Specifying a Calculus curriculum

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Abstract

This paper addresses the question of how to effectively specify a curriculum. Often this is done by providing a syllabus for the course, or a short one paragraph blurb. In this paper the thesis is that a currisulum is effectively specified by making a list of questions that could appear on the exam. Advantages of this approach are clarity, consistency, and freedom of teaching methodology.

Key words— Calculus, Mathematics education, curriculum design, course design

Contents

AMS Subject Classifications: Primary ????; Secondary ????.

1 Introduction

This paper provides an answer to the question:

What is an effective way to specify a curriculum?

The answer which this paper puts forward is: An effective way to specify a curriculum is

by making a list of the problems that could appear on the exam.

A common answer to the question is to write a syllabus. Our experience is that the problem list provides a more effective specification.

This paper intends to provide a proof of concept for our thesis by providing a list of questions that could appear on a calculus exam for a Calculus course. This list was used at University of Wisconsin–Madison in the years 1999-2008.

The list contains around 100 questions per week and these were assigned as homework (14 weeks, a total of about 1400 questions). Although this may seem like a large amount of HW, initial appearances can be deceiving. Every in class example was taken from the problem list for that week. Arun Ram held office hours each Sunday afternoon for 3 hours and did problems from the list for that week as requested by students. In a normal session we would often work through and explain how to do about 30 HW problems of their choice. The TAs were told not to hesitate to do any problem that the students requested for them. Our idea was that if we did many problems for them then the students could learn how to write mathematics well. We were actually doing about half the homework for them (for those questions, the students only had to record, in their own handwriting, what we told them in what they turned in).

Even a reading of the table of contents of this article gives a very direct and complete statement on the topics and skills that were covered in this course. The problems themselves give an exact formulation of what the goals of the course are and what skills the students have been exposed to in the course.

As is the case in most Universities, there weren't sufficient resources to do careful grading of HWs for a large Calculus class. The TAs were asked to spot check for worrisome issues and completeness and assign a grade between 0 and 3 for that week. If a student needed further specifics on any given problem they were encouraged to ask to have that problem done in the discussion section or in office hours. In sections and office hours we did problems from the list thoroughly with careful explanation on methodology and execution of quality mathematical writing (to "optimize your grade on the exam"). Since these were large classes, the students had office hours of about 10 TAs as well as the office hours of the lecturer available to them.

The students understood quickly how the HWs would be graded and that HW was 20% of the total grade. That means that each assignment was worth about 1.2%. If a student was stressed about a HW assignment in some particular week it was easy to argue that skipping the assignment altogether would only reduced their grade from 98 to 96 and they would still receive an A for the course. There was no pretence that grading of HW was the right way to receive specific feedback on mathematical writing and the students understood that that feedback would be provided in those times that we were doing homework problems for them. We were honest about the fact that the grades for the HW were primarily a carrot to engage them in the problems lists and the class.

The exam questions were taken randomly from the list. When needed, random numbers were generated quickly by using the tools at random.com.

Advantages of the problem lists:

(1) The problem lists provided a good balance of consistency and freedom. The list provided vivid clarity on what skills were to be demonstrated on the exam. The role of the teachers became simply to do their best to impart these skills to the students. There was freedom for different teaching methods. This freedom was healthy and inspiring and engaged both the teacher (to find their best way of explaining it to students) and the students (to find the teacher that explained it the way that made them capable of doing that question).

(2) The problem list provided vivid clarity for the students and the TAs and other staff and administrators. The existence and focus on the problems lists greatly reduced the administrative tasks necessary for running the course. Each problem on the exam was taken verbatim from the list. No student could claim that a question was unexpected.

(3) If faculty from other departments claimed that the mathematics department was not covering the topics they needed for their discipline, then we simply invited the those faculty to contribute problems to the list. A bit of rearrangement would balance the number of problems for each week and then there was no question that the material requested by that discipline was a part of our course. Of course, as usual, their students would claim that they hadn't learned some topic in our course, and we would very easily address this by pulling out the list and verifying that the summer vacation had affected the student's memory.

(4) There was no need to prepare sample exams. Any sample exam could be prepared simply by having random.com produce numbers. In some years we specified that would prepare the exam by choosing one random problem from each HW assignment. This meant that the exam had exactly 14 questions. The student found that knowing that the exam would have exactly 14 questions reduced stress.

Diadvantages of the problem lists:

(1) The biggest disadvantage was having to manage student complaints that there was too much HW.

