d = t + xi + yj + zk$ then

$$d\overline{d} = t^2 + x^2 + y^2 + z^2.$$

HW:. Show that if $d \in \mathbb{D}$ and $d \neq 0$ then

$$\frac{1}{\|d\|^2}\overline{d} = d^{-1}.$$

5.5 The favourite space $\mathbb{R}^2$

The favourite example is $\mathbb{R}^2 = \{(x_1, x_2) \mid x_1, x_2 \in \mathbb{R}\}$ with *addition* and *scalar multiplication* given by

$$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2) \qquad \text{and} \qquad c(x_1, x_2) = (cx_1, cx_2), \quad \text{for } c \in \mathbb{R},$$

with *inner product*

$$\begin{matrix} \mathbb{R}^2 \times \mathbb{R}^2 & \longrightarrow & \mathbb{R}_{\ge 0} \\ (x, y) & \longmapsto & \langle x, y\rangle \end{matrix} \qquad \text{given by} \qquad \langle (x_1, x_2), (y_1, y_2)\rangle = x_1y_1 + x_2y_2,$$

with *norm*

$$\begin{matrix} \mathbb{R}^2 & \longrightarrow & \mathbb{R}_{\ge 0} \\ x & \longmapsto & \|x\| \end{matrix} \qquad \text{given by} \qquad \|(x_1, x_2)\| = \sqrt{x_1^2 + x_2^2},$$

with *metric* $d\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}_{\ge 0}$ given by

$$d((x_1, x_2), (y_1, y_2)) = \|(x_1, x_2) - (y_1, y_2)\| = \|(x_1 - y_1, x_2 - y_2)\| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2},$$

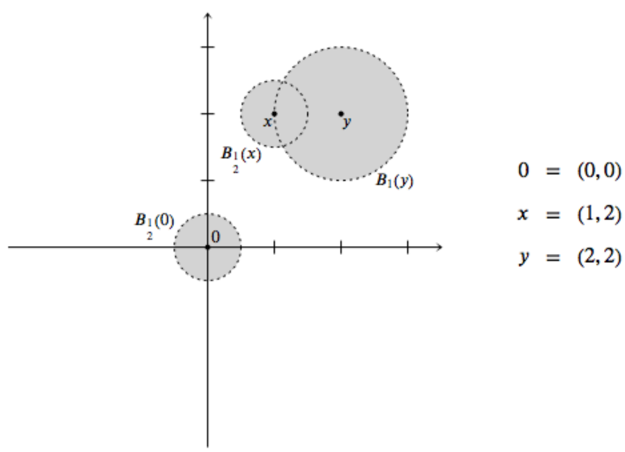
with *angle function* $\theta\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}_{[0,2\pi)}$ given by

$$\theta((x_1, x_2), (y_1, y_2)) = \arccos\Big(\frac{\langle (x_1, x_2), (y_1, y_2)\rangle}{\|(x_1, x_2)\| \cdot \|(y_1, y_2)\|}\Big),$$

and

$$B_\epsilon(x) = \{y \in \mathbb{R}^2 \mid d(y, x) < \epsilon\}$$

is the *ball of radius ϵ centered at x* (yes, to stress, strongly, that we normally assume that the set $\mathbb{R}^2$ is endowed with lots of extra structures this is, intentionally, a very run-on sentence).



Open balls in $\mathbb{R}^2$.

5.6 The favourite spaces $\mathbb{R}^n$

5.6.1 n-tuples are functions

Let $n \in \mathbb{Z}_{>0}$. Identify n-tuples $(x_1, \ldots, x_n)$ of elements of $\mathbb{R}$ with functions $\vec{x}\colon \{1, 2, \ldots, n\} \to \mathbb{R}$ so that

$$\text{the } n\text{-tuple } (x_1, \ldots, x_n) \qquad \text{is identified with the function} \qquad \begin{matrix} \vec{x}\colon & \{1, \ldots, n\} & \to & \mathbb{R} \\ & i & \mapsto & x_i \end{matrix}$$

5.6.2 The vector space $\mathbb{R}^n$

Let $n \in \mathbb{Z}_{\geq 0}$. The space of functions from $\{1, 2, \ldots, n\}$ to $\mathbb{R}$ is

$$\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) \mid x_i \in \mathbb{R}\} = \{\text{functions } \vec{x} \colon \{1, \ldots, n\} \to \mathbb{R}\}.$$

with *addition and scalar multiplication* given by

$$(x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) \quad \text{and}$$
$$c(x_1, x_2, \ldots, x_n) = (cx_1, cx_2, \ldots, x_n), \qquad \text{for } c \in \mathbb{R},$$

and with *inner product* $\langle\, , \,\rangle \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ given by

$$\langle (x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \rangle = x_1 y_1 + x_2 y_2 + \cdots x_n y_n,$$

with *norm*

$$\begin{matrix} \mathbb{R}^n & \longrightarrow & \mathbb{R}_{\geq 0} \\ x & \longmapsto & \|x\| \end{matrix} \qquad \text{given by} \qquad \|(x_1, x_2, \ldots, x_n)\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2},$$

with *metric* $d \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ given by

$$\begin{aligned} d((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)) &= \|(x_1, x_2, \ldots, x_n) - (y_1, y_2, \ldots, y_n)\| \\ = \|(x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n)\| &= \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}, \end{aligned}$$

with *angle function* $\theta \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{[0, 2\pi)}$ given by

$$\theta((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)) = \arccos\Big(\frac{\langle (x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \rangle}{\|(x_1, x_2, \ldots, x_n)\| \cdot \|(y_1, y_2, \ldots, y_n)\|} \Big),$$

and

$$B_\epsilon(x) = \{y \in \mathbb{R}^n \mid d(y, x) < \epsilon\}$$

is the *ball of radius ϵ centered at x* (yes, to stress, strongly, that we normally assume that the set $\mathbb{R}^n$ is endowed with lots of extra structures this is, intentionally, a very run-on sentence).

5.7 Matrices

Let $\mathbb{F}$ be a field. Let $m, n \in \mathbb{Z}_{>0}$.

- An $m \times n$ matrix with entries in $\mathbb{F}$ is a table of elements of $\mathbb{F}$ with m rows and n columns. More precisely, an $m \times n$ matrix with entries in $\mathbb{F}$ is a function

$$A\colon \{1, \ldots, m\} \times \{1, \ldots, n\} \longrightarrow \mathbb{F}.$$

- A *column vector of length* n is an $n \times 1$ matrix.
- A *row vector of length* n is an $1 \times n$ matrix.
- The (i,j) *entry of a matrix* A is the element $A(i,j)$ in row i and column j of A.

$$A = \begin{pmatrix} A(1,1) & A(1,2) & \cdots & A(1,m) \\ A(2,1) & A(2,2) & \cdots & A(2,m) \\ \vdots & & & \vdots \\ A(n,1) & A(n,2) & \cdots & A(n,m) \end{pmatrix}$$

Let $M_{m\times n}(\mathbb{F})$ be the set of $m \times n$ matrices with entries in $\mathbb{F}$.
Let $M_n(\mathbb{F}) = M_{n\times n}(\mathbb{F})$ be the set of $n \times n$ matrices with entries in $\mathbb{F}$.

- The *sum* of $m \times n$ matrices A and B is the $m \times n$ matrix $A + B$ given by

$$(A+B)(i,j) = A(i,j) + B(i,j), \qquad \text{for } i \in \{1, \ldots, m\} \text{ and } j \in \{1, \ldots, n\}.$$

- The *scalar multiplication* of an element $c \in \mathbb{F}$ with an $m \times n$ matrix A is the $m \times n$ matrix $c \cdot A$ given by

$$(c \cdot A)(i,j) = c \cdot A(i,j), \qquad \text{for } i \in \{1, \ldots, m\} \text{ and } j \in \{1, \ldots, n\}.$$

- The *product* of an $m \times n$ matrix A and an $n \times p$ matrix B is the $m \times p$ matrix AB given by

$$\begin{aligned}(AB)(i,k) &= \sum_{j=1}^{n} A(i,j)B(j,k) \\ &= A(i,1)B(1,k) + A(i,2)B(2,k) + \cdots + A(i,n)B(n,k),\end{aligned}$$

  for $i \in \{1, \ldots, m\}$ and $k \in \{1, \ldots, p\}$.

The *zero matrix* is the $m \times n$ matrix $0 \in M_{m\times n}(\mathbb{F})$ given by

$$0(i,j) = 0, \qquad \text{for } i \in \{1, \ldots, m\} \text{ and } j \in \{1, \ldots, n\}.$$

The *negative* of a matrix $A \in M_{m\times n}(\mathbb{F})$ is the matrix $-A \in M_{m\times n}(\mathbb{F})$ given by

$$(-A)(i,j) = -A(i,j), \qquad \text{for } i \in \{1, \ldots, m\} \text{ and } j \in \{1, \ldots, n\}.$$

For $k \in \{1, \ldots, m\}$ and $\ell \in \{1, \ldots, n\}$ let $E_{k\ell} \in M_{m\times n}(\mathbb{F})$ be the matrix given by

$$E_{k\ell}(i,j) = \begin{cases} 1, & \text{if } i = k \text{ and } j = \ell, \\ 0, & \text{otherwise,} \end{cases}$$

so that $E_{k\ell}$ has a 1 in the (k,ℓ) entry and all other entries 0.

Proposition 5.1. *Let $m, n \in \mathbb{Z}_{>0}$ and let $M_{m\times n}(\mathbb{F})$ be the set of $m \times n$ matrices with entries in $\mathbb{F}$.*

(a) If $A, B, C \in M_{m\times n}(\mathbb{F})$ then $A + (B + C) = (A + B) + C$.

(b) If $A, B \in M_{m\times n}(\mathbb{F})$ then $A + B = B + A$.

(c) If $A \in M_{m\times n}(\mathbb{F})$ then $0 + A = A$ and $A + 0 = A$.

(d) If $A \in M_{m\times n}(\mathbb{F})$ then $(-A) + A = 0$ and $A + (-A) = 0$.

(e) If $A \in M_{m\times n}(\mathbb{F})$ and $c_1, c_2 \in \mathbb{F}$ then $c_1 \cdot (c_2 \cdot A) = (c_1c_2) \cdot A$.

(f) If $A \in M_{m\times n}(\mathbb{F})$ and $1 \in \mathbb{F}$ is the identity in $\mathbb{F}$ then $1 \cdot A = A$.

The *Kronecker delta* is given by

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases}$$

The *identity matrix* is the $n \times n$ matrix $1 \in M_{n\times n}(\mathbb{F})$ given by

$$1(i, j) = \delta_{ij}, \qquad \text{for } i \in \{1, \ldots, m\} \text{ and } j \in \{1, \ldots, n\}.$$

Proposition 5.2. *Let $n \in \mathbb{Z}_{>0}$ and let $M_n(\mathbb{F})$ be the set of $n \times n$ matrices in $\mathbb{F}$.*

(a) If $A, B, C \in M_n(\mathbb{F})$ then $A + (B + C) = (A + B) + C$.

(b) If $A, B \in M_n(\mathbb{F})$ then $A + B = B + A$.

(c) If $A \in M_n(\mathbb{F})$ then $0 + A = A$ and $A + 0 = A$.

(d) If $A \in M_n(\mathbb{F})$ then $(-A) + A = 0$ and $A + (-A) = 0$.

(e) If $A, B, C \in M_n(\mathbb{F})$ then $A(BC) = (AB)C$.

(f) If $A, B, C \in M_n(\mathbb{F})$ then $(A + B)C = AC + BC$ and $C(A + B) = CA + CB$.

(g) If $A \in M_n(\mathbb{F})$ then $1A = A$ and $A1 = A$.

The *transpose* of an $m \times n$ matrix A is the $n \times m$ matrix A^t given by

$$A^t(i, j) = A(j, i), \qquad \text{for } i \in \{1, \ldots, n\} \text{ and } j \in \{1, \ldots, m\}.$$

Proposition 5.3. *Let $m, n \in \mathbb{Z}_{>0}$, let $M_{m\times n}(\mathbb{F})$ be the set of $m \times n$ matrices with entries in $\mathbb{F}$, and let $M_n(\mathbb{F})$ be the set of $n \times n$ matrices in $\mathbb{F}$.*

(a) If $A, B \in M_{m\times n}(\mathbb{F})$ then $(A + B)^t = A^t + B^t$,

(b) If $A \in M_{m\times n}(\mathbb{F})$ and $c \in \mathbb{F}$ then $(c \cdot A)^t = c \cdot A^t$,

(c) If $A, B \in M_n(\mathbb{F})$ then $(AB)^t = B^tA^t$.

(d) If $A \in M_n(\mathbb{F})$ then $(A^t)^t = A$.

Proposition 5.4. *Let $m, n \in \mathbb{Z}_{>0}$, let $M_{m\times n}(\mathbb{F})$ be the set of $m \times n$ matrices with entries in $\mathbb{F}$. Then*

(a) (span) $M_{m\times n}(\mathbb{F}) = \left\{ \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}E_{ij} \mid c_{ij} \in \mathbb{F} \right\}$.

(b) (linear independence) *If $c_{11}, \ldots, c_{mn} \in \mathbb{F}$ and*

$$\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}E_{ij} = 0 \qquad \textit{then} \qquad \textit{if } k \in \{1, \ldots, m\} \textit{ and } \ell \in \{1, \ldots, n\} \textit{ then } c_{k\ell} = 0.$$

5.8 Fields

A *field* is a set $\mathbb{F}$ with functions

$$\begin{array}{ccc} \mathbb{F}\times\mathbb{F} & \longrightarrow & \mathbb{F} \\ (a,b) & \longmapsto & a+b \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \mathbb{F}\times\mathbb{F} & \longrightarrow & \mathbb{F} \\ (a,b) & \longmapsto & ab \end{array}$$

such that

(Fa) If $a,b,c\in\mathbb{F}$ then $(a+b)+c=a+(b+c)$,

(Fb) If $a,b\in\mathbb{F}$ then $a+b=b+a$,

(Fc) There exists $0\in\mathbb{F}$ such that

$$\text{if } a\in\mathbb{F} \quad \text{then} \quad 0+a=a \text{ and } a+0=a,$$

(Fd) If $a\in\mathbb{F}$ then there exists $-a\in\mathbb{F}$ such that $a+(-a)=0$ and $(-a)+a=0$,

(Fe) If $a,b,c\in\mathbb{F}$ then $(ab)c=a(bc)$,

(Ff) If $a,b,c\in\mathbb{F}$ then

$$(a+b)c=ac+bc \qquad \text{and} \qquad c(a+b)=ca+cb,$$

(Fg) There exists $1\in\mathbb{F}$ such that

$$\text{if } a\in\mathbb{F} \quad \text{then} \quad 1\cdot a=a \text{ and } a\cdot 1=a,$$

(Fh) If $a\in\mathbb{F}$ and $a\neq 0$ then there exists $a^{-1}\in\mathbb{F}$ such that $aa^{-1}=1$ and $a^{-1}a=1$,

(Fi) If $a,b\in\mathbb{F}$ then $ab=ba$.

Proposition 5.5. *Let $\mathbb{F}$ be a field.*

(a) If $a\in\mathbb{F}$ then $a\cdot 0=0$.

(b) If $a\in\mathbb{F}$ then $-(-a)=a$.

(c) If $a\in\mathbb{F}$ and $a\neq 0$ then $(a^{-1})^{-1}=a$.

(d) If $a\in\mathbb{F}$ then $a(-1)=-a$.

(e) If $a,b\in\mathbb{F}$ then $(-a)b=-ab$.

(f) If $a,b\in\mathbb{F}$ then $(-a)(-b)=ab$.

5.9 $\mathbb{Z}$-algebras

A $\mathbb{Z}$-*algebra* is a set $\mathbb{A}$ with functions

$$\begin{array}{ccc} \mathbb{A}\times\mathbb{A} & \longrightarrow & \mathbb{A} \\ (a,b) & \longmapsto & a+b \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \mathbb{A}\times\mathbb{A} & \longrightarrow & \mathbb{A} \\ (a,b) & \longmapsto & ab \end{array}$$

such that

(Aa) If $a,b,c\in\mathbb{F}$ then $(a+b)+c=a+(b+c)$,

(Ab) If $a,b\in\mathbb{F}$ then $a+b=b+a$, (Ac) There exists $0 \in \mathbb{F}$ such that

$$\text{if } a \in \mathbb{F} \quad \text{then} \quad 0 + a = a \text{ and } a + 0 = a,$$

(Ad) If $a \in \mathbb{F}$ then there exists $-a \in \mathbb{F}$ such that $a + (-a) = 0$ and $(-a) + a = 0$,

(Ae) If $a, b, c \in \mathbb{F}$ then $(ab)c = a(bc)$,

(Af) If $a, b, c \in \mathbb{F}$ then

$$(a + b)c = ac + bc \qquad \text{and} \qquad c(a + b) = ca + cb,$$

(Ag) There exists $1 \in \mathbb{F}$ such that

$$\text{if } a \in \mathbb{F} \quad \text{then} \quad 1 \cdot a = a \text{ and } a \cdot 1 = a,$$

Proposition 5.6. *Let $\mathbb{A}$ be a $\mathbb{Z}$-algebra.*

(a) If $a \in \mathbb{A}$ then $a \cdot 0 = 0$.