This problem was primarily a psychological problem. Every in class example was taken from the lists. Arun Ram held office hours on a Sunday afternoon for 3 hours, and did problems from the list for that week as requested by students. The TAs were told not to hesitate to do any problem that the students requested for them. Our idea was that if we did many problems for them then the students could learn how to write mathematics well. We were doing about half the homework for them and this had an incredible psychological effect: the students were interested in class, they were interested in tutorials and they were interested in office hours. They had to do the other half, which was much easier as they had already been engaged in similar problems and they could do the remaining problems in a similar manner and in a similar amount of time to the time it took us to do the first half of the problems for them. Of course, they often claimed that it took much longer, but on a more careful analysis they usually admitted that in their time tracking they had included some time spent comparing the attributes of Leonardo DiCaprio and Matt Damon. So there are many psychologies at play in the measurement of the quantity of homework.

The net effect was that the students were engaged, and learned.

Of course, some students declared that they would hate me for life for giving so much homework. But not all students – some students were cautiously appreciative at the time. Below is a report from a student which was received 15 years later (thank you to Rachel Weber for giving permission to reproduce this message here).

On 31 Aug 2020, at 06:03, Rachel Weber wrote:

Hello Professor Ram,

Maybe you get these types of thank you emails all the time, but I did want to reach out and say thank you for being an excellent and challenging professor for me... all the way back in 2004 at the University of Wisconsin.

Being home during the pandemic, I've been thinking about life a lot more, and one of the topics has been - what things had the greatest influence on me in my career as an Industrial Engineer in healthcare systems performance improvement? Funnily enough, one of the answers is Calc I freshman year.

I remember hearing the first week how much work your class was, and wanting to switch out of it to be in a different class with some of my newfound friends. Luckily, that did not happen. I remember first getting introduced to the idea of hours long office hours, on Sundays, and really having to put in the time to learn and get better. After challenging myself to meet the expectations you set, honestly all my classes after that seemed so much easier.

But it wasn't just whipping me into shape for college... even more so, I think you unlocked my ability to visualize. I remember your blackboard drawings and finding the area under the curve by imagining all the rectangular slices, getting the area of those pieces and adding them up. In my work life now, I am always drawing out what is happening in our healthcare processes for patients. It took me a while to realize that most people don't see what happens in a process in their heads, and drawing is one of my most important communication tools. Where did I get that from? Some of that came from you and your class.

And finally... "Ask me a question." You started every class that way, and I like to imagine you still do. Somewhere in my memory floating around are a few of your personal stories, like your father saying - I have one son that is rich, and another that is successful.

I see that you are at the University of Melbourne now, and I hope this email finds you well. Thank you - my time in your class had a very positive impact on me.

Sincerely,

Rachel Weber

Acknowledgments. Most of all I thank the many students who have put up with me (or not) over the years. I've have learned immense amounts from you and have enjoyed your wonderful personalites and interaction. In so many ways this has been one of the most enriching aspects of my life path. Second I would like to thank all the amazing TAs that have lived and breathed these teaching experiences with me. Your energy and companionship was always refreshing and stimulating. I would particularly like to thank Zajj Daugherty for so many conversations and discussions about teaching and, in particular, about this problem list method for structuring teaching. Finally I am grateful to my teachers, who taught me Calculus, mathematics, and also the art of teaching itself.

Many problems on the list below were originally taken from resources from which I learned Calculus, in particular Thomas and Finney 5th Edition, and P.N. Arora, Senior Secondary Mathematics. I gratefully acknowledge these resources and the wondrous world that they opened up for me.

2 HW assignments Fall 2006

2.1 HW1 Fall 2006: Due September 11, 2006 (Numbers, functions, identities)

Problem A. Numbers

(1) What are the positive integers and why do we care?

(2) What are the nonnegative integers and why do we care?

(3) What are the rational numbers and why do we care?

(4) What are the real numbers and why do we care?

(5) What are the complex numbers and why do we care?

(6) What do $2+3$, $2+\frac{5}{7}$, $\frac{4}{9}+\frac{5}{7}$, $2+1.4$ and $2+\sqrt{2}$ mean?

(7) What do x^2, $\frac{1}{x}$ and $\sqrt{x}$ mean?

(8) What do $a+b$, $a+\frac{b}{c}$, $\frac{a}{b}+\frac{c}{d}$ mean?