(b) If $a \in \mathbb{A}$ then $-(-a) = a$.

(d) If $a \in \mathbb{A}$ then $a(-1) = -a$.

(e) If $a, b \in \mathbb{A}$ then $(-a)b = -ab$.

(f) If $a, b \in \mathbb{A}$ then $(-a)(-b) = ab$.

5.10 Ordered fields

An *ordered field* is a field $\mathbb{F}$ with a total order $\le$ such that

(OFa) If $a, b, c \in \mathbb{F}$ and $a \le b$ then $a + c \le b + c$,

(OFb) If $a, b \in \mathbb{F}$ and $a \ge 0$ and $b \ge 0$ then $ab \ge 0$.

Proposition 5.7. *Let $\mathbb{F}$ be an ordered field.*

(a) If $a \in \mathbb{F}$ and $a > 0$ then $-a < 0$.

(b) If $a \in \mathbb{F}$ and $a \ne 0$ then $a^2 > 0$.

(c) $1 \ge 0$.

(d) If $a \in \mathbb{F}$ and $a > 0$ then $a^{-1} > 0$.

(e) If $a, b \in \mathbb{F}$ and $a \ge 0$ and $b \ge 0$ then $a + b \ge 0$.

(f) If $a, b \in \mathbb{F}$ and $0 < a < b$ then $b^{-1} < a^{-1}$.

Proposition 5.8. *Let $\mathbb{F}$ be an ordered field. Let $x, y \in \mathbb{F}$ with $x \ge 0$ and $y \ge 0$. Then*

$$x \le y \qquad \textit{if and only if} \qquad x^2 \le y^2.$$

5.11 Limits, continuity, sequences and series

5.11.1 Limits and Continuity

Let f be a function, say $f\colon \mathbb{R} \to \mathbb{R}$ or $f\colon \mathbb{C} \to \mathbb{C}$. Write

$$\lim_{x\to a} f(x) = \ell \qquad \text{if } f(x) \text{ gets closer and closer to } \ell \quad \text{as} \quad x \text{ gets closer and closer to } a.$$

Write

$$\lim_{x\to \infty} f(x) = \ell \qquad \text{if } f(x) \text{ gets closer and closer to } \ell \quad \text{as} \quad x \text{ gets larger and larger.}$$

Let $s, t \in \mathbb{Z}_{>0}$ and let $p \in \mathbb{R}$. A function $f\colon \mathbb{R} \to \mathbb{R}$ is *continuous at p* if

$$\lim_{x\to p} f(x) = f(p).$$

A function $f\colon \mathbb{R} \to \mathbb{R}$ is *continuous* if f satisfies: if $p \in \mathbb{R}$ then f is continuous at p.

Theorem 5.9. *Let $c \in \mathbb{R}$. The functions*

$$\begin{matrix} \mathbb{R}\times\mathbb{R} & \to & \mathbb{R} \\ (a_1,a_2) & \mapsto & a_1+a_2 \end{matrix} \quad \textit{and} \quad \begin{matrix} \mathbb{R}\times\mathbb{R} & \to & \mathbb{R} \\ (a_1,a_2) & \mapsto & a_1a_2 \end{matrix} \quad \textit{and} \quad \begin{matrix} \mathbb{R} & \to & \mathbb{R} \\ a & \mapsto & ca \end{matrix}$$

are continuous.

Theorem 5.10. *Let $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$.*

$$\textit{If } f \textit{ and } g \textit{ are continuous then} \qquad g\circ f\colon \begin{matrix} \mathbb{R} & \to & \mathbb{R} \\ a & \mapsto & g(f(a)) \end{matrix} \quad \textit{is continuous.}$$

5.11.2 x^n and e^x are continuous

Proposition 5.11.

(a) Let $n \in \mathbb{Z}_{>0}$. The function $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(a) = a^n$ is continuous.

(b) The function $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(a) = e^a$ is continuous.

Since addition, multiplication and scalar multiplication in $\mathbb{R}$ are continuous (Theorem 5.9) then all polynomial functions are continuous, the trigonometric functions sin and cos are continuous, and the hyperbolic functions sinh and cosh are continuous.

5.11.3 Sequences

A *sequence* $(a_1, a_2, \ldots)$ in $\mathbb{C}$ is a function $\begin{matrix} \mathbb{Z}_{>0} & \to & \mathbb{C} \\ n & \mapsto & a_n. \end{matrix}$

Write

$$\lim_{x\to \infty} a_n = \ell \qquad \text{if } a_n \text{ gets closer and closer to } \ell \quad \text{as} \quad n \text{ gets larger and larger.}$$

The fact that $\mathbb{R}$ is ordered and consists of all infinite decimal expansions gives two very useful statements:

1. (Cauchy criterion) If $(a_1, a_2, a_3, \ldots)$ is a sequence in $\mathbb{R}$ which satisfies

   if $k \in \mathbb{Z}_{>0}$ then there exists $N_k \in \mathbb{Z}_{>0}$ such that if $m \in \mathbb{Z}_{>N_k}$ then $|a_m - a_{N_k}| < 10^{-k}$

   then
$$\ell = \lim_{n\to\infty} a_n \quad \text{exists in } \mathbb{R}$$
   (the first k decimal places of ℓ are the same as the first k decimal places of a_{N_k}).

2. (bounded increasing sequences have limits) If $(a_1, a_2, a_3, \ldots)$ is a sequence in $\mathbb{R}$ such that there exists $b \in \mathbb{R}$ such that
$$\text{if } n \in \mathbb{Z}_{>9} \text{ then } a_n \le a_{n+1} \text{ and } a_n \le b$$
   then
$$\ell = \lim_{n\to\infty} a_n \quad \text{exists in } \mathbb{R},$$
   (the limit ℓ is the smallest real number such that if $n \in \mathbb{Z}_{>0}$ then $a_n \le \ell$).

5.11.4 Series

Let $(a_1, a_2, a_3, \ldots)$ be a sequence in $\mathbb{R}$ or $\mathbb{C}$. Write
$$\sum_{n=1}^{\infty} a_n = \ell \qquad \text{if} \qquad \lim_{r\to\infty}\Big(\sum_{n=1}^{r} a_n\Big) = \ell,$$
where $\sum_{n=1}^{r} a_n = a_1 + a_2 + \cdots + a_r$. In other words,
$$\text{the } \textit{series } \sum_{n=1}^{\infty} a_n \quad \text{is the sequence } (s_1, s_2, s_3, \ldots), \qquad \text{where} \quad \begin{array}{l} s_1 = a_1, \\ s_2 = a_1 + a_2, \\ s_3 = a_1 + a_2 + a_3, \\ \vdots \end{array}$$

Let $s \in \mathbb{C}$.

The *constant series at* s is $\quad c(s) = \sum_{n=1}^{\infty} s,$

the *geometric series at* s is $\quad g(s) = \sum_{n=0}^{\infty} s^n,$

the *exponential function at* s is $\quad e^s = \sum_{n=0}^{\infty} \frac{1}{n!} s^n,$

the *Riemann zeta function at* s is $\quad \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$

HW:. Show that $c(0) = 0$ and if $s \ne 0$ then $c(s)$ does not exist in $\mathbb{C}$.

HW:. Show that if $|s| < 1$ then $g(s) = \frac{1}{1-s}$ and if $|s| > 1$ then $g(s)$ does not exist in $\mathbb{C}$.

HW:. Show that if $s \in \mathbb{C}$ then e^s exists in $\mathbb{C}$.

HW:. Assume that $p \in \mathbb{R}_{>0}$. Show that if $p < 1$ tnen $\zeta(p)$ exists in $\mathbb{R}$ and if $p \ge 1$ then $\zeta(p)$ does not exist in $\mathbb{C}$.

6 Proof machine

This section is the salvation for a student of mathematics.

6.1 Andante cantabile

6.1.1 Memories

It was in the second semester of my undergraduate education at MIT that I first met pure mathematics, open and closed sets, the book "Baby Rudin", and Warren Ambrose. The course was '18.100 Mathematical Analysis'. Warren Ambrose had a great effect on me. Somehow we had a one-to-one conversation where we both confessed that our true love was music and that we were doing math only as a backup. At the time, I was still far from being a professional mathematician and he was a famous geometer nearing the end of his career and his life (it was 1984 and he died in 1995 at the age of 81). He told me that he had been a jazz trumpet player but an accident had made him unable to play properly and so he had pursued mathematics for a profession. His exams (two midterm exams and a final) were all 24 hour open-book closed-friend take-home tests: 10 questions, true or false, graded 1 if correct, -1 if incorrect, and 0 if not answered. The average score across the class (about 20 students) was often around 0. But this mechanism taught you better than any other what proof meant – if you were unable to provide a proof you believed in then you risked getting -1 for that question. The questions were always very interesting. I carried those questions around for years until sometime in 2012 when I accidentally left them in a classroom and, when I came back to find them an hour later, they were gone.

6.1.2 Assume the Ifs and To show the Thens: "Proof machine"

The first courses I had that required me to start constructing proofs (Mathematical Analysis, Abstract algebra, Topology) were tough for me. I couldn't figure out the magic trick that made some people able to do this. By the time I started graduate school I still hadn't figured out this magic and I thought it likely that without it it would be impossible for me to succeed in obtaining a PhD in mathematics. On the other hand I began to notice that, in combinatorics particularly, if I knew that I could make some bijection or other then I was *absolutely sure* that I could make it and there was something more than just wishy-washy hand waving that I was doing to have this certainty. I was just starting to get the hang of it.

It was when I was a postdoc that I realized that most of mathematics is just mechanical work, and the bright ideas that are needed are few and far between. This gave me confidence as I was sure that I had the diligence and endurance to do mechanical work, and I was also pretty certain that if any actual "talent" was going to be required then I wasn't going to be a successful mathematician.

Just at that moment I got assigned to teach the undergraduate Abstract Algebra course (at Univ. of Wisconsin–Madison) and so I needed to figure out how to explain to my students how they too could do the necessary proofs. That was the catalyst for me to formulate the mechanism that I now call "proof machine".

As I have progressed in a career as a professional research mathematician I have been amazed to observe how many times "proof machine" has saved me, provided the direction, guided me to where I might have to think, clarified where I didn't need to waste effort thinking, provided the proof and protected me from making mistakes.

"Proof machine" was also the key that unlocked the mysterious world of writing and changed me from a teenager who hated English class, any kind of writing and especially term papers, into a versatile writer (at least in the cases when I do the writing carefully and thoroughly and with the same structural framework that I use when I do a proof in "proof machine" in mathematics). I am always struck by how helpful "proof machine" is for getting out good writing (letters, reports, reviews, papers, memos, emails, etc).

I am continually amazed at how useful "proof machine" is in my daily life and meetings, in helping me be organised and efficient, helping me to get to the core of the issue as necessary, and helping me to optimize impact and productivity for effort expended. "Proof machine" is a skill (not a talent) which is learned by practice (and more practice and more practice) in the same way that one develops skill and facility on a musical instrument by lots of practice.

My hope is that I can teach "proof machine" to as many of my students as I can so that they can also benefit from this wonderful tool in their lives and careers. After all, it is really easy: To prove "If A then B", *Assume* the ifs and *To show* the thens, and that's about all there is to it. The rest is just practice.

6.2 The grammar of mathematics

- **Definitions** are the foundation of mathematics.
- **Theorems** are the landmarks of mathematics.
- **Proofs** are the explanation of mathematics.

Learning to read, write and speak mathematics is a skill that anyone can learn. Like all languages, it requires lots of practice to use it fluently.

Like all languages, the grammar of quality mathematical communication is very rigid.

It is **impossible** to prove a statement without being able to write down the definitions of all the terms in the statement.

The grammar of a definition is:

```
A noun is a _______ such that

(a) If _______ then _______, and
(b) If _______ then _______, and
(c) If _______ then _______, and ...
```

Let $\mathbb{F}$ be field and let V and W be $\mathbb{F}$-vector spaces.
A linear transformation from V to W is a function $f: V \to W$ such that

(a) If $v_1, v_2 \in V$ then $f(v_1 + v_2) = f(v_1) + f(v_2)$,
(b) If $c \in \mathbb{F}$ and $v \in V$ then $f(cv) = cf(v)$.

An adjective is most conveniently defined by putting it in the form of a noun:

```
A adjective noun is a noun such that

(a) If _______ then _______, and
(b) If _______ then _______, and
(c) If _______ then _______, and ...
```

An injective function is a function $f: S \to T$ such that

(a) If $s_1, s_2 \in S$ and $s_1 \neq s_2$ then $f(s_1) \neq f(s_2)$.

Sometimes definitions of adjectives take the form:

```
Let S be a noun.
A noun S is adjective if S satisfies

(a) If _______ then _______, and
(b) If _______ then _______, and
(c) If _______ then _______, and ...
```

Let $f: S \to T$ be a function.
A function $f: S \to T$ is injective if f satisfies

(a) If $s_1, s_2 \in S$ and $s_1 \neq s_2$ then $f(s_1) \neq f(s_2)$.

The words "let" and "assume" are synonyms for "if". The grammar of a lemma, proposition or theorem (or any other statement) is:

```
If _______ then _______.
```

Two special constructions in mathematical language are:

```
There exists _______ such that _______.
```

and

```
There exists a unique _______ such that _______.
```

6.3 How to do Proofs: "Proof Machine"

There *is* a certain "formula" or method to doing proofs. Some of the guidelines are given below. The most important factor in learning to do proofs is practice, just as when one is learning a new language.

1. There are very few words needed in the structure of a proof. Organized in rows by synonyms they are:

   ```
   To show
   Assume, Let, Suppose, Define, If
   Since, Because, By
   Then, Thus, So
   There exists, There is
   Recall, We know, But
   ```

   Do not use '`for all`' or '`for each`'. These can always be replaced by '`if`' to achieve greater clarity, accuracy and efficiency.

   Do not use the phrase '`for some`'. It can always be replaced by '`There exists`' to achieve greater clarity, accuracy and efficiency.

2. The overall structure of a proof is a block structure like an outline. For example:

   ```
   To show:  If __A__ then __B__ and __C__.
       Assume:  __A__.
   [itemsep=-0.2em]
   To show:  (a) __B__.
             (b) __C__.
   (a) To show:  __B__.
           _______.
           _______.
           _______.
       Thus __B__.
   (b) To show:  __C__.
           _______.
           _______.
           _______.
       Thus __C__.
   So __B__ and __C__.
   So, if __A__ then __B__ and __C__.
   ```

3. Any proof or section of proof begins with one of the following:

   (a) `To show:  If __A__ then __B__.`
   (b) `To show:  There exists __C__ such that __D__.`
   (c) `To show:  __E__.`

4. Immediately following this, the next step is

Case (a) Assume the ifs and 'To show' the thens. The next lines are

- `Assume __A__.`
- `To show: __B__.`

Case (b) To show an object exists you must find it. The next lines are

- `Define __xxx = ________.`
- `To show: __xxx__ satisfies __D__.`

Case (c) Rewrite the statement in __E__ by using a definition. The next line is

- `To show: __E′__.`

There are some kinds of proofs which have a special structure.

(E) Proofs of equality: LHS=RHS.

```
To show: __A=B__.
    Left Hand Side: __A= ···______
                    __ = ···______
                    __ = ···______
                    __ = ···______
                    __ = expression____
    Right Hand Side: __B= ···______
                     __ = ···______
                     __ = ···______
                     __ = ···______
                     __ = THE SAME expression____
```

(F) Counterexamples: Proofs of falseness

To show that a statement, "If ____ then ____", is false you *must* give an example.

`To show: There exists a __xxx__ such that`
(a) `xxx` satisfies the `if`s of the statement that you are showing is false,
(a) `xxx` satisfies the opposite of some assertion in the `then`s of the statement that you are showing is false.