(9) What do 2^3, $2^{\frac{5}{7}}$, $\left(\frac{2}{3}\right)^{\frac{5}{7}}$, 2^x, $2^{1.4}$ and $2^{\sqrt{2}}$ mean?

(10) What do a^b, $a^{\frac{b}{c}}$, $\left(\frac{a}{b}\right)^{\frac{c}{d}}$, 2^x, and x^2 mean?

(11) What do x^x and $x^{\sqrt{x}}$ mean?

Problem B. Computing with complex numbers

(1) Find a complex number z such that $z+w=w$ for all other complex numbers w.

(2) Find a complex number x such that $xw=w$ for all other complex numbers w.

(3) Compute $(3-7i)+(2+5i)$ and graph the result.

(4) Compute $(-12+3i)-(7-5i)$ and graph the result.

(5) Compute $(4+8i)(2-3i)$ and graph the result.

(6) Compute $\dfrac{-15+i}{4+2i}$ and graph the result.

(7) Compute $(3-2i)^3$ and graph the result.

(8) Compute $\sqrt{2i}$ and graph the result.

(9) Compute $\dfrac{1}{a+bi}$ and graph the result, where $a,b\in\mathbb{R}$.

(10) Compute $(3-5i)+(7+2i)$ and graph the result.

(11) Compute $(5-2i)-(3-6i)$ and graph the result.

(12) Compute $(2-4i)(3+2i)$ and graph the result.

(13) Compute $\dfrac{6-i}{4+2i}$ and graph the result.

(14) Compute $1^{1/4}$ and graph the result.

(15) Compute $16^{1/4}$ and graph the result.

(16) Compute $(27^{1/3})^4$ and $27^{(4+1/3)}$ graph the result.

(17) Compute $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\cdots$.

(18) Compute $1\cdot 2$, $1\cdot 2\cdot 3$, $1\cdot 2\cdot 3\cdot 4$, $1\cdot 2\cdot 3\cdot 4\cdot 5$ and $1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6$.

(19) Compute $1+\frac{1}{1}+\frac{1}{1\cdot 2}+\frac{1}{1\cdot 2\cdot 3}+\frac{1}{1\cdot 2\cdot 3\cdot 4}+\frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5}+\cdots$.

Problem C. Functions

(1) What is x^2?

(2) What is e^x?

(3) What is $\sin x$?

(4) What is $\cos x$?

(5) What is $\tan x$?

(6) What is $\cot x$?

(7) What is $\sec x$?

(8) What is $\csc x$?

(9) What is $\sinh x$?

(10) What is $\cosh x$?

(11) What is $\tanh x$?

(12) What is $\coth x$?

(13) What is $\mathrm{sech} x$?

(14) What is $\mathrm{csch} x$?

(15) What is $\sqrt{x}$?

(16) What is $\ln x$?

(17) What is $\sin^{-1} x$?

(18) What is $\cos^{-1} x$?

(19) What is $\tan^{-1} x$?

(20) What is $\cot^{-1} x$?

(21) What is $\sec^{-1} x$?

(22) What is $\csc^{-1} x$?

(23) What is $\sinh^{-1} x$?

(24) What is $\cosh^{-1} x$?

(25) What is $\tanh^{-1} x$?

(26) What is $\coth^{-1} x$?

(27) What is $\mathrm{sech}^{-1}x$?

(28) What is $\mathrm{csch}^{-1}x$?

Problem D. Function identities

(1) Explain why $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$.

(2) Explain why $\dfrac{x^n - 1}{x-1} = 1 + x + x^2 + x^3 + \cdots + x^{n-1}$.

(3) Find all possibilities for $c_0, c_1, c_2 \ldots$ so that $f(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots$ satisfies $f(x+y) = f(x)f(y)$.

(4) Explain why $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \dfrac{x^5}{5!} + \dfrac{x^6}{6!} + \cdots$.

(5) Explain why $\ln x$ is the inverse function to e^x.

(6) Verify the identity $e^{x+y} = e^x e^y$.

(7) Verify the identity $e^{-x} = \dfrac{1}{e^x}$.

(8) Verify the identity $(e^x)^n = e^{nx}$.

(9) Verify the identity $e^0 = 1$.

(10) Verify the identity $\ln(xy) = \ln x + \ln y$.

(11) Verify the identity $-\ln x = \ln(1/x)$.

(12) Verify the identity $\ln x^n = n \ln x$.

(13) Verify the identity $\ln 1 = 0$.

(14) Explain why $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$.

(15) Explain why $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$.

(16) Verify the identity $e^{ix} = \cos x + i \sin x$.

(17) Verify the identity $\cos^2 x + \sin^2 x = 1$.

(18) Verify the identity $\sin(-x) = -\sin x$.

(19) Verify the identity $\cos(-x) = \cos x$.

(20) Verify the identity $\sin(x+y) = \sin x \cos y + \cos x \sin y$.

(21) Verify the identity $\cos(x+y) = \cos x \cos y - \sin x \sin y$.

(22) Verify the identity $\cos x = \frac{e^{ix} + e^{-ix}}{2}$.

(23) Verify the identity $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$.

(24) Explain why $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$.

(25) Explain why $\sin x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$.

(26) Verify the identity $e^{ix} = \cosh x + \sinh x$.

(27) Verify the identity $\cosh^2 x - \sinh^2 x = 1$.

(28) Verify the identity $\sinh(-x) = -\sinh x$.

(29) Verify the identity $\cosh(-x) = \cosh x$.

(30) Verify the identity $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$.

(31) Verify the identity $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$.

(32) Verify the identity $\cosh x = \frac{e^x + e^{-x}}{2}$.

(33) Verify the identity $\sinh x = \dfrac{e^x - e^{-x}}{2}$.

Problem E. Trigonometric function identities

(1) Verify the identity $\tan(x+y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$.

(2) Verify the identity $\sin(x/2) = \pm\sqrt{\dfrac{1-\cos x}{2}}$.

(3) Verify the identity $\cos 3x = \cos^3 x - 3\cos x \sin^2 x$.

(4) Verify the identity $\sin 3x = 3\cos^2 x \sin x - \sin^3 x$.

(5) Verify the identity $\sin^2 A \cot^2 A = (1 - \sin A)(1 + \sin A)$.

(6) Verify the identity $\tan B = \dfrac{\cos B}{\sin B \cot^2 B}$.

(7) Verify the identity $\dfrac{\tan V \cos V}{\sin V} = 1$.

(8) Verify the identity $\sin E \cot E + \cos E \tan E = \sin E + \cos E$.

(9) Verify the identity $\dfrac{1}{\sec^2 x} + \dfrac{1}{\csc^2 x} - 1 = 0$.

2.2 HW2 Fall 2006: Due September 18, 2006 (Angles, trig identities, derivatives)

Problem A. Angles

(1) What is π and where did it come from?

(2) Explain how to measure angles in radians, in degrees, and how convert from degrees to radians.

(3) What is the connection between measuring angles in radians and measuring distances?

(4) What is the circumference of a circle of radius r? How do you know?

(5) What is the length of an arc of angle θ on the boundary of a circle of radius r? How do you know?

(6) What is the area of a circle of radius r? How do you know?

(7) What is the area of a sector of angle θ in a circle of radius r? How do you know?

(8) Using angles, what is $\sin x$?

(9) Using angles, what is $\cos x$?

(10) Using angles, show that $\sin(-x) = -\sin x$.

(11) Using angles, show that $\cos(-x) = \cos x$.

(12) Using angles, show that $\sin^2 x + \cos^2 x = 1$.

(13) Using angles, show that $\sin(x + y) = \sin x \cos y + \cos x \sin y$.

(14) Using angles, show that $\cos(x + y) = \cos x \cos y - \sin x \sin y$.

Problem B. Computing trigonometric functions

(1) Explain how to derive $\sin\frac{\pi}{6}$, $\cos\frac{\pi}{6}$, $\tan\frac{\pi}{6}$, $\cot\frac{\pi}{6}$, $\sec\frac{\pi}{6}$ and $\csc\frac{\pi}{6}$ in radical form.

(2) Explain how to derive $\sin\frac{\pi}{3}$, $\cos\frac{\pi}{3}$, $\tan\frac{\pi}{3}$, $\cot\frac{\pi}{3}$, $\sec\frac{\pi}{3}$ and $\csc\frac{\pi}{3}$ in radical form.

(3) Explain how to derive $\sin\frac{\pi}{4}$, $\cos\frac{\pi}{4}$, $\tan\frac{\pi}{4}$, $\cot\frac{\pi}{4}$, $\sec\frac{\pi}{4}$ and $\csc\frac{\pi}{4}$ in radical form.