(U) Proofs of uniqueness.

To show that an object is unique you must show that if there are two of them then they are really the same.

`To show:` A THING `is unique.`
`Assume` X_1 `and` X_2 `are both` THINGs.
`To show:` $X_1 = X_2$.

(I) Proofs by induction.

A statement to be proved by induction *must* have the form

`If` n `is a positive integer then` __A__.

The proof by induction should have the form

`Proof by induction.`
`Base case:`
`To show: If` $n = 1$ `then` __A__.
______.
______.
______.
`Thus, if` $n = 1$ `then` __A__.
`Induction step:`
`Let` ℓ `be a positive integer and assume that if` n `is a positive integer and` $n < \ell$ `then` __A__.
`To show:` __A__.

The mechanics of proof by induction is an unwinding of the *definition* of $\mathbb{Z}_{>0}$.

(CP) Proofs by contrapositive.

`To show: If` __A__ `then` __B__.
`To show: If` __not B__ `then` __not A__.

(BAD) Proofs by contradiction.

(*) `Assume` the opposite of what you want to show.
________.
________.
________.
End up showing the opposite of some assumption (not necessarily the (*) assumption).
`Contradiction to` <u>specify exactly what assumption is being contradicted</u>.
`Thus` assumption (*) is wrong and what you want to show is true.

PROOFS BY CONTRADICTION ARE STRONGLY DISCOURAGED. In all known cases they can be replaced by a proof by contrapositive for greater clarity, direction and efficiency.

6.4 Example proofs

The following example proofs have been chosen because they are results that are often assumed, are needed for many topics in calculus, algebra and analysis and topology and are rarely proved carefully in an undergraduate curriculum; facts like, if $a \neq 0$ then $a^2 > 0$. These often seem “obvious”, until you meet that first example, like a field witih 5 elements, where $2 \neq 0$ and $2^2 = -1$. After getting over the initial shock, then one begins to wonder why such a fact might ever be true, and how it might be proved when it is. It is proved in Proposition 6.5(b), below.

6.4.1 An inverse function to f exists if and only if f is bijective.

Functions are for comparing sets.

Let S and T be sets. A *function from S to T* is a subset $\Gamma_f \subseteq S \times T$ such that

$$\text{if } s \in S \quad \text{then} \quad \text{there exists a unique } t \in T \text{ such that } (s,t) \in \Gamma_f.$$

Write

$$\Gamma_f = \{(s, f(s)) \mid s \in S\}$$

so that the function Γ_f can be expressed as

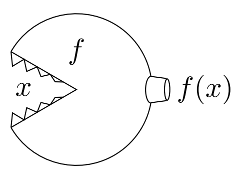
$$\text{an “assignment”} \qquad \begin{array}{rccc} f\colon & S & \to & T \\ & s & \mapsto & f(s) \end{array}$$

which must satisfy

(a) If $s \in S$ then $f(s) \in T$, and

(b) If $s_1, s_2 \in S$ and $s_1 = s_2$ then $f(s_1) = f(s_2)$.

In other words, a **function** is a creature that eats an element of S, chews on it, and spits out an element of T.

elements of S  elements of T

What a function spits out depends only on what goes in.
Let S and T be sets.

- Two functions $f\colon S \to T$ and $g\colon S \to T$ are *equal* if they satisfy

$$\text{if } s \in S \quad \text{then} \quad f(s) = g(s).$$

- A function $f\colon S \to T$ is *injective* if f satisfies the condition

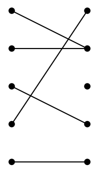
$$\text{if } s_1, s_2 \in S \text{ and } f(s_1) = f(s_2) \quad \text{then} \quad s_1 = s_2.$$

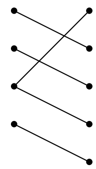
- A function $f\colon S \to T$ is *surjective* if f satisfies the condition

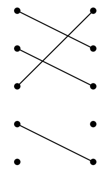
$$\text{if } t \in T \quad \text{then} \quad \text{there exists } s \in S \text{ such that } f(s) = t.$$

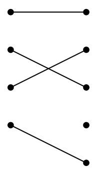
- A function $f: S \to T$ is *bijective* if f is both injective and surjective.

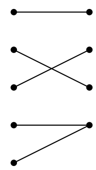
Examples. It is useful to visualize a function $f: S \to T$ as a graph with edges $(s, f(s))$ connecting elements $s \in S$ and $f(s) \in T$. With this in mind the following are examples:

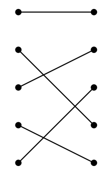

(a) a function


(b) not a function


(c) not a function


(d) an injective function


(e) a surjective function


(f) a bijective function

In these pictures the elements of the left column are the elements of the set S and the elements of the right column are the elements of the set T. In order to be a function the graph must have exactly one edge adjacent to each point in S. The function is injective if there is at most one edge adjacent to each point in T. The function is surjective if there is at least one edge adjacent to each point in T.

Composition of functions

Let $f: S \to T$ and $g: T \to U$ be functions. The *composition* of f and g is the function

$$g \circ f \quad \text{given by} \quad \begin{array}{rccc} g \circ f: & S & \to & U \\ & s & \mapsto & g(f(s)) \end{array}$$

Let S be a set. The *identity map on* S is the function given by

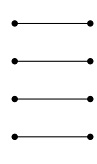
$$\begin{array}{rccc} \mathrm{id}_S: & S & \to & S \\ & s & \mapsto & s \end{array}$$

Let $f: S \to T$ be a function. The *inverse function to* f is a function

$$f^{-1}: T \to S \qquad \text{such that} \qquad f \circ f^{-1} = \mathrm{id}_T \quad \text{and} \quad f^{-1} \circ f = \mathrm{id}_S.$$

Theorem 6.1. *Let $f: S \to T$ be a function. An inverse function to f exists if and only if f is bijective.*

Representing functions as graphs, the identity function id_S looks like


(a) the identity function id_S

In the pictures below, if the left graph is a pictorial representation of a function $f\colon S \to T$ then the inverse function to f, $f^{-1}\colon T \to S$, is represented by the graph on the right; the graph for f^{-1} is the mirror-image of the graph for f.

(b) the function f (c) the function f^{-1}

Graph (d) below, represents a function $g\colon S \to T$ which is not bijective. The inverse function to g does not exist in this case: the graph (e) of a possible candidate, is not the graph of a function.

(d) the function g (e) not a function

Theorem 6.2. *Let $f\colon S \to T$ be a function. The inverse function to f exists if and only if f is bijective.*

Proof.

$\Rightarrow$: Assume $f\colon S \to T$ has an inverse function $f^{-1}\colon T \to S$.

To show: (a) f is injective.

(b) f is surjective.

(a) Assume $s_1, s_2 \in S$ and $f(s_1) = f(s_2)$.

To show: $s_1 = s_2$.

$$s_1 = f^{-1}f(s_1)) = f^{-1}f(s_2)) = s_2.$$

So f is injective.

(b) Let $t \in T$.

To show: There exists $s \in S$ such that $f(s) = t$.

Let $s = f^{-1}(t)$.

Then

$$f(s) = f(f^{-1}(t)) = t.$$

So f is surjective.

So f is bijective.

$\Leftarrow$: Assume $f\colon S \to T$ is bijective.

To show: f has an inverse function.

We need to define a function $\varphi\colon T \to S$.

Let $t \in T$.

Since f is surjective there eists $s \in S$ such that $f(s) = t$.

Define $\varphi(t) = s$.

To show: (a) φ is well defined.
(b) φ is an inverse function to f.

(a) To show: (aa) If $t \in T$ then $\varphi(t) \in S$.
(ab) If $t_1, t_2 \in T$ and $t_1 = t_2$ then $\varphi(t_1) = \varphi(t_2)$.

(aa) This follows from the definition of φ.
(ab) Assume $t_1, t_2 \in T$ and $t_1 = t_2$.
Let $s_1, s_2 \in S$ such that $f(s_1) = t_1$ and $f(s_2) = t_2$.
Since $t_1 = t_2$ then $f(s_1) = f(s_2)$.
Since f is injective this implies that $s_1 = s_2$.
So $\varphi(t_1) = s_1 = s_2 = \varphi(t_2)$.

So φ is well defined.

(b) To show: (ba) If $s \in S$ then $\varphi(f(s)) = s$.
(bb) If $t \in T$ then $f(\varphi(t)) = t$.

(ba) This follows from the definition of φ.
(bb) Assume $t \in T$.
Let $s \in S$ be such that $f(s) = t$.
Then

$$f(\varphi(t)) = f(s) = t.$$

So $\varphi \circ f$ and $f \circ \varphi$ are the identity functions on S and T, respectively.

So φ is an inverse function to f.

□

6.4.2 An equivalence relation on S and a partition of S are the same data.

Let S be a set.

- A *relation* $\sim$ *on* S is a subset $R_\sim$ of $S \times S$. Write $s_1 \sim s_2$ if the pair (s_1, s_2) is in the subset $R_\sim$ so that

$$R_\sim = \{(s_1, s_2) \in S \times S \mid s_1 \sim s_2\}.$$

- An *equivalence relation* on S is a relation $\sim$ on S such that

  (a) if $s \in S$ then $s \sim s$,
  (b) if $s_1, s_2 \in S$ and $s_1 \sim s_2$ then $s_2 \sim s_1$,
  (c) if $s_1, s_2, s_3 \in S$ and $s_1 \sim s_2$ and $s_2 \sim s_3$ then $s_1 \sim s_3$.

Let $\sim$ be an equivalence relation on a set S and let $s \in S$. The *equivalence class of* s is the set

$$[s] = \{t \in S \mid t \sim s\}.$$

A *partition of a set* S is a collection $\mathcal{P}$ of subsets of S such that

(a) If $s \in S$ then there exists $P \in \mathcal{P}$ such that $s \in P$, and
(b) If $P_1, P_2 \in \mathcal{P}$ and $P_1 \cap P_2 \neq \emptyset$ then $P_1 = P_2$.

Theorem 6.3.

(a) If S is a set and let $\sim$ be an equivalence relation on S then

$$\text{the set of equivalence classes of } \sim \qquad \text{is a partition of } S.$$

(b) If S is a set and $\mathcal{P}$ is a partition of S then

$$\text{the relation defined by} \qquad s \sim t \quad \text{if } s \text{ and } t \text{ are in the same } P \in \mathcal{P}$$

is an equivalence relation on S.

Proof.

(a) To show: (aa) If $s \in S$ then s is in some equivalence class.
(ab) If $[s] \cap [t] \neq \emptyset$ then $[s] = [t]$.

(aa) Let $s \in S$.
Since $s \sim s$ then $s \in [s]$.

(ab) Assume $[s] \cap [t] \neq \emptyset$.
To show: $[s] = [t]$.
Since $[s] \cap [t] \neq \emptyset$ then there is an $r \in [s] \cap [t]$.
So $s \sim r$ and $r \sim t$.
By transitivity, $s \sim t$.
To show: (aba) $[s] \subseteq [t]$.
(abb) $[t] \subseteq [s]$.

(aba) Assume $u \in [s]$.
Then $u \sim s$.
We know $s \sim t$.
So, by transitivity, $u \sim t$.
Therefore $u \in [t]$.

So $[s] \subseteq [t]$.

(aba) Assume $v \in [t]$.
Then $v \sim t$.
We know $t \sim s$.
So, by transitivity, $v \sim s$.
Therefore $v \in [s]$.

So $[t] \subseteq [s]$.
So $[s] = [t]$.

So the equivalence classes partition S.

(b) To show: $\sim$ is an equivalence relation, i.e. that $\sim$ is reflexive, symmetric and transitive.
To show: (ba) If $s \in S$ then $s \sim s$.
(bb) If $s \sim t$ then $t \sim s$.
(bc) If $s \sim t$ and $t \sim u$ then $s \sim u$.

(ba) Since s and s are in the same S_α then $s \sim s$.

(bb) Assume $s \sim t$.
Then s and t are in the same S_α.
So $t \sim s$.

(bb) Assume $s \sim t$ and $t \sim u$.
Then s and t are in the same S_α and t and u are in the same S_α.
So $s \sim u$.

So $\sim$ is an equivalence relation.

□

6.4.3 Identities in a field

A *field* is a set $\mathbb{F}$ with functions

$$\begin{array}{ccc} \mathbb{F}\times\mathbb{F} & \longrightarrow & \mathbb{F} \\ (a,b) & \longmapsto & a+b \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \mathbb{F}\times\mathbb{F} & \longrightarrow & \mathbb{F} \\ (a,b) & \longmapsto & ab \end{array}$$

such that

(Fa) If $a,b,c\in\mathbb{F}$ then $(a+b)+c=a+(b+c)$,

(Fb) If $a,b\in\mathbb{F}$ then $a+b=b+a$,

(Fc) There exists $0\in\mathbb{F}$ such that

$$\text{if } a\in\mathbb{F} \quad \text{then} \quad 0+a=a \text{ and } a+0=a,$$

(Fd) If $a\in\mathbb{F}$ then there exists $-a\in\mathbb{F}$ such that $a+(-a)=0$ and $(-a)+a=0$,

(Fe) If $a,b,c\in\mathbb{F}$ then $(ab)c=a(bc)$,

(Ff) If $a,b,c\in\mathbb{F}$ then

$$(a+b)c=ac+bc \qquad \text{and} \qquad c(a+b)=ca+cb,$$

(Fg) There exists $1\in\mathbb{F}$ such that

$$\text{if } a\in\mathbb{F} \quad \text{then} \quad 1\cdot a=a \text{ and } a\cdot 1=a,$$

(Fh) If $a\in\mathbb{F}$ and $a\neq 0$ then there exists $a^{-1}\in\mathbb{F}$ such that $aa^{-1}=1$ and $a^{-1}a=1$,

(Fi) If $a,b\in\mathbb{F}$ then $ab=ba$.

Proposition 6.4. *Let $\mathbb{F}$ be a field.*

(a) If $a\in\mathbb{F}$ then $a\cdot 0=0$.

(b) If $a\in\mathbb{F}$ then $-(-a)=a$.

(c) If $a\in\mathbb{F}$ and $a\neq 0$ then $(a^{-1})^{-1}=a$.

(d) If $a\in\mathbb{F}$ then $a(-1)=-a$.

(e) If $a,b\in\mathbb{F}$ then $(-a)b=-ab$.

(f) If $a,b\in\mathbb{F}$ then $(-a)(-b)=ab$.

Proof.

(a) Assume $a\in\mathbb{F}$.

$$\begin{aligned} a\cdot 0 &= a\cdot(0+0), \quad \text{by (Fc)},\\ &= a\cdot 0+a\cdot 0, \quad \text{by (Ff)}. \end{aligned}$$

Add $-a\cdot 0$ to each side and use (Fd) to get $0=a\cdot 0$.

(b) Assume $a\in\mathbb{F}$.
By (Fd),

$$-(-a)+(-a)=0=a+(-a).$$

Add $-a$ to each side and use (Fd) to get $-(-a)=a$.

(c) Assume $a\in\mathbb{F}$ and $a\neq 0$.

By (Fh),
$$(a^{-1})^{-1}\cdot a^{-1} = 1 = a\cdot a^{-1}.$$
Multiply each side by a and use (Fh) and (Fg) to get $(a^{-1})^{-1} = a$.

(d) Assume $a \in \mathbb{F}$.
By (Ff),
$$a(-1) + a\cdot 1 = a(-1+1) = a\cdot 0 = 0,$$
where the last equality follows from part (a).
So, by (Fg), $a(-1) + a = 0$.
Add $-a$ to each side and use (Fd) and (Fc) to get $a(-1) = -a$.

(e) Assume $a, b \in \mathbb{F}$.
$$\begin{aligned}(-a)b + ab &= (-a+a)b, \quad \text{by (Ff)},\\ &= 0\cdot b, \quad \text{by (Fd)},\\ &= 0, \quad \text{by part (a)}.\end{aligned}$$
Add $-ab$ to each side and use (Fd) and (Fc) to get $(-a)b = -ab$.

(f) Assume $a, b \in \mathbb{F}$.
$$\begin{aligned}(-a)(-b) &= -(a(-b)), \quad \text{by (e)},\\ &= -(-ab), \quad \text{by (e)},\\ &= ab, \quad \text{by part (b)}.\end{aligned}$$

□

6.4.4 Identities in an ordered field

An *ordered field* is a field $\mathbb{F}$ with a total order $\le$ such that

(OFa) If $a, b, c \in \mathbb{F}$ and $a \le b$ then $a + c \le b + c$,

(OFb) If $a, b \in \mathbb{F}$ and $a \ge 0$ and $b \ge 0$ then $ab \ge 0$.