(4) Explain how to derive $\sin\frac{\pi}{2}$, $\cos\frac{\pi}{2}$, $\tan\frac{\pi}{2}$, $\cot\frac{\pi}{2}$, $\sec\frac{\pi}{2}$ and $\csc\frac{\pi}{2}$ in radical form.

(5) Explain how to derive $\sin 0$, $\cos 0$, $\tan 0$, $\cot 0$, $\sec 0$ and $\csc 0$ in radical form.

(6) Explain how to derive $\sin\frac{3\pi}{4}$, $\cos\frac{3\pi}{4}$, $\tan\frac{3\pi}{4}$, $\cot\frac{3\pi}{4}$, $\sec\frac{3\pi}{4}$ and $\csc\frac{3\pi}{4}$ in radical form.

(7) Explain how to derive $\sin\frac{-2\pi}{3}$, $\cos\frac{-2\pi}{3}$, $\tan\frac{-2\pi}{3}$, $\cot\frac{-2\pi}{3}$, $\sec\frac{-2\pi}{3}$ and $\csc\frac{-2\pi}{3}$ in radical form.

(8) Compute $\sin\frac{\pi}{6} + \cos\frac{\pi}{6}$ in radical form.

(9) Compute $\left(\sin\frac{\pi}{6}\right)\left(\cos\frac{\pi}{6}\right)$ in radical form.

(10) Compute $\left(\tan\frac{\pi}{3}\right)\left(\cot\frac{\pi}{3}\right)$ in radical form.

Problem C. Trigonometric function identities

(1) Verify the identity $\dfrac{\sec A - 1}{\sec A + 1} + \dfrac{\cos A - 1}{\cos A + 1} = 0$.

(2) Verify the identity $\sin V(1 + \cot^2 V) = \csc V$.

(3) Verify the identity $\dfrac{\sin(\pi/2 - w)}{\cos(\pi/2 - w)} = \cot w$.

(4) Verify the identity $\sec(\pi/2 - z) = \dfrac{1}{\sin z}$.

(5) Verify the identity $1 + \tan^2(\pi/2 - x) = \dfrac{1}{\cos^2(\pi/2 - x)}$.

(6) Verify the identity $\dfrac{\sin A}{\csc A} + \dfrac{\cos A}{\sec A} = 1$.

(7) Verify the identity $\dfrac{\sec B}{\cos B} - \dfrac{\tan B}{\cot B} = 1$.

(8) Verify the identity $\dfrac{1}{\csc^2 w} + \sec^2 w + \dfrac{1}{\sec^2 w} = 2 + \dfrac{\sec^2 w}{\csc^2 w}$.

(9) Verify the identity $\sec^4 V - \sec^2 V = \dfrac{1}{\cot^4 V} + \dfrac{1}{\cot^2 V}$.

(10) Verify the identity $\sin^4 x + \cos^2 x = \cos^4 x + \sin^2 x$.

(11) Verify the identity $\tan 3\alpha = \dfrac{3\tan\alpha - \tan^3\alpha}{1 - 3\tan^2\alpha}$.

(12) Verify the identity $\cot(\alpha/2) = \dfrac{\sin\alpha}{1 - \cos\alpha}$.

(13) Verify the identity $\cos(\pi/6 - x) + \cos(\pi/6 + x) = \sqrt{3}\cos x$.

(14) Verify the identity $\sin(\alpha + \beta)\sin(\alpha - \beta) = \sin^2\alpha - \sin^2\beta$.

(15) Verify the identity $\sin(\pi/3 - x) + \sin(\pi/3 + x) = \sqrt{3}\cos x$.

(16) Verify the identity $\cos(\pi/4 - x) - \cos(\pi/4 + x) = \sqrt{2}\sin x$.

(17) Verify the identity $2\sin\alpha\cos\beta = \sin(\alpha + \beta) + \sin(\alpha - \beta)$.

(18) Verify the identity $2\sin\alpha\sin\beta = \cos(\alpha - \beta) - \cos(\alpha + \beta)$.

Problem D. Fun trigonometric function identities

(1) Verify the identity $\cos 2\theta = 2\sin(\pi/4+\theta)\sin(\pi/4-\theta)$.

(2) Verify the identity $(1/2)\sin 2A = \dfrac{\tan A}{1+\tan^2 A}$.

(3) Verify the identity $\cot(x/2) = \dfrac{1+\cos x}{\sin x}$.