Proposition 6.5. *Let $\mathbb{F}$ be an ordered field.*

(a) If $a \in \mathbb{F}$ and $a > 0$ then $-a < 0$.

(b) If $a \in \mathbb{F}$ and $a \ne 0$ then $a^2 > 0$.

(c) $1 \ge 0$.

(d) If $a \in \mathbb{F}$ and $a > 0$ then $a^{-1} > 0$.

(e) If $a, b \in \mathbb{F}$ and $a \ge 0$ and $b \ge 0$ then $a + b \ge 0$.

(f) If $a, b \in \mathbb{F}$ and $0 < a < b$ then $b^{-1} < a^{-1}$.

Proof.

(a) Assume $a \in \mathbb{F}$ and $a > 0$.
Then $a + (-a) > 0 + (-a)$, by (OFb).
So $0 > -a$, by (Fd) and (Fc).

(b) Assume $a \in \mathbb{F}$ and $a \ne 0$.

Case 1: $a > 0$.
Then $a \cdot a > a \cdot 0$, by (OFb).
So $a^2 > 0$, by part (a).
Case 2: $a < 0$.
Then $-a > 0$, by part (a).
Then $(-a)^2 > 0$, by Case 1.
So $a^2 > 0$, by Proposition 6.4 (f).

(c) To show: $1 \geq 0$.
$1 = 1^2 \geq 0$, by part (b).

(d) Assume $a \in \mathbb{F}$ and $a > 0$.
By part (b), $a^{-2} = (a^{-1})^2 > 0$.
So $a(a^{-1})^2 > a \cdot 0$, by (OFb).
So $a^{-1} > 0$, by (Fh) and Proposition 6.4 (a).

(e) Assume $a, b \in \mathbb{F}$ and $a \geq 0$ and $b \geq 0$.

$$\begin{aligned} a + b &\geq 0 + b, \quad \text{by (OFa)}, \\ &\geq 0 + 0, \quad \text{by (OFa)}, \\ &= 0, \quad \text{by (Fc)}. \end{aligned}$$

(f) Assume $a, b \in \mathbb{F}$ and $0 < a < b$.
So $a > 0$ and $b > 0$.
Then, by part (d), $a^{-1} > 0$ and $b^{-1} > 0$.
Thus, by (OFb), $a^{-1}b^{-1} > 0$.
Since $a < b$, then $b - a > 0$, by (OFa).
So, by (OFb), $a^{-1}b^{-1}(b - a) > 0$.
So, by (Fh), $a^{-1} - b^{-1} > 0$.
So, by (OFa), $a^{-1} > y^{-1}$.

□

Proposition 6.6. *Let $\mathbb{F}$ be an ordered field and let $x, y \in \mathbb{F}$ with $x \geq 0$ and $y \geq 0$. Then*

$$x \leq y \qquad \text{if and only if} \qquad x^2 \leq y^2.$$

Proof. Assume $x, y \in S$ and $x \geq 0$ and $y \geq 0$.
To show: (a) If $x \leq y$ then $x^2 \leq y^2$.
(b) If $x^2 \leq y^2$ then $x \leq y$.

(b) Assume $x^2 \leq y^2$.
Adding $(-x^2)$ to each side and using (OFa) gives $y^2 + (-x^2) \geq x^2 + (-x^2) = 0$.
So $y^2 - x^2 \geq 0$.
Using Proposition 6.4(e) and axioms (Ff) and (Fi),

$$\begin{aligned} (y - x)(y + x) &= yy + (-x)y + yx + (-x)x = y^2 + (-xy) + xy + (-xx) \\ &= y^2 + 0 - x^2 = y^2 - x^2. \end{aligned}$$

So $(y - x)(y + x) \geq 0$.

By Proposition 6.5(e) and Proposition 6.5 (d),
since $x \geq 0$ and $y \geq 0$ then $x + y \geq 0$ and $(x+y)^{-1} > 0$ (or $x = 0$ and $y = 0$).
So, by (OFb), $(y-x)(y+x)(x+y)^{-1} \geq 0$.
Using (Fg), then $y - x \geq 0$.
Adding x to both sides and using (OFa) gives $y \geq x$.

(a) Assume $y \geq x$.
Then $y - x \geq 0$.
Since $y \geq 0$ and $x \geq 0$ then, by (OFa), $(y+x) \geq y + 0 = y \geq 0$.
So, by (OFb), $(y-x)(y+x) \geq 0$.
So $y^2 - x^2 \geq 0$.
So $y^2 \geq x^2$.

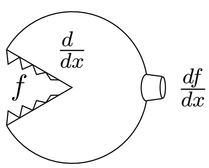
□

6.4.5 The power rule and the chain rule for the derivative

The first two of these proofs provide examples of proofs by induction.

There are **different kinds of derivatives**:

Derivative with respect to x



The derivative $\frac{d}{dx}$ satisfies

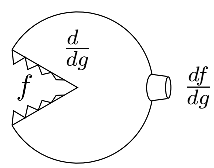
$$\frac{dx}{dx} = 1,$$

$$\frac{d(cf)}{dx} = c\frac{df}{dx}, \quad \text{if } c \text{ is a constant},$$

$$\frac{d(y+z)}{dx} = \frac{dy}{dx} + \frac{dz}{dx},$$

$$\frac{d(yz)}{dx} = y\,\frac{dz}{dx} + \frac{dy}{dx}\,z.$$

Derivative with respect to g



The derivative $\frac{d}{dg}$ satisfies

$$\frac{dg}{dg} = 1,$$

$$\frac{d(cf)}{dg} = c\frac{df}{dg}, \quad \text{if } c \text{ is a constant},$$

$$\frac{d(y+z)}{dg} = \frac{dy}{dg} + \frac{dz}{dg},$$

$$\frac{d(yz)}{dg} = y\,\frac{dz}{dg} + \frac{dy}{dg}\,z.$$

Proposition 6.7. *If* $n \in \mathbb{Z}_{\geq 0}$ *then* $\dfrac{dx^n}{dx} = nx^{n-1}$.

Proof. The base case $n = 0$*:*

$$\frac{d1}{dx} = \frac{d1 \cdot 1}{dx} = 1 \cdot \frac{d1}{dx} + \frac{d1}{dx} \cdot 1 = \frac{d1}{dx} + \frac{d1}{dx}.$$

Subtract $\frac{d1}{dx}$ from each side to get $0 = \frac{d1}{dx}$. So

$$\frac{dx^0}{dx} = \frac{d1}{dx} = 0 = 0x^{0-1}.$$

The base case $n = 1$:. By definition of $\frac{d}{dx}$ then $\frac{dx}{dx} = 1$. So

$$\frac{dx^1}{dx} = \frac{dx}{dx} = 1 = 1 \cdot 1 = 1x^0 = 1x^{1-1}.$$

Induction step. Assume that if $m, n \in \mathbb{Z}_{\geq 0}$ and $m < n$ then $\frac{dx^m}{dx} = mx^{m-1}$. Then

$$\frac{dx^n}{dx} = \frac{x \cdot x^{n-1}}{dx} = x\frac{dx^{n-1}}{dx} + \frac{dx}{dx} \cdot x^{n-1} = x(n-1)x^{n-2} + 1 \cdot x^{n-1} = (n-1)x^{n-1} + x^{n-1} = nx^{n-1}.$$

Hence, if $n \in \mathbb{Z}_{\geq 0}$ then $\frac{dx^n}{dx} = nx^{n-1}$. □

Proposition 6.8. *If $n \in \mathbb{Z}_{\geq 0}$ then $\frac{dg^n}{dx} = ng^{n-1}\frac{dg}{dx}$.*

Proof. The base case $n = 0$:

$$\frac{dg^0}{dx} = \frac{d1}{dx} = 0 = 0g^{0-1}\frac{dg}{dx}.$$

The base case $n = 1$: Since $g^0 = 1$ then

$$\frac{dg}{dx} = 1 \cdot g^0 \frac{dg}{dx}.$$

Induction step. Assume that if $m, n \in \mathbb{Z}_{\geq 0}$ and $m < n$ then $\frac{dg^m}{dx} = mg^{m-1}\frac{dg}{dx}$. Then

$$\begin{aligned}\frac{dg^n}{dx} &= \frac{g \cdot g^{n-1}}{dx} = g\frac{dg^{n-1}}{dx} + \frac{dg}{dx} \cdot g^{n-1} \\ &= g(n-1)g^{n-2}\frac{dg}{dx} + g^{n-1}\frac{dg}{dx} = (n-1)g^{n-1}\frac{dg}{dx} + g^{n-1}\frac{dg}{dx} = ng^{n-1}\frac{dg}{dx}.\end{aligned}$$

Hence, if $n \in \mathbb{Z}_{\geq 0}$ then $\frac{dg^n}{dx} = ng^{n-1}\frac{dg}{dx}$.

□

Theorem 6.9. *If $p = c_0 + c_1 g + c_2 g^2 + \cdots$ with $c_0, c_1, c_2 \ldots \in \mathbb{C}$ then*

$$\frac{dp}{dx} = \frac{dp}{dg} \cdot \frac{dg}{dx}.$$

Proof. Assume $p = c_0 + c_1 g + c_2 g^2 + \cdots$ with $c_0, c_1, c_2 \ldots \in \mathbb{C}$.
To show: $\frac{dp}{dx} = \frac{dp}{dg} \cdot \frac{dg}{dx}$.

Using Proposition 6.8,

$$\begin{aligned}\frac{dp}{dx} &= \frac{d}{dx}(c_0 + c_1 g + c_2 g^2 + \cdots) \\ &= \frac{dc_0}{dx} + \frac{dc_1 g}{dx} + \frac{dc_2 g^2}{dx} + \frac{dc_3 g^3}{dx} + \cdots \\ &= c_0 \frac{d1}{dx} + c_1 \frac{dg}{dx} + c_2 \frac{dg^2}{dx} + c_3 \frac{dg^3}{dx} + \cdots \\ &= c_0 \cdot 0 + c_1 \frac{dg}{dx} + c_2 \cdot 2g \frac{dg}{dx} + c_3 \cdot 3g_2 \frac{dg}{dx} + \cdots \\ &= \big(c_0 \cdot 0 + c_1 + c_2 \cdot 2g + c_3 \cdot 3g_2 + \cdots\big) \frac{dg}{dx} \\ &= \frac{dp}{dg} \cdot \frac{dg}{dx},\end{aligned}$$

where the last line follows from Proposition 6.7. □

6.4.6 Proof that $\sup(E)$ is unique

The following proof gives an example of a proof of uniqueness. The definitions of supremum and infimum are examples of what are called 'universal objectts'.

Let S be a set. A *relation* on S is a subset $\angle$ of $S \times S$. If $x, y \in \angle$ write $x \angle y$.
A *poset*, or *partially ordered set*, is a set with a relation $\le$ on S such that

(a) If $s, y, z \in S$ and $x \le y$ and $y \le z$ then $x \le z$. and

(b) If $x, y \in S$ and $x \le y$ and $y \le x$ then $x = y$.

Let $(S, \le)$ be a poset. Let E be a subset of S.
A *supremum of* E, or *least upper bound of* E, is $\sup(E)$ such that
(a) $\sup(E) \in S$ and $\sup(E)$ satisfies the condition: if $x \in E$ then $x \le \sup(E)$, and
(b) If $b \in S$ satisfies the condition: if $x \in E$ then $x \le b$, then $\sup(E) \le b$.
A *infimum of* E, or *greatest lower bound of* E, is $\inf(E)$ such that
(a) $\inf(E) \in S$ and $\inf(E)$ satisfies the condition: if $x \in E$ then $\inf(E) \le x$, and
(b) If $b \in S$ satisfies the condition: if $x \in E$ then $x \le b$, then $b \le \inf(E)$.

Proposition 6.10. *Let P be a poset and let $E \subseteq P$. If $\sup(E)$ exists in P then $\sup(E)$ is unique.*

Proof. Assume that E has a supremum in P.
Assume that $\sup_1(E) \in P$ and $\sup_2(E)$ are supremums of E in P.
To show: $\sup_1(E) = \sup_2(E)$.
To show: (a) $\sup_1(E) \le \sup_2(E)$.
(b) $\sup_2(E) \le \sup_1(E)$.

(a) Since $\sup_1(E)$ is a supremum of E the $\sup_1(E)$ satisfies: if $x \in E$ then $x \le \sup_1(E)$.
So $\sup_1(E) \le \sup_2(E)$.
(b) Since $\sup_2(E)$ is a supremum of E the $\sup_2(E)$ satisfies: if $x \in E$ then $x \le \sup_2(E)$.
So $\sup_2(E) \le \sup_1(E)$.
So $\sup_1(E) = \sup_2(E)$. □

6.4.7 Completing the square is the quadratic formula

Example 6.1. Let $a, b, c \in \mathbb{C}$ with $a \neq 0$. Show that if $ax^2 + bx + c = 0$ then

$$x = \frac{-b + (b^2 - 4ac)^{\frac{1}{2}}}{2a},$$

where the right hand side is actually a set of values since

$$(b^2 - 4ac)^{\frac{1}{2}} = \{x \in \mathbb{C} \mid x^2 = b^2 - 4ac\}.$$

Proof. Let $a, b, c \in \mathbb{C}$. Then

$$\begin{aligned} ax^2 + bx + c &= a(x^2 + \frac{b}{a}x + \frac{c}{a}) = a\Big(x^2 + 2\tfrac{1}{2}\frac{b}{a} + (\tfrac{1}{2}\frac{b}{a})^2 + (\frac{c}{a} - (\tfrac{1}{2}\frac{b}{a})^2)\Big) \\ &= a\Big(\big(x + \frac{b}{2a}\big)^2 + (\frac{c}{a} - (\tfrac{1}{2}\frac{b}{a})^2)\Big) = a\Big(\big(x + \frac{b}{2a}\big)^2 - (\frac{b^2 - 4ac}{4a^2})\Big). \end{aligned}$$

Hence, if $ax^2 + bx + c = 0$ and $a \neq 0$ then

$$0 = a\Big(\big(x + \frac{b}{2a}\big)^2 - (\frac{b^2 - 4ac}{4a^2})\Big), \qquad \text{which gives} \qquad 0 = \big(x + \frac{b}{2a}\big)^2 - (\frac{b^2 - 4ac}{4a^2}).$$

$$\text{So} \quad \frac{b^2 - 4ac}{4a^2} = \big(x + \frac{b}{2a}\big)^2, \qquad \text{which gives} \qquad \Big(\frac{b^2 - 4ac}{4a^2}\Big)^{\frac{1}{2}} = x + \frac{b}{2a}.$$

$$\text{So} \quad \frac{(b^2 - 4ac)^{\frac{1}{2}}}{2a} = x + \frac{b}{2a}, \qquad \text{which gives} \qquad x = \frac{-b + (b^2 - 4ac)^{\frac{1}{2}}}{2a}.$$

□

6.4.8 Binomial theorem proof

Theorem 6.11. *Let $n, k \in \mathbb{Z}_{\geq 0}$ with $k \in \{0, 1, \ldots, n\}$. Assume $xy = yx$.*

(a) If $k \in \{1, \ldots, n-1\}$ then

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}, \qquad \text{and} \qquad \binom{n}{0} = 1 \quad \text{and} \quad \binom{n}{n} = 1.$$

(b) $\binom{n}{k}$ is the coefficient of $x^{n-k}y^k$ in $(x+y)^n$.

(c) Let S be a set with cardinality n.
Then $\binom{n}{k}$ is the number of subsets of S with cardinality k.

(d) $e^{(x+y)} = e^x e^y$.

Proof. (a) $\binom{n}{0} = \frac{n!}{0!(n-0)!} = \frac{n!}{1 \cdot n!} = 1$ and $\binom{n}{n} = \frac{n!}{n!(n-n)!} = \frac{n!}{n!0!} = \frac{n!}{n! \cdot 1} = 1$.
If $k \in \{1, \ldots, n-1\}$ then

$$\begin{aligned}
\binom{n-1}{k-1} + \binom{n-1}{k-1} &= \frac{(n-1)!}{(k-1)!(n-1-(k-1))!} + \frac{(n-1)!}{k!(n-1-k)!} \\
&= \frac{(n-1)!}{(k-1)!(n-1-k)!}\Big(\frac{1}{n-k} + \frac{1}{k}\Big) \\
&= \frac{(n-1)!}{(k-1)!(n-1-k)!}\frac{n}{k(n-k)} = \frac{n!}{k!(n-k)!} = \binom{n}{k}.
\end{aligned}$$

(b) The base cases are

$$(x+y)^0 = 1 = \binom{0}{0}x^0y^0 \qquad \text{and} \qquad (x+y)^1 = x + y = \binom{1}{0}x^1y^0 + \binom{1}{1}x^0y^1.$$

Then, by induction,

$$\begin{aligned}
(x+y)^n &= (x+y)^{n-1}(x+y) \\
&= \Big(\binom{n-1}{0}x^{n-1}y^0 + \binom{n-1}{1}x^{n-2}y^1 + \cdots + \binom{n-1}{n-2}x^1y^{n-2} + \binom{n-1}{n-1}x^0y^{n-1}\Big)(x+y) \\
&= \binom{n-1}{0}x^ny^0 + \binom{n-1}{1}x^{n-1}y^1 + \cdots + \binom{n-1}{n-2}x^2y^{n-2} + \binom{n-1}{n-1}x^1y^{n-1} \\
&\qquad + \binom{n-1}{0}x^{n-1}y^1 + \cdots + \binom{n-1}{n-2}x^1y^{n-1} + \binom{n-1}{n-1}x^0y^n \\
&= \binom{n}{0}x^ny^0 + \binom{n}{1}x^{n-1}y^1 + \cdots + \binom{n}{n-1}x^1y^{n-1} + \binom{n}{n}x^0y^n,
\end{aligned}$$

where the last equality follows from part (a).
(c) Since

$$\begin{aligned}
(x+y)^n = \underbrace{(x+y)\cdots(x+y)}_{n \text{ factors}} &= \sum_{k=0}^{n} \sum_{\substack{J \subseteq \{1,\ldots,n\} \\ \mathrm{Card}(J)=k}} \Big(\prod_{\substack{i \in \{1,\ldots,n\} \\ i \notin J}} x\Big)\Big(\prod_{\substack{j \in \{1,\ldots,n\} \\ j \in J}} y\Big) \\
&= \sum_{k=0}^{n} \mathrm{Card}(\{J \subseteq \{1, \ldots, n\} \mid \mathrm{Card}(J) = k\})\, x^{n-k}y^k,
\end{aligned}$$

the coefficient of $x^{n-k}y^k$ is the number of ways of choosing k factors (each of which comtributes a y to $x^{n-k}y^k$) from the n factors in $(x+y)\cdots(x+y)=(x+y)^n$.

(d)

$$\begin{aligned}
e^{(x+y)} &= 1+(x+y)+\frac{1}{2!}(x+y)^2+\frac{1}{3!}(x+y)^3+\cdots \\
&= \begin{matrix} 1 \\ +(x+y) \\ +\frac{1}{2!}(x^2+2xy+y^2) \\ +\frac{1}{3!}(x^3+3x^2y+3xy^2+y^3) \\ +\frac{1}{4!}(x^4+4x^3y+6x^2y^2+4xy^3+y^4) \\ +\frac{1}{5!}(x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5) \\ \vdots \end{matrix} \\
&= \begin{matrix} 1 \\ +x+y \\ +\frac{1}{2!}x^2+xy+\frac{1}{2!}y^2 \\ +\frac{1}{3!}x^3+\frac{1}{2!}x^2y+x\frac{1}{2!}y^2+\frac{1}{3!}y^3 \\ +\frac{1}{4!}x^4+\frac{1}{3!}x^3y+\frac{1}{2!}x^2\frac{1}{2!}y^2+x\frac{1}{3!}y^3+\frac{1}{4!}y^4 \\ +\frac{1}{5!}x^5+\frac{1}{4!}x^4y+\frac{1}{3!}x^3\frac{1}{2!}y^2+\frac{1}{2!}x^2\frac{1}{3!}y^3+x\frac{1}{4!}y^4+\frac{1}{5!}y^5 \\ \vdots \end{matrix} \\
&= e^x+e^xy+e^x\frac{1}{2!}y^2+e^x\frac{1}{3!}y^3+\cdots \\
&= e^xe^y,
\end{aligned}$$

where the next to last equality follows by adding up the diagonals. □

7 Some fun lectures

7.1 Numbers, functions, inverse functions and the exponential

7.1.1 Numbers

The **positive integers** are $1, 2, 3, 4, 5, 6, \ldots$.

The **nonnegative integers** are $0, 1, 2, 4, 5, 6, \ldots$.

The **rational numbers** are

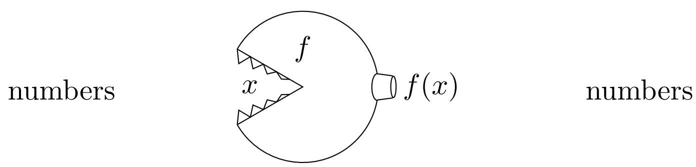
$$\frac{a}{b}\,, \qquad a \text{ an integer, } b \text{ an integer, } b \neq 0.$$

The **real numbers** are all possible decimal expansions.

The **complex numbers** are $a + bi$, where a and b are real numbers and i is a number such that $i^2 = -1$.

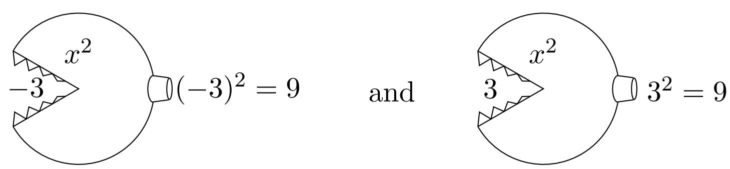
7.1.2 Functions

A **function** is a creature that eats a number, chews on it, and spits out a new number.



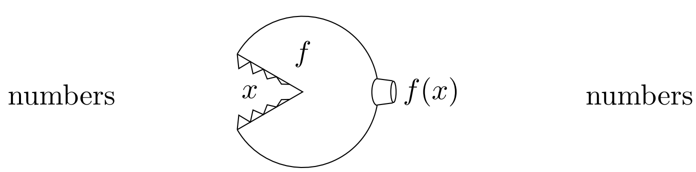
What a function spits out depends only on what goes in.

Example. The function $f(x) = x^2$ has



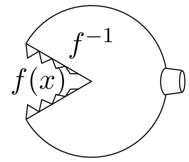
7.1.3 Inverse 'function's

A **function** is a creature that eats a number, chews on it, and spits out a new number.



What a function spits out depends only on what goes in.

The **inverse 'function'** to a function f is backwards of f. The inverse 'function' undoes what f did.

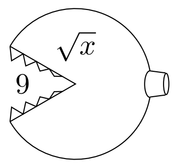
numbers  numbers

The inverse 'function' is usually *not a function* because what it spits out depends on its mood. For example:

$$\sqrt{x} \text{ is the inverse 'function' to } x^2$$

and

$$\sqrt{9} = 3 \text{ on good Mondays,} \quad \text{and} \quad \sqrt{9} = -3 \text{ on bad Tuesdays.}$$



-3? or 3? make up your mind!

7.1.4 The exponential and the log

If n is a positive integer then n**-factorial** is

$$n! = 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n.$$

The **exponential**

$$e^x = 1 + x + \tfrac{1}{2!}x^2 + \tfrac{1}{3!}x^3 + \cdots \qquad \text{is the most important function in mathematics.}$$

If n is a positive integer then

$$a^n = \underbrace{a \cdot a \cdot a \cdots a}_{n \text{ factors}}.$$

This satisfies

$$a^{x+y} = a^x a^y \qquad \text{and} \qquad a^1 = a,$$

which forces

$$a^x = e^{x\log(a)}, \qquad \text{where } \log(a) \text{ is the number such that } e^{\log(a)} = a.$$

7.2 The binomial theorem

Let $k \in \mathbb{Z}_{\ge 0}$. Define k **factorial** by

$$0! = 1 \qquad \text{and} \qquad k! = k\cdot(k-1)\cdots 3\cdot 2\cdot 1 \quad \text{if } k \in \mathbb{Z}_{>0}.$$

Let $n, k \in \mathbb{Z}_{\ge 0}$ with $k \le n$. Define

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Theorem 7.1. *Let $n, k \in \mathbb{Z}_{\ge 0}$ with $k \le n$.*

(a) Let S be a set with cardinality n. Then
$\binom{n}{k}$ is the number of subsets of S with cardinality k.

(b) $\binom{n}{k}$ is the coefficient of $x^{n-k}y^k$ in $(x+y)^n$.

(c) If $k \in \{1, \ldots, n-1\}$ then

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}, \qquad \textit{and} \qquad \binom{n}{0} = 1 \quad \textit{and} \quad \binom{n}{n} = 1.$$

This theorem says that the table of numbers

$$\begin{array}{ccccccccccc}
 & & & & & \binom{0}{0} & & & & & \\
 & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\
 & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\
 & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\
 & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\
\binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} \\
\ddots & & & & & \vdots & & & & & \ddots
\end{array}$$

are the numbers in **Pascal's triangle**

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
\ddots & & & & & \vdots & & & & & \ddots
\end{array}$$

and that

$$\begin{array}{rcc}
(x+y)^0 & = & 1, \\
(x+y)^1 & = & x+y, \\
(x+y)^2 & = & x^2+2xy+y^2, \\
(x+y)^3 & = & x^3+3x^2y+3xy^2+y^3, \\
(x+y)^4 & = & x^4+4x^3y+6x^2y^2+4xy^3+y^4, \\
(x+y)^5 & = & x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5, \\
\vdots & & \vdots
\end{array}$$

7.3 Partial fractions is backwards of common denominator

Partial fractions is the name for the backward of making a common denominator.

$$\frac{5x+22}{(x+2)(x+6)} = \frac{3}{x+2} + \frac{2}{x+6} \qquad \text{or} \qquad \frac{31}{33} = \frac{3}{11} + \frac{2}{3}.$$

A better terminology for partial fractions might be 'rotanimoned nommoc' (backwards of 'common denominator').

Splitting.

$$\text{If} \qquad 1 = pr + qs \qquad \text{then} \quad \frac{1}{pq} = \frac{r}{q} + \frac{s}{p} \quad \text{and} \quad \frac{a}{pq} = \frac{ar}{q} + \frac{as}{p}.$$

Powers.. Suppose a_1, a_2, a_3 are not divisible by p.

$$\frac{a_1p^2 + a_2p + a_3}{p^3} = \frac{a_1}{p} + \frac{a_2}{p^2} + \frac{a_3}{p^3}.$$

Remainders.

$$\text{If} \qquad a = bq + r \qquad \text{then} \quad \frac{a}{q} = b + \frac{r}{q}.$$

HW: Find a, b such that $\dfrac{2x^4 + 3x^2}{(x^2+1)^2(x^2+2)} = \dfrac{a}{(x^2+1)^2} + \dfrac{b}{x^2+2}$.

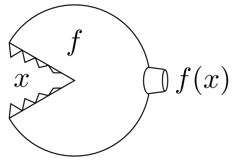
HW: Find a, b, c such that $\dfrac{3x^2 - 2x + 1}{(x+1)(x^2+2x+2)} = \dfrac{a}{x+1} - \dfrac{b(2x+2)}{x^2+2x+2} - \dfrac{c}{(x+1)^2+1}$.

HW: Find a, b such that $\dfrac{9x+1}{(x-3)(x+1)} = \dfrac{a}{x-3} + \dfrac{b}{x+1}$.

HW: Find a, b, c such that $\dfrac{4}{x^2(x+2)} = \dfrac{a}{x+2} + \dfrac{b}{x} + \dfrac{c}{x^2}$.

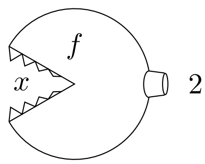
7.4 Derivatives

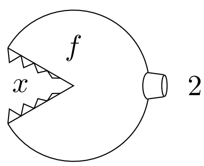
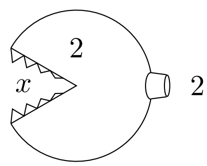
A **function** eats a number, chews on it, and spits out another number.

numbers  numbers

A **constant function** always spits out the same number, no matter what the input is.

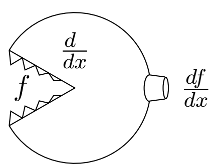
Example: $f(x) = 2$.

numbers  numbers

We call this function 2;  is 

So, 2 *sometimes means the number* 2, and *sometimes means the function* 2.

A **derivative** eats a function, chews on it, and spits out another function.

functions  functions

The derivative $\frac{d}{dx}$ knows what to spit out by always following the rules:

(1) $\frac{dx}{dx} = 1$,

(2) $\frac{d(cf)}{dx} = c\frac{df}{dx}$, if c is a constant,

(3) $\frac{d(f+g)}{dx} = \frac{df}{dx} + \frac{dg}{dx}$,

(4) $\frac{d(fg)}{dx} = f\,\frac{dg}{dx} + \frac{df}{dx}\,g$.

Example 7.1. Prove that $\frac{d1}{dx} = 0$.

Example 7.2. Find $\frac{dy}{dx}$ if $y = x^2$.

Example 7.3. Find $\frac{dy}{dx}$ if $y = x^3$.

Example 7.4. Find $\frac{dy}{dx}$ if $y = x^4$.

. . . and we keep on going . . .

Example 7.5. Find $\frac{dy}{dx}$ if $y = x^{6342}$.

. . . and we keep on going . . .

Example 7.6. Find $\frac{dx^n}{dx}$ for $n \in \{1, 2, 3, \ldots\}$.

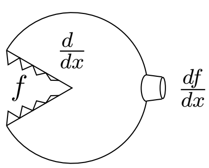
Example 7.7. Find $\frac{dx^{-6342}}{dx}$.

Example 7.8. Find $\frac{dx^{-n}}{dx}$ for $n \in \{1, 2, 3, \ldots\}$.

... and thus we have found $\quad \frac{dx^n}{dx} = nx^{n-1}, \quad$ for *all* integers n. (AMAZING!)

There are **different kinds of derivatives**:

Derivative with respect to x



The derivative $\frac{d}{dx}$ satisfies

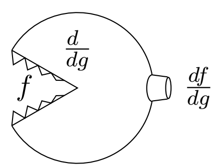
$$\frac{dx}{dx} = 1,$$

$$\frac{d(cf)}{dx} = c\frac{df}{dx}, \quad \text{if } c \text{ is a constant,}$$

$$\frac{d(y+z)}{dx} = \frac{dy}{dx} + \frac{dz}{dx},$$

$$\frac{d(yz)}{dx} = y\,\frac{dz}{dx} + \frac{dy}{dx}\,z.$$

Derivative with respect to g



The derivative $\frac{d}{dg}$ satisfies

$$\frac{dg}{dg} = 1,$$

$$\frac{d(cf)}{dg} = c\frac{df}{dg}, \quad \text{if } c \text{ is a constant,}$$

$$\frac{d(y+z)}{dg} = \frac{dy}{dg} + \frac{dz}{dg},$$

$$\frac{d(yz)}{dg} = y\,\frac{dz}{dg} + \frac{dy}{dg}\,z.$$

What is the relation between $\quad \frac{df}{dx} \quad$ **and** $\quad \frac{df}{dg} \quad$ **?**

$$\frac{dg^0}{dg} = \frac{d1}{dg} = 0, \qquad \frac{dg^0}{dx} = \frac{d1}{dx} = 0,$$

$$\frac{dg}{dg} = 1, \qquad \frac{dg}{dx} = \frac{dg}{dx},$$

$$\begin{aligned}\frac{dg^2}{dg} &= \frac{dg \cdot g}{dg} \\ &= g\frac{dg}{dg} + \frac{dg}{dg}g \\ &= g + g = 2g,\end{aligned} \qquad \begin{aligned}\frac{dg^2}{dx} &= \frac{dg \cdot g}{dx} \\ &= g\frac{dg}{dx} + \frac{dg}{dx}g \\ &= 2g\frac{dg}{dx},\end{aligned}$$

$$\begin{aligned}\frac{dg^3}{dg} &= \frac{dg^2 \cdot g}{dg} \\ &= g^2\frac{dg}{dg} + \frac{dg^2}{dg}g \\ &= g^2 + 2g \cdot g = 3g^2,\end{aligned} \qquad \begin{aligned}\frac{dg^3}{dx} &= \frac{dg^2 \cdot g}{dx} \\ &= g^2\frac{dg}{dx} + \frac{dg^2}{dx}g \\ &= g^2\frac{dg}{dx} + 2g\frac{dg}{dx}g \\ &= g^2\frac{dg}{dx} + 2g^2\frac{dg}{dx} \\ &= 3g^2\frac{dg}{dx},\end{aligned}$$

$$\begin{aligned}\frac{dg^4}{dg} &= \frac{dg^3 \cdot g}{dg} \\ &= g^3\frac{dg}{dg} + \frac{dg^3}{dg}g \\ &= g^3 + 3g^2 \cdot g = 4g^3,\end{aligned} \qquad \begin{aligned}\frac{dg^4}{dx} &= \frac{dg^3 \cdot g}{dx} \\ &= g^3\frac{dg}{dx} + \frac{dg^3}{dx}g \\ &= g^3\frac{dg}{dx} + 3g^2\frac{dg}{dx}g \\ &= g^3\frac{dg}{dx} + 3g^3\frac{dg}{dx} \\ &= 4g^3\frac{dg}{dx},\end{aligned}$$

$$\vdots \qquad \vdots$$

$$\frac{dg^{6342}}{dg} = 6342g^{6341}, \qquad \frac{dg^{6342}}{dx} = 6342g^{6341}\frac{dg}{dx},$$

$$\begin{aligned}\frac{d(3g^2+2g+7)}{dg} &= \frac{d(3g^2)}{dg} + \frac{d(2g)}{dg} + \frac{d7}{dg} \\ &= 3\frac{dg^2}{dg} + 2\frac{dg}{dg} + 0 \\ &= 3 \cdot 2g + 2 \cdot 1 \\ &= 6g + 2,\end{aligned} \qquad \begin{aligned}\frac{d(3g^2+2g+7)}{dx} &= \frac{d(3g^2)}{dx} + \frac{d(2g)}{dx} + \frac{d7}{dx} \\ &= 3\frac{dg^2}{dx} + 2\frac{dg}{dx} + 0 \\ &= 3 \cdot 2g\frac{dg}{dx} + 2\frac{dg}{dx} \\ &= (6g+2)\frac{dg}{dx},\end{aligned}$$

Thus, we are seeing that

$$\frac{df}{dx} = \frac{df}{dg}\frac{dg}{dx}, \qquad \text{which is the } \textbf{chain rule}.$$

7.5 Limits

Write

$$\lim_{x\to 2} f(x) = 10$$

if $f(x)$ gets closer and closer to 10 as x gets closer and closer to 2.