(4) Verify the identity $\sin 2B(\cot B + \tan B) = 2$.

(5) Verify the identity $\dfrac{1-\tan^2\theta}{1+\tan^2\theta} = \cos 2\theta$.

(6) Verify the identity $1+\cos 2A = \dfrac{2}{1+\tan^2 A}$.

(7) Verify the identity $\tan 2x \tan x + 2 = \dfrac{\tan 2x}{\tan x}$.

(8) Verify the identity $\csc A \sec A = 2\csc 2A$.

(9) Verify the identity $\cot x = \dfrac{\sin 2x}{1-\cos 2x}$.

(10) Verify the identity $1-\sin A = \left(\sin\dfrac{A}{2} - \cos\dfrac{A}{2}\right)^2$.

(11) Verify the identity $\cos^4 A = \dfrac{2\cos 2A + \cos^2 2A + 1}{4}$.

(12) Verify the identity $\dfrac{\sin A + \sin B}{\sin A - \sin B} = \dfrac{\tan\left(\frac{A+B}{2}\right)}{\tan\left(\frac{A-B}{2}\right)}$.

(13) Verify the identity $\dfrac{\sin\alpha + \sin 3\alpha}{\cos\alpha + \cos 3\alpha} = \tan 2\alpha$.

(14) Verify the identity $\dfrac{\cos 2A}{1+\sin 2A} = \dfrac{\cot A - 1}{\cot A + 1}$.

(15) Verify the identity $\dfrac{\cos A + \sin A}{\cos A - \sin A} = \dfrac{1+\sin 2A}{\cos 2A}$.

(16) Verify the identity $\cot\alpha - \cot\beta = \dfrac{\sin(\beta-\alpha)}{\sin\alpha\sin\beta}$.

(17) Verify the identity $\tan\theta\csc\theta\cos\theta = 1$.

(18) Verify the identity $\cos^2\theta = \dfrac{\cot^2\theta}{1+\cot^2\theta}$.

(19) Verify the identity $\dfrac{1-\sin A}{1+\sin A} = (\sec A - \tan A)^2$.

(20) Verify the identity $(\tan A - \cot A)^2 + 4 = \sec^2 A + \csc^2 A$.

(21) Verify the identity $\cos B\cos(A+B)+\sin B\sin(A+B)=\cos A$.

(22) Verify the identity $\dfrac{\tan A-\sin A}{\sec A}=\dfrac{\sin^3 A}{1+\cos A}$.

(23) Verify the identity $\dfrac{2\tan^2 A}{1+\tan^2 A}=1-\cos 2A$.

(24) Verify the identity $\tan 2A=\tan A+\dfrac{\tan A}{\cos 2A}$.

(25) Verify the identity $\sin 2A=\dfrac{2\tan A}{1+\tan^2 A}$.

(26) Verify the identity $\dfrac{4\sin A}{1-\sin^2 A}=\dfrac{1+\sin A}{1-\sin A}-\dfrac{1-\sin A}{1+\sin A}$.

(27) Verify the identity $\tan A+\sin A=\dfrac{\csc A+\cot A}{\csc A\cot A}$.

Problem E. Inverse trig function identities

(1) Verify the identity $\cos(\tan^{-1}x)=\dfrac{1}{\sqrt{1+x^2}}$.

(2) Verify the identity $\sin(\tan^{-1}x)=\dfrac{x}{\sqrt{1+x^2}}$.

(3) Verify the identity $\sin(\cos^{-1}x)=\sqrt{1-x^2}$.

(4) Verify the identity $\tan(\cos^{-1}x)=\dfrac{\sqrt{1-x^2}}{x}$.

(5) Verify the identity $\cos(\sin^{-1}x)=\sqrt{1-x^2}$.

(6) Verify the identity $\tan(\cot^{-1}x)=1/x$.

(7) Verify the identity $\cot(\cot^{-1}2)=2$.

(8) Verify the identity $\sin(\cot^{-1}x)=\dfrac{1}{\sqrt{1+x^2}}$.

(9) Verify the identity $\cos(\cot^{-1}x)=\dfrac{x}{\sqrt{1+x^2}}$.

(10) Verify the identity $\sin^{-1}(-x)=-\sin^{-1}x$.

(11) Verify the identity $\tan^{-1}(-x)=-\tan^{-1}x$.

(12) Verify the identity $\tan^{-1}x=\cot^{-1}(1/x)$.