Example: Evaluate $\lim_{x\to 2} \frac{3x^2+8}{x^2-x}$.

When $x = 1.5$, $\quad \frac{3x^2+8}{x^2-x} = 19.66....$

When $x = 1.9$, $\quad \frac{3x^2+8}{x^2-x} = 11.011....$

When $x = 1.99$, $\quad \frac{3x^2+8}{x^2-x} = 10.091....$

When $x = 1.999$, $\quad \frac{3x^2+8}{x^2-x} = 10.00901....$

When $x = 1.9999$, $\quad \frac{3x^2+8}{x^2-x} = 10.0009001....$

So $\lim_{x\to 2} \frac{3x^2+8}{x^2-x} = 10$.

Usually determining the limit is straightforward.

Example 7.9. $\lim_{x\to 1} 6x^2 - 4x + 3 = 5$.

But sometimes ...

Example: $\lim_{x\to 0} \frac{\sqrt{1+x}-1}{x} \stackrel{?}{=} \frac{0}{0}$.

$$\frac{0}{0} \quad \text{MAKES NO SENSE.}$$

Example: $\lim_{x\to 0} \frac{5x}{x} \stackrel{?}{=} \frac{0}{0}$.

$$\lim_{x\to 0} \frac{5x}{x} = \lim_{x\to 0} 5 = 5.$$

Example: $\lim_{x\to 0} \frac{17x}{2x} \stackrel{?}{=} \frac{0}{0}$.

$$\lim_{x\to 0} \frac{17x}{2x} = \lim_{x\to 0} \frac{17}{2} = \frac{17}{2}.$$

Let's go back to

Example: $\lim_{x\to 0} \frac{\sqrt{1+x}-1}{x} \stackrel{?}{=} \frac{0}{0}$.

$$\begin{aligned}
\lim_{x\to 0} \frac{\sqrt{1+x}-1}{x} &= \lim_{x\to 0} \frac{\sqrt{1+x}-1}{x} \cdot \frac{(\sqrt{1+x}+1)}{(\sqrt{1+x}+1)} \\
&= \lim_{x\to 0} \frac{1+x-1}{x(\sqrt{1+x}+1)} = \lim_{x\to 0} \frac{x}{x(\sqrt{1+x}+1)} \\
&= \lim_{x\to 0} \frac{1}{\sqrt{1+x}+1} = \frac{1}{\sqrt{1+0}+1} = \frac{1}{2}.
\end{aligned}$$

So, whenever a limit *looks* like it is coming out to $\frac{0}{0}$ it needs to be looked at in a different way to see what it is *really* getting closer and closer to.

Example 7.10. Evaluate $\lim_{x\to 7} \frac{x^2-49}{x-7}$.

Example 7.11. Evaluate $\lim_{x\to 5} \frac{x^5-3125}{x-5}$.

Example 7.12. Evaluate $\lim_{x\to a} \frac{x^{5/2}-a^{5/2}}{x-a}$.

Particularly useful limits

Example 7.13. Evaluate $\lim_{x\to 0} \frac{\sin(x)}{x}$.

Example 7.14. Evaluate $\lim_{x\to 0} \frac{\cos(x)-1}{x}$.

Example 7.15. Evaluate $\lim_{x\to 0} \frac{e^x-1}{x}$.

Example 7.16. Evaluate $\lim_{x\to 0} \frac{\log(1+x)}{x}$.

Example 7.17. Evaluate $\lim_{x\to 0} (1+x)^{1/x}$.

Note: $n\to\infty$ means as n gets larger and larger.

Example 7.18. Evaluate $\lim_{n\to\infty} (1+\frac{1}{n})^n$.

Example 7.19. Evaluate $\lim_{x\to \pi} \frac{\sin(x)}{x-\pi}$.

Example 7.20. Evaluate $\lim_{x\to \infty} \frac{x^2-7x+11}{3x^2+10}$.

Example 7.21. Evaluate $\lim_{x\to 0} \frac{\sin(3x)}{\sin(5x)}$.

Example 7.22. Evaluate $\lim_{x\to 1} \frac{1-x}{(\cos^{-1}(x))^2}$.

Example 7.23. Evaluate $\lim_{\Delta x\to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$ when $f(x)=\sin(2x)$.

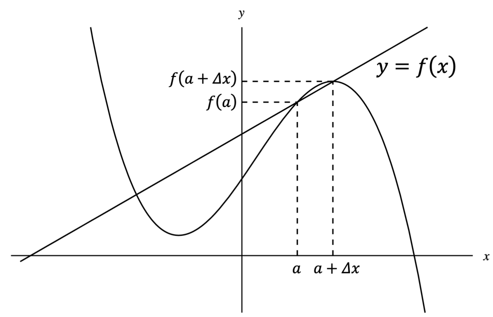
Example 7.24. Evaluate $\lim_{\Delta x\to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$ when $f(x)=\cos(x^2)$.

Example 7.25. Evaluate $\lim_{\Delta x\to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$ when $f(x)=x^x$.

7.5.1 Limits and derivatives

Example 7.26. (The fundamental theorem of change) For a smooth continuous function $f: \mathbb{R} \to \mathbb{R}$ let

$$D_f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$



So that $D_f(a)$ is the slope of f at $x = a$ (the rate of change of f with respect to x at $x = a$).

Let c be a constant and let f and g be functions and assume that D_f and D_g exist. Show that

(a) $D_x = 1$,

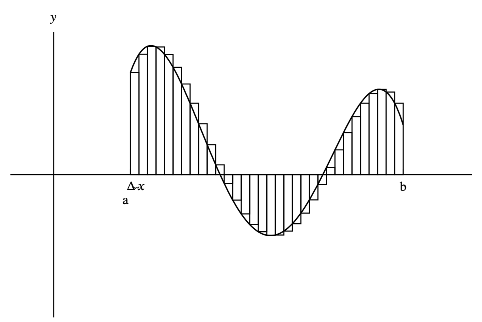
(b) $D_{cf} = cD_f$,

(c) $D_{f+g} = D_f + D_g$,

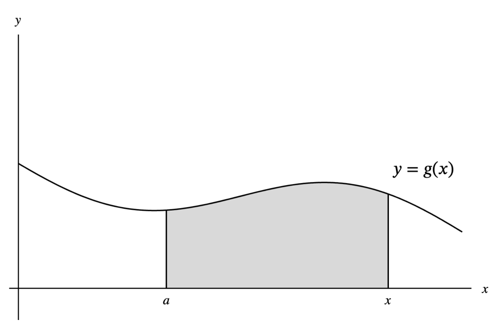
(d) $D_{fg} = D_f \cdot g + f \cdot D_g$.

Example 7.27. (The fundamental theorem of measure) For $a, b \in \mathbb{R}$ with $p < a < b$ and a smooth continuous function $f: \mathbb{R} \to \mathbb{R}$ let

$$\int_a^b f\,dx = \lim_{h \to 0} \Big(\sum_{j=0}^{\lfloor \frac{b-a}{h} \rfloor} f(a + jh)h \Big)$$



If $f: \mathbb{R} \to \mathbb{R}_{\geq 0}$ then $\int_a^b f\,dx$ is the area under f between $x = a$ and $x = b$.



Let $p \in \mathbb{R}$ with $p < a < b$ and let $A: \mathbb{R}_{[p,b]} \to \mathbb{R}_{\geq 0}$ be the function given by

$$A(x) = \int_p^x f\,dx.$$

Then
$$\lim_{h\to 0} \frac{A(x+h)-A(x)}{h} = f(x) \qquad \text{and} \qquad A(b)-A(a) = \int_a^b f\,dx.$$

7.6 Calculus, Functions and inverse functions

Calculus is the study of

(1) Derivatives
(2) Integrals
(3) Applications of derivatives
(4) Applications of integrals

A *derivative* is a creature you put a function into, it chews on it, and spits out a new function.

$$f \to \boxed{\frac{d}{dx}} \to \frac{df}{dx}.$$

The *integral* is the derivative backwards:

$$f \leftarrow \boxed{\int \quad dx} \leftarrow \frac{df}{dx} \qquad \text{or} \qquad \frac{df}{dx} \to \boxed{\int \quad dx} \to f.$$

A *function* is one down on the food chain.

$$\begin{matrix} \text{input} & & & & \text{output} \\ \text{number} & \to & \boxed{f} & \to & \text{number} \\ x & & & & f(x) \end{matrix}$$

Functions take a number as input, chew on it a bit, and spit out a new number.

The *inverse function* to f is f backwards:

$$x \leftarrow \boxed{f^{-1}} \leftarrow f(x) \qquad \text{or} \qquad \begin{matrix} f(x) \to & & \to x \\ & \boxed{f^{-1}} & \\ z \to & & \to f^{-1}(z) \end{matrix}$$

Example.

$$\begin{array}{rcl} x \to & & \to x^2 \\ 1 \to & f(x) = x^2 & \to 1 \\ 2 \to & & \to 4 \\ 3 \to & & \to 9 \\ -3 \to & & \to 9 \\ \pi \to & & \to \pi^2 \\ \sqrt{7} \to & & \to 7 \end{array}$$

The inverse function is

$$\begin{array}{rcl} x^2 \to & & \to x \\ 1 \to & f^{-1}(x) = \sqrt{x} & \to 1 \\ 4 \to & & \to 2 \\ 9 \to & & \to 3 \\ 9 \to & & \to -3 \\ \pi^2 \to & & \to \pi \\ 7 \to & & \to \sqrt{7} \end{array}$$

The inverse function is not always a function because there might be some uncertainty about what the inverse function will spit out:

$$9 \to \boxed{f^{-1}(x) = \sqrt{x}} \to 3 \qquad \text{or} \qquad 9 \to \boxed{f^{-1}(x) = \sqrt{x}} \to -3.$$

Numbers are at the very bottom of the food chain.

7.6.1 And so we discovered ... Numbers

At some point humankind wanted to count things and discovered the **positive integers**,

$$1,\ 2,\ 3,\ 4,\ 5,\ \ldots.$$

GREAT for counting something,
BUT what if you don't have anything? How do we talk about nothing, nulla, zilch?
... and so we discovered the **nonnegative integers**,

$$0,\ 1,\ 2,\ 3,\ 4,\ 5,\ \ldots.$$

GREAT for adding,

$$5+3=8,\quad 0+10=10,\quad 21+37=48,$$

BUT not so great for subtraction,

$$5-3=2,\quad 2-0=2,\quad 12-34=???.$$

... and so we discovered the **integers**

$$\ldots,-3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3,\ \ldots\ .$$

GREAT for adding, subtracting and multiplying,

$$3\cdot 6=18,\quad -3\cdot 2=-6,\quad 0\cdot 7=0,$$

BUT not so great if you only want part of the sausage ...,
... and so we discovered the **rational numbers**,

$$\frac{a}{b}\,,\qquad a \text{ an integer, } b \text{ an integer, } b\neq 0.$$

GREAT for addition, subtraction, multiplication, and division,
BUT not so great for finding $\sqrt{2}=$????,
... and so we discovered the **real numbers**,

all decimal expansions.

Examples:

$$\begin{aligned} \pi &= 3.1415926\ldots\ , \\ e &= 2.71828\ldots\ , \\ \sqrt{2} &= 1.414\ldots\ , \\ 10 &= 10.0000\ldots\ , \end{aligned} \qquad \begin{aligned} \tfrac{1}{3} &= .3333\ldots\ , \\ \tfrac{1}{8} &= .125 = .125000000\ldots\ , \end{aligned}$$

GREAT for addition, subtraction, multiplication, and division,
BUT not so great for finding $\sqrt{-9}=$????,
... and so we discovered the **complex numbers**,

$$a+bi,\qquad a \text{ a real number, } b \text{ a real number, } i=\sqrt{-1}.$$

7.6.2 Operations on complex numbers

Examples of complex numbers: $3+\sqrt{2}i$, $6 = 6+0i$, $\pi+\sqrt{7}i$,
and

$$\sqrt{-9} = \sqrt{9(-1)} = \sqrt{9}\sqrt{-1} = 3i.$$

GREAT.

Addition: $(3+4i)+(7+9i) = 3+7+4i+9i = 10+13i.$

Subtraction: $(3+4i)-(7+9i) = 3-7+4i-9i = -4-5i.$

Multiplication:

$$\begin{aligned}(3+4i)(7+9i) &= 3(7+9i)+4i(7+9i)\\ &= 21+27i+28i+36i^2\\ &= 21+55i-36\\ &= -15+55i.\end{aligned}$$

Division:

$$\begin{aligned}\frac{3+4i}{7+9i} = \frac{(3+4i)}{(7+9i)}\frac{(7-9i)}{(7-9i)} &= \frac{21-27i+28i+36}{49-63i+63i+81}\\ &= \frac{57+i}{130} = \frac{57}{130}+\frac{1}{130}i.\end{aligned}$$

Square Roots: We want $\sqrt{-3+4i}$ to be some $a+bi$.

$$\text{If} \qquad \sqrt{-3+4i} = a+bi$$

then

$$\begin{aligned}-3+4i = (a+bi)^2 &= a^2+abi+abi+b^2i^2\\ &= a^2-b^2+2abi.\end{aligned}$$

So

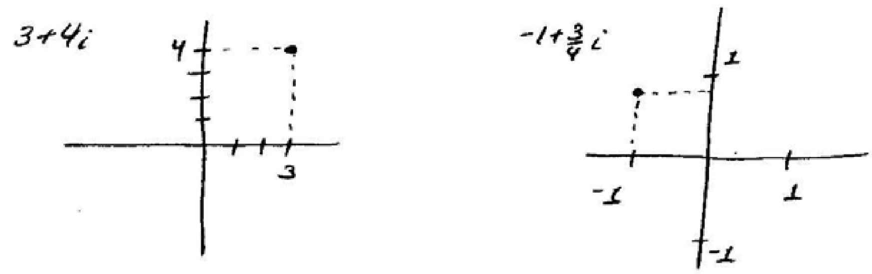
$$a^2-b^2 = -3 \qquad \text{and} \qquad 2ab = 4.$$

Solve for a and b.

$$b = \frac{4}{2a} = \frac{2}{a}. \qquad \begin{aligned}&\text{So} \quad a^2-\left(\frac{2}{a}\right)^2 = -3.\\ &\text{So} \quad a^2-\frac{4}{a^2} = -3.\\ &\text{So} \quad a^4-4 = -3a^2.\\ &\text{So} \quad a^4+3a^2-4 = 0.\\ &\text{So} \quad (a^2+4)(a^2-1) = 0.\end{aligned}$$

So $a^2 = -4$ or $a^2 = 1$.
So $a = \pm 1$, and $b = \frac{2}{\pm 1} = 2$ or -2.
So $a+bi = 1+2i$ or $a+bi = -1-2i$.
So $\sqrt{-3+4i} = \pm(1+2i)$.

Graphing:



Really, the i-axis and not-i-axis should be properly labeled

Factoring:

$$x^2+5 = (x+\sqrt{5}\,i)(x-\sqrt{5}\,i),$$
$$x^2+x+1 = \Big(x-\Big(-\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}i\Big)\Big)\Big(x-\Big(-\tfrac{1}{2}-\tfrac{\sqrt{3}}{2}i\Big)\Big)$$

This is REALLY why we like the complex numbers.
The **fundamental theorem of algebra** says that ANY POLYNOMIAL

$$\text{(for example, } x^{12673}+2563x^{159}+\pi x^{121}+\sqrt{7}\,x^{23}+9621\tfrac{1}{2}\text{)}$$

can be factored completely as

$$(x-u_1)(x-u_2)\cdots(x-u_n)$$

where $u_1, u_2, \ldots, u_n$ are complex numbers.

7.7 The interest sequence

Example. If you borrow \$500 on your credit card at 14% interest, find the amounts due at the end of two years if the interest is compounded

(a) annually,
(b) quarterly,
(c) monthly,
(d) daily,
(e) hourly,
(f) every second,
(g) every nanosecond,
(h) continuously.

(a) You owe

$$500 + 500(.14) = 500(1 + .14) \text{ after one year} \qquad \text{and} \qquad 500(1 + .14)(1 + .14) \text{ after two years.}$$

(b) You owe

$$500 + 500\Big(\frac{.14}{12}\Big) = 500\Big(1 + \frac{.14}{12}\Big) \quad \text{after one month.}$$

You owe

$$500\Big(1 + \frac{.14}{12}\Big)\Big(1 + \frac{.14}{12}\Big) \quad \text{after two months.}$$

You owe

$$500\Big(1 + \frac{.14}{12}\Big)^{24} \quad \text{after two years.}$$

(f) You owe

$$500 + 500\Big(\frac{.14}{365 \cdot 24 \cdot 3600}\Big) \quad \text{after one second.}$$

and

$$500\Big(1 + \frac{.14}{365 \cdot 24 \cdot 3600}\Big)^{2\cdot 365\cdot 24\cdot 3600} \quad \text{after two years.}$$

In fact,

$$\begin{aligned} \lim_{n\to\infty} 500\Big(1 + \frac{.14}{n}\Big)^{2n} &= 500 \lim_{n\to\infty} \left(e^{\log\left(1+\frac{.14}{n}\right)}\right)^{2n} \\ &= 500 \lim_{n\to\infty} e^{2n\log\left(1+\frac{.14}{n}\right)} \\ &= 500 \lim_{n\to\infty} e^{2\cdot .14\frac{\log(1+\frac{.14}{n})}{\frac{.14}{n}}} \\ &= 500 \lim_{n\to\infty} e^{.28\frac{\log(1+\frac{.14}{n})}{\frac{.14}{n}}} = 500e^{.28}, \end{aligned}$$

since

$$\lim_{x\to 0} \frac{\log(1+x)}{x} = 1.$$

So you owe $500e^{.28}$ after two years if the interest is compounded continuously.

Note: $500(1+.14)^2 = 649.80$, $\quad 500\Big(1 + \frac{.14}{12}\Big)^{24} \approx 660.49$, $\quad$ and $\quad 500e^{.28} \approx 661.58$.

7.8 Limits

The *tolerance set* is

$$\mathbb{E} = \{10^{-1}, 10^{-2}, \ldots\}.$$

For $n \in \mathbb{Z}_{>0}$ define $d\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$

$$d(x,y) = \sqrt{(y_1 - x_1)^2 + \cdots + (y_n - x_n)^2}, \qquad \text{if } x = (x_1, \ldots, x_n) \text{ and } y = (y_1, \ldots, y_n).$$

Let $m, n \in \mathbb{Z}_{>0}$ and let $f\colon \mathbb{R}^m \to \mathbb{R}^n$. Let $a \in \mathbb{R}^m$ and $\ell \in \mathbb{R}^n$.

$$\lim_{x\to a} f(x) = \ell \qquad \text{means}$$

$$\text{if } \varepsilon \in \mathbb{E} \text{ then there exists } \delta \in \mathbb{E} \text{ such that} \quad \text{if } 0 < d(x,a) < \delta \text{ then } d(f(x), \ell) < \varepsilon.$$

Here is a translation into the language of "English":

In English	In Math
The client has a machine f that produces steel rods of length ℓ for sales.	Let $f\colon X \to \mathbb{R}$ and let $\ell \in \mathbb{R}$.
The output of f gets closer and closer to ℓ as the input gets closer and closer to a means	$\lim_{x\to a} f(x) = \ell$ means
if you give me a tolerance the client needs, in other words, the number of decimal places of accuracy the client requires	if $\varepsilon \in \mathbb{E}$
then my business will tell you	then there exists
the accuracy you need on the dials of the machine so that	$\delta \in \mathbb{E}$ such that
if the dials are set within δ of a	if $0 < d(x,a) < \delta$
then the output of the machine will be within ε of ℓ.	then $d(f(x), \ell) < \epsilon$.

Let $n \in \mathbb{Z}_{>0}$ and let $a_1, a_2, \ldots$ be a sequence in $\mathbb{R}^n$. Let $\ell \in \mathbb{R}$.

$$\lim_{n\to\infty} a_n = \ell \qquad \text{means}$$

$$\text{if } \varepsilon \in \mathbb{E} \text{ then there exists } N \in \mathbb{Z}_{>0} \text{ such that} \quad \text{if } n \in \mathbb{Z}_{\geq N} \text{ then } d(a_n, \ell) < \varepsilon.$$

7.9 Formula 1

Let $D = \frac{d}{dx}$ so that $D^2 f = \frac{d^2 f}{dx^2}$ and $D^7 f = \frac{d^7 f}{dx^7}$. Then

$$f(x) = f(a) + (Df)(a)(x-a) + \frac{1}{2!}(D^2 f)(a)(x-a)^2 + \frac{1}{3!}(D^3 f)(a)(x-a)^3 + \cdots .$$

If $a = 0$ then

$$f(x) = f(0) + (Df)(0)x + \frac{1}{2!}(D^2 f)(0)x^2 + \frac{1}{3!}(D^3 f)(0)x^3 + \cdots .$$

So if we want to find the series expansion for e^x then

$$\begin{aligned} e^x &= e^0 + \big(\frac{de^x}{dx}\big)(0)x + \frac{1}{2!}\big(\frac{d^2 e^x}{dx^2}\big)(0)x^2 + \frac{1}{3!}\big(\frac{d^3 e^x}{dx^3}\big)(0)x^3 + \cdots \\ &= e^0 + (e^x)(0)x + \frac{1}{2!}(e^x)(0)x^2 + \frac{1}{3!}(e^x)(0)x^3 + \cdots \\ &= e^0 + e^0 x + \frac{1}{2!}e^0 x^2 + \frac{1}{3!}e^0 x^3 + \cdots \\ &= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots \end{aligned}$$

If we want to find

$$\begin{aligned} \lim_{x\to 0} \frac{e^x - 1}{x} &= \lim_{x\to 0} \frac{\big(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{4!}x^4 + \cdots\big) - 1}{x} \\ &= \lim_{x\to 0} \frac{x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{4!}x^4 + \cdots}{x} \\ &= \lim_{x\to 0} 1 + \frac{1}{2!}x + \frac{1}{3!}x^2 + \cdots + \frac{1}{4!}x^3 + \cdots \\ &= 1 + 0 + 0 + 0 + \cdots \\ &= 1. \end{aligned}$$

If we want to find the series expansion for $\sin(x)$ then

$$\begin{aligned} \sin(x) &= \sin(0) + \big(\frac{d\sin(x)}{dx}\big)(0)x + \frac{1}{2!}\big(\frac{d^2\sin(x)}{dx^2}\big)(0)x^2 + \frac{1}{3!}\big(\frac{d^3\sin(x)}{dx^3}\big)(0)x^3 + + \frac{1}{4!}\big(\frac{d^4\sin(x)}{dx^4}\big)(0)x^4 + \cdots \\ &= \sin(0) + \cos(0)x + \frac{1}{2!}(-\sin(0))x^2 + \frac{1}{3!}(-\cos(0))x^3 + \frac{1}{4!}(\sin(0))x^4 + \cdots \\ &= 0 + x - 0 - \frac{1}{3!}x^3 + 0 + \frac{1}{5!}x^5 - 0 - \frac{1}{7!}x^7 + 0 + \cdots \\ &= x - -\frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots . \end{aligned}$$

If we want to find

$$\begin{aligned} \lim_{x\to 0} \frac{\sin(x)}{x} &= \lim_{x\to 0} \frac{x - -\frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{}{7!}x^7 + \cdots}{x} \\ &= \lim_{x\to 0} \big(1 - -\frac{1}{3!}x^2 + \frac{1}{5!}x^4 - \frac{}{7!}x^6 + \cdots\big) \\ &= 1 - 0 + 0 - 0 + 0 - \cdots \\ &= 1. \end{aligned}$$

If we want to find the series expansion for $\frac{1}{1-x}$ then

$$\begin{aligned}\frac{1}{1-x} &= \frac{1}{1-x}\Big]_{x=0} + \Big(\frac{d\frac{1}{1-x}}{dx}\Big]_{x=0}\Big)x + \frac{1}{2!}\Big(\frac{d^2\frac{1}{1-x}}{dx^2}\Big]_{x=0}\Big)x^2 + \frac{1}{3!}\Big(\frac{d^3\frac{1}{1-x}}{dx^3}\Big]_{x=0}\Big)x^3 + \cdots \\ &= 1 + \Big(\frac{1}{(1-x)^2}\Big]_{x=0}\Big)x + \frac{1}{2!}\Big(\frac{2}{(1-x)^3}\Big]_{x=0}\Big)x^2 + \frac{1}{3!}\Big(\frac{3\cdot 2}{(1-x)^4}\Big]_{x=0}\Big)x^3 + \cdots \\ &= 1 + x + x^2 + x^3 + x^4 + x^5 + \cdots.\end{aligned}$$

If we want to find the series expansion for $\frac{1}{1+x}$ then

$$\begin{aligned}\frac{1}{1+x} = \frac{1}{1-(-x)} &= 1 + (-x) + (-x)^2 + (-x)^3 + (-x)^4 + \cdots \\ &= 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots.\end{aligned}$$

If we want to find the series expansion for $\log(1+x)$ then

$$\begin{aligned}\int \frac{1}{1+x}\,dx &= \log(1+x) \\ &= \int \big(1 - x + x^2 - x^3 + x^4 - x^5 + \cdots\big)\,dx \\ &= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \frac{1}{6}x^6 + \cdots.\end{aligned}$$

So, in particular,

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots = \log(1+1) = \log(2).$$

Also

$$\begin{aligned}\lim_{x\to 0}\frac{\log(1+x)}{x} &= \lim_{x\to 0}\frac{x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \frac{1}{6}x^6 + \cdots}{x} \\ &= \lim_{x\to 0}\big(1 - \frac{1}{2}x + \frac{1}{3}x^2 - \frac{1}{4}x^3 + \frac{1}{5}x^4 - \frac{1}{6}x^5 + \cdots\big) \\ &= 1 - 0 + 0 - 0 + \cdots \\ &= 1.\end{aligned}$$

If we want to find the series expansion for $\tan^{-1}(x)$ then

$$\begin{aligned}\tan^{-1}(x) &= \int \frac{1}{1+x^2}\,dx \\ &= \int \big(1 - x^2 + x^4 - x^6 + x^8 - \cdots\big)\,dx \\ &= x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9 - \frac{1}{11}x^{11} + \cdots\end{aligned}$$

If we want to find π then

$$\begin{aligned}\pi &= 4\cdot\big(\tfrac{\pi}{4}\big) = 4\tan^{-1}(1) \\ &= 4\big(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots\big) \\ &= 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} - \frac{4}{11} + \frac{4}{13} - \cdots\big).\end{aligned}$$

7.10 Series expansions and derivatives

The derivative of f with respect to x is $\dfrac{df}{dx}$. It is common to write $f'(x)$ in place of $\dfrac{df}{dx}$.

$$f'(x) = \frac{df}{dx}.$$

The *second derivative of f with respect to x* is

$$f''(x) = \frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right),$$

the derivative of the derivative of f. Both $\dfrac{d^2 f}{dx^2}$ and $f''(x)$ are notations for the same thing, the second derivative of f.

The *third derivative of f with respect to x* is

$$f'''(x) = \frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right),$$

the derivative of the second derivative of f. Use the notations $\dfrac{d^3 f}{dx^3}$ and $f'''(x)$ interchangeably for the third derivative of f.

The *fourth derivative of f with respect to x* is

$$f^{(4)}(x) = \frac{d^4 f}{dx^4} = \frac{d}{dx}\left(\frac{d^3 f}{dx^3}\right),$$

the derivative of the third derivative of f.

Let a be a number. Then f *evaluated at a* is

$$f(a) = f\big|_{x=a} = c_0 + c_1 a + c_2 a^2 + c_3 x^3 + \cdots,$$

if $f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$. Use both notations, $f(a)$ and $f\big|_{x=a}$, interchangeably, for f evaluated at a.

Example: If $f(x) = 7x^3 + 3x^2 + 5x + 12$ and $a = 3$ then

$$\begin{aligned} f(3) &= 7\cdot 3^3 + 3\cdot 3^2 + 5\cdot 3 + 12 = 8\cdot 3^3 + 27 = 9\cdot 3^3 = 3^5, \\ f\big|_{x=3} &= 7\cdot 3^3 + 3\cdot 3^2 + 5\cdot 3 + 12 = 8\cdot 3^3 + 27 = 9\cdot 3^3 = 3^5. \end{aligned}$$

$$\begin{aligned} \frac{df}{dx} &= 21x^2 + 6x + 5, & \frac{df}{dx}\Big|_{x=3} &= 21\cdot 3^2 + 6\cdot 3 + 5 = 189 + 23 = 202, \\ f' &= 21x^2 + 6x + 5, & f'(3) &= 21\cdot 3^2 + 6\cdot 3 + 5 = 189 + 23 = 202, \end{aligned}$$

$$\begin{aligned} \frac{d^2 f}{dx^2} &= 42x + 6, & \frac{d^2 f}{dx^2}\Big|_{x=3} &= 42\cdot 3 + 6 = 132, \\ f'' &= 42x + 6, & f''(3) &= 42\cdot 3 + 6 = 132, \end{aligned}$$

$$\begin{aligned} \frac{d^3 f}{dx^3} &= 42, & \frac{d^3 f}{dx^3}\Big|_{x=3} &= 42, \\ f''' &= 42, & f'''(3) &= 42, \end{aligned}$$

$$\begin{aligned} \frac{d^4 f}{dx^4} &= 0, & \frac{d^4 f}{dx^4}\Big|_{x=3} &= 0, \\ f^{(4)} &= 0, & f^{(4)}(3) &= 0. \end{aligned}$$

Series expansions and the limit formula for the derivative

If $\quad f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + c_5(x-a)^5 + \cdots$

then

$$f(a) = c_0,$$

$$\frac{df}{dx}\Big|_{x=a} = \big(c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + 5c_5(x-a)^4 + \cdots\big)\Big|_{x=a} \;=\; c_1,$$

$$\frac{d^2f}{dx^2}\Big|_{x=a} = \big(2c_2 + 3\cdot 2c_3(x-a) + 4\cdot 3c_4(x-a)^2 + 5\cdot 4c_5(x-a)^3 + \cdots\big)\Big|_{x=a} \;=\; 2c_2,$$

$$\frac{d^3f}{dx^3}\Big|_{x=a} = \big(3\cdot 2c_3 + 4\cdot 3\cdot 2c_4(x-a) + 5\cdot 4\cdot 3c_5(x-a)^2 + 6\cdot 5\cdot 4c_6(x-a)^3 + \cdots\big)\Big|_{x=a} \;=\; 3\cdot 2c_3,$$

$$\frac{d^4f}{dx^4}\Big|_{x=a} = \big(4\cdot 3\cdot 2c_4 + 5\cdot 4\cdot 3\cdot 2c_5(x-a) + 6\cdot 5\cdot 4\cdot 3c_4(x-a)^2 + \cdots\big)\Big|_{x=a} \;=\; 4\cdot 3\cdot 2c_4,$$

and we can continue this process to find

$$\frac{d^kf}{dx^k}\Big|_{x=a} = k!\,c_k, \qquad \text{for } k = 1, 2, 3, \ldots.$$

Dividing both sides by $k!$ gives

$$c_k = \frac{1}{k!}\left(\frac{d^kf}{dx^k}\Big|_{x=a}\right).$$

So

$$f(x) = f(a) + \left(\frac{df}{dx}\Big|_{x=a}\right)(x-a) + \frac{1}{2!}\left(\frac{d^2f}{dx^2}\Big|_{x=a}\right)(x-a)^2 + \frac{1}{3!}\left(\frac{d^3f}{dx^3}\Big|_{x=a}\right)(x-a)^3 + \cdots,$$

or, equivalently,

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \frac{1}{3!}f'''(a)(x-a)^3 + \frac{1}{4!}f^{(4)}(a)(x-a)^4 + \cdots.$$

Now subtract $f(a)$ from both sides:

$$f(x) - f(a) = f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \frac{1}{3!}f'''(a)(x-a)^3 + \frac{1}{4!}f^{(4)}(a)(x-a)^4 + \cdots.$$

Divide both sides by $x-a$.

$$\frac{f(x)-f(a)}{x-a} = f'(a) + \frac{1}{2!}f''(a)(x-a) + \frac{1}{3!}f'''(a)(x-a)^2 + \frac{1}{4!}f^{(4)}(a)(x-a)^3 + \cdots.$$

Evaluate both sides at $x = a$.

$$\frac{f(x)-f(a)}{x-a}\Big|_{x=a} = f'(a) + 0 + 0 + 0 + 0 + \cdots.$$

So $\qquad f'(a) = \frac{f(x)-f(a)}{x-a}\Big|_{x=a}$.

Let $x = a+h$. Then $\qquad f'(a) = \frac{f(a+h)-f(a)}{a+h-a}\Big|_{a+h=a}$.

So $\qquad \frac{df}{dx}\Big|_{x=a} = \frac{f(a+h)-f(a)}{h}\Big|_{h=0}$.

Another way to write this is

$$\frac{df}{dx}\Big|_{x=a} = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}.$$

Example: Suppose you want to know what f I'm thinking of and I refuse to tell you.

You ask me what $f(0)$ is and I say "6".
You ask me what $f'(0)$ is and I say "10".
You ask me what $f''(0)$ is and I say "31".
You ask me what $f'''(0)$ is and I say "5".
You ask me what $f^{(4)}(0)$ is and I say "7".
You ask me what $f^{(5)}(0)$ is and I say "0".
You ask me what $f^{(6)}(0)$ is and I say "0".
You ask me what $f^{(7)}(0)$ is and I say "0, they are all coming out to 0 now.".

At this point you win because you know that

$$\begin{aligned}
f(x) &= f(0) + f'(0)(x-0) + \frac{1}{2!}f''(0)(x-0)^2 + \frac{1}{3!}f'''(0)(x-0)^3 + \cdots \\
&= 6 + 10(x-0) + \frac{1}{2!}31(x-0)^2 + \frac{1}{3!}5(x-0)^3 + \frac{1}{4!}7(x-0)^4 \\
&\qquad + \frac{1}{5!}\cdot 0(x-0)^5 + \frac{1}{6!}\cdot 0(x-0)^6 + \frac{1}{7!}\cdot 0(x-0)^7 + 0 + 0 + \cdots \\
&= 6 + 10x + \frac{31}{2}x^2 + \frac{5}{6}x^3 + \frac{7}{24}x^4,
\end{aligned}$$

and so you have found out what f is.

Example: Suppose you want to know what f I'm thinking of and I refuse to tell you.

You ask me what $f(0)$ is and I say "I won't tell you, but $f(3) = 4$".

You ask me what $f'(0)$ is and I say "I won't tell you, but $\frac{df}{dx}\Big|_{x=3} = 2$".

You ask me what $f''(0)$ is and I say "I won't tell you, but $\frac{d^2f}{dx^2}\Big|_{x=3} = 5$".

You ask me what $f'''(0)$ is and I say "I won't tell you, but $\frac{d^3f}{dx^3}\Big|_{x=3} = 0$ and all the rest of the $\frac{d^kf}{dx^k}\Big|_{x=3}$ are coming out to 0".

At this point you win because you know that

$$
\begin{aligned}
f(x) &= f\Big|_{x=3} + \left(\frac{df}{dx}\Big|_{x=3}\right)(x-3) + \frac{1}{2!}\left(\frac{d^2 f}{dx^2}\Big|_{x=3}\right)(x-3)^2 + \frac{1}{3!}\left(\frac{d^3 f}{dx^3}\Big|_{x=3}\right)(x-3)^3 + \cdots \\
&= 2 + 5(x-3) + \frac{1}{2!}5(x-3)^2 + \frac{1}{3!}\cdot 0(x-3)^3 + 0 + 0 + \cdots \\
&= 2 + 5x - 15 + \frac{5}{2}(x^2 - 6x + 9) + 0 + 0 + \cdots \\
&= -13 + 5x + \frac{5}{2}x^2 - 15x + \frac{45}{2} \\
&= \frac{5}{2}x^2 - 10x + \frac{19}{2},
\end{aligned}
$$

and so you know what f is.

7.11 The genesis lecture

The function god(t)

There is one function that

(a) in the Beginning, created something from nothing, and

(b) is "unchanging", or rather, its change is itself.

Through the ages thinkers have contemplated this function and nowadays it is common to write (a) and (b) in abbreviated form,

$$(\mathrm{a}')\quad \mathrm{god}(0)=1,\qquad\qquad \text{and}\qquad\qquad (\mathrm{b}')\quad \frac{d\ \mathrm{god}(t)}{dt}=\mathrm{god}(t),$$

but the meaning is still the same.

Two of the children of god are eve and adam:

$$\mathrm{god}(it)=\mathrm{adam}(t)+i\ \mathrm{eve}(t).$$

Trying to understand god(t)

If we try to "understand" god in "normal" terms,

$$\mathrm{god}(t)=a_0+a_1t+a_2t^2+a_3t^3+\cdots,$$

then

$$\text{since}\qquad \mathrm{god}(0)=1,\qquad\qquad a_0=1,\qquad \text{and}$$

$$\text{since}\qquad \frac{d\mathrm{god}(t)}{dt}=\mathrm{god}(t),\qquad \begin{aligned} a_1&=a_0, &&\text{and}\\ 2a_2&=a_1, &&\text{and}\\ 3a_3&=a_2, &&\text{and}\\ 4a_4&=a_3, &&\text{and}\\ 5a_5&=a_4, &&\ldots,\text{ etc.,}\end{aligned}$$

and so

$$\mathrm{god}(t)=1\ +\ t\ +\ \frac{1}{2!}t^2\ +\ \frac{1}{3!}t^3\ +\ \frac{1}{4!}t^4\ +\ \cdots,$$

which illustrates that god(t) exists everywhere and goes on forever.

An amazing thing about god(t)

One of the amazing things about god is that

$$\mathrm{god}(t+s)=\mathrm{god}(t)\ \mathrm{god}(s).$$

To see why god is this way suppose that there is a "different" function such that

$$(\mathrm{a}'')\ \text{ is "unchanging"}\qquad \left(\text{i.e. } \frac{d\ \widetilde{\mathrm{god}}(t)}{dt}=\widetilde{\mathrm{god}}(t)\right),\qquad \text{and}$$

(b″) in the Beginning, was the way that god is after s millenia (i.e. $\widetilde{\mathrm{god}}(0) = \mathrm{god}(s)$).

By the chain rule,

$$\frac{d\ \mathrm{god}(t+s)}{dt} = \mathrm{god}(t+s) \qquad \text{and} \qquad \mathrm{god}(0+s) = \mathrm{god}(s),$$

and so

$$\mathrm{god}(t+s) = \widetilde{\mathrm{god}}(t).$$

Also,

$$\frac{d\ (\mathrm{god}(t)\mathrm{god}(s))}{dt} = \mathrm{god}(t)\mathrm{god}(s), \qquad \text{and} \qquad \mathrm{god}(0)\mathrm{god}(s) = \mathrm{god}(s),$$

and so

$$\mathrm{god}(t)\mathrm{god}(s) = \widetilde{\mathrm{god}}(t) = \mathrm{god}(t+s).$$

What about adam(t) **and** eve(t)**?**

$$\begin{aligned}
\mathrm{god}(it) &= 1 + it + \frac{(it)^2}{2!} + \frac{(it)^3}{3!} + \frac{(it)^4}{4!} + \frac{(it)^5}{5!} + \cdots \\
&= 1 + \frac{i^2t^2}{2!} + \frac{i^4t^4}{4!} + \frac{i^6t^6}{6!} + \cdots \\
&\quad + it + \frac{i^3t^3}{3!} + \frac{i^5t^5}{5!} + \frac{i^7t^7}{7!} + \cdots \\
&= 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \cdots \\
&\quad + it - \frac{it^3}{3!} + \frac{it^5}{5!} - \frac{it^7}{7!} + \cdots \\
&= \left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \frac{t^8}{8!} - \cdots\right) + i\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots\right)
\end{aligned}$$

and, since adam and eve are the children of god,

$$\text{i.e.} \qquad \text{because} \qquad \mathrm{god}(it) = \mathrm{adam}(t) + i\ \mathrm{eve}(t)\ ,$$

we see that

$$\mathrm{adam}(t) = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \frac{t^8}{8!} + \cdots, \qquad \text{and}$$

$$\mathrm{eve}(t) = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \frac{t^9}{9!} + \cdots,$$

from which it follows that

$$\mathrm{adam}(0) = 0, \qquad \mathrm{eve}(0) = 1,$$

$$\mathrm{adam}(-t) = -\mathrm{adam}(t), \qquad \mathrm{eve}(-t) = \mathrm{eve}(t),$$

$$\frac{d\ \mathrm{adam}(t)}{dt} = \mathrm{eve}(t), \qquad \frac{d\ \mathrm{eve}(t)}{dt} = -\mathrm{adam}(t).$$

So, adam and eve are complete opposites and identical twins at the same time.

Complete opposites and identical twins at the same time, another manifestation

$$\begin{aligned} 1 &= \mathrm{god}(0) = \mathrm{god}(it - it) = \mathrm{god}(it + i(-t)) = \mathrm{god}(it)\mathrm{god}(i(-t)) \\ &= (\mathrm{adam}(t) + i\ \mathrm{eve}(t))(\mathrm{adam}(-t) + i\ \mathrm{eve}(-t)) \\ &= (\mathrm{adam}(t) + i\ \mathrm{eve}(t))(\mathrm{adam}(t) - i\ \mathrm{eve}(t)) \\ &= (\mathrm{adam}(t))^2 + (\mathrm{eve}(t))^2, \end{aligned}$$

$$\text{i.e.} \qquad 1 = (\mathrm{adam}(t))^2 + (\mathrm{eve}(t))^2.$$

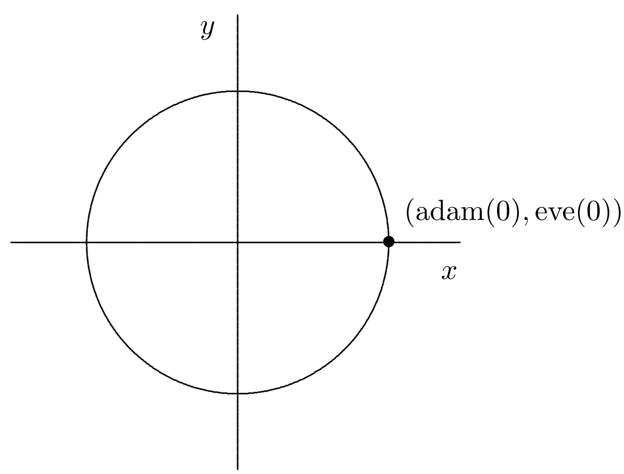
Through the ages: where are we now?

Let $x = \mathrm{eve}(t)$ and $y = \mathrm{adam}(t)$.

(A) In the Beginning the point (x, y) was at $(\mathrm{adam}(0), \mathrm{eve}(0)) = (1, 0)$, and

since $1 = \mathrm{adam}(t))^2 + (\mathrm{eve}(t))^2$, $\quad x^2 + y^2 = 1$, and

(B) adam and eve travel through the ages on a circle of radius 1.



Where are they after d millenia?

$$\begin{array}{c}\text{The distance traveled}\\ \text{after } d \text{ millenia}\end{array} = \int_{t=0}^{t=d} ds = \int_{t=0}^{t=d} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_{t=0}^{t=d} \sqrt{\left(\frac{d\ \mathrm{adam}(t)}{dt}\right)^2 + \left(\frac{d\ \mathrm{eve}(t)}{dt}\right)^2}\, dt$$

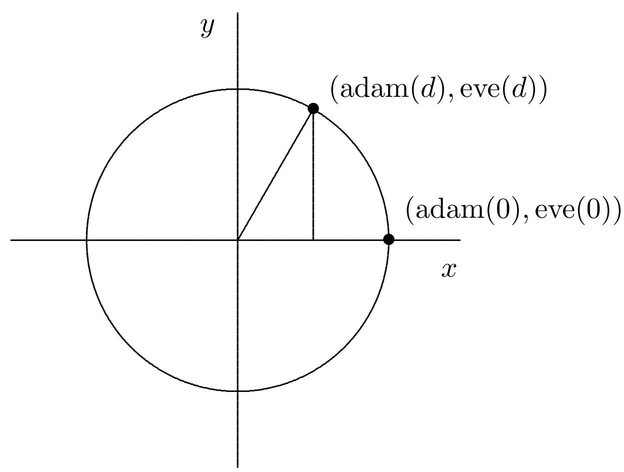
$$= \int_{t=0}^{t=d} \sqrt{(\mathrm{eve})^2 + (-\mathrm{adam}(t))^2}\, dt$$

$$= \int_{t=0}^{t=d} \sqrt{1}\, dt = \int_{t=0}^{t=d} dt = t\,\Big|_{t=0}^{t=d} = d - 0 = d,$$

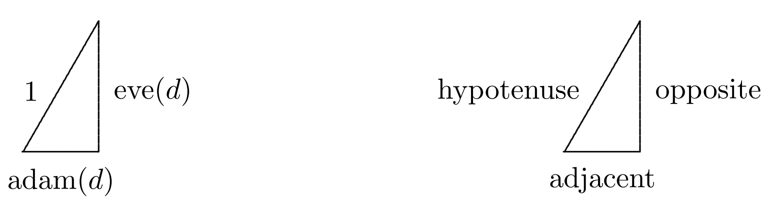
and so

$$\mathrm{adam}(t) = x\text{-coordinate of the point on a circle of radius 1 which is distance } d \text{ from the point } (1,0),$$ and

$$\mathrm{eve}(t) = y\text{-coordinate of the point on a circle of radius 1 which is distance } d \text{ from the point } (1,0).$$



The triangle in this picture is



and so

$$\mathrm{adam}(d) = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \text{and} \qquad \mathrm{eve}(d) = \frac{\text{adjacent}}{\text{hypotenuse}}$$

for a right triangle with angle d.

Some remarks on society

1. It is interesting to note that our school systems like to introduce our children to adam(t) and eve(t) but prefer to hide from my child how close they really are to god(t).

2. Mathematicians are a cloistered group and prefer to study god, adam, and eve in anonymity. In the mathematical literature

god(t)	is usually called	e^t ,	
adam(t)	is usually termed	$\cos t$,	and
eve(t)	is usually called	$\sin t$.	

References

[Ra1] A. Ram, *Calculus: Student information*, preprint 2025.

[Ra2] A. Ram, *Calculus: TA information*, preprint 2025.

[Ra3] A. Ram, *Specifying a calculus curriculum*, preprint 2025.

[Ra4] A. Ram, *Math 221: Calculus Exams, Fall 1999-2006*, preprint 2025.