(13) Verify the identity $\tan^{-1}x=\sin^{-1}\left(\dfrac{x}{\sqrt{1+x^2}}\right)$.

(14) Verify the identity $\sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)$.

Problem F. Basic derivatives

(1) What is $\frac{d}{dx}$?

(2) Explain why $\frac{d1}{dx} = 0$.

(3) Explain why $\frac{da}{dx} = 0$ if a is a number.

(4) Explain why $\frac{dx}{dx} = 1$.

(5) Explain why $\frac{dx^2}{dx} = 2x$.

(6) Explain why $\frac{dx^3}{dx} = 3x^2$.

(7) Explain why $\frac{dx^{-1}}{dx} = -x^{-2}$.

(8) Explain why $\frac{dx^{-2}}{dx} = -2x^{-3}$.

(9) Explain why $\frac{dx^{-3}}{dx} = -3x^{-4}$.

(10) Explain why $\frac{d(3x^2+2x)^{-1}}{dx} = \frac{-(6x+2)}{(3x^2+2x)^2}$.

(11) Explain why $\frac{dx^{1/2}}{dx} = \frac{1}{2}x^{-1/2}$.

(12) Explain why $\frac{dx^{1/3}}{dx} = \frac{1}{3}x^{-2/3}$.

(13) Explain why $\frac{dx^{3/5}}{dx} = \frac{3}{5}x^{-2/5}$.

(14) Explain why $\frac{dx^n}{dx} = nx^{n-1}$, for all positive integers n.

(15) Explain why $\frac{dx^n}{dx} = nx^{n-1}$, for $n = 0$.

(16) Explain why $\frac{dx^n}{dx} = nx^{n-1}$, for all negative integers n.

(17) Explain why $\frac{dx^{m/n}}{dx} = (m/n)x^{(m/n)-1}$, for all integers m and n, with $n \neq 0$.

2.3 HW3 Fall 2006: September 25, 2006 (Derivatives, chain rule)

Problem A. The chain rule

(1) Let g be a function. Show that $\frac{dg^0}{dx} = 0\frac{dg}{dx}$.

(2) Let g be a function. Show that $\frac{dg^1}{dx} = 1g^0\frac{dg}{dx}$.

(3) Let g be a function. Show that $\frac{dg^2}{dx} = 2g^1\frac{dg}{dx}$.

(4) Let g be a function. Show that $\frac{dg^3}{dx} = 3g^2\frac{dg}{dx}$.

(5) Let g be a function. Show that $\frac{dg^4}{dx} = 4g^3\frac{dg}{dx}$.

(6) Let g be a function. Show that $\frac{dg^5}{dx} = 5g^4\frac{dg}{dx}$.

(7) Let g be a function. Show that $\frac{dg^n}{dx} = ng^{n-1}\frac{dg}{dx}$ for any positive integer n.

(8) Let $f(y) = 4y^3 + 7y^2 + 2y - 13$ and let g be a function.
Show that $\frac{d(f(g))}{dx} = (12g^2 + 14g + 2)\frac{dg}{dx}$.

(9) Let f be a polynomial and let g be a function. Show that $\frac{d(f(g))}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx}$.

Problem B. Derivatives of the basic functions.

(1) Explain why $\frac{de^x}{dx} = e^x$.

(2) Explain why $\frac{d\sin x}{dx} = \cos x$.

(3) Explain why $\frac{d\cos x}{dx} = -\sin x$.

(4) Explain why $\frac{d\tan x}{dx} = \sec^2 x$.

(5) Explain why $\frac{d\cot x}{dx} = -\csc^2 x$.

(6) Explain why $\frac{d\sec x}{dx} = \tan x \sec x$.

(7) Explain why $\frac{d\csc x}{dx} = -\cot x \csc x$.

(8) Explain why $\frac{d\ln x}{dx} = \frac{1}{x}$.

(9) Explain why $\frac{d\sin^{-1} x}{dx} = \frac{1}{\sqrt{1-x^2}}$.

(10) Explain why $\frac{d\cos^{-1} x}{dx} = -\frac{1}{\sqrt{1-x^2}}$.

(11) Explain why $\frac{d\tan^{-1} x}{dx} = \frac{1}{1+x^2}$.

(12) Explain why $\frac{d\cot^{-1} x}{dx} = -\frac{1}{1+x^2}$.

(13) Explain why $\frac{d\csc^{-1} x}{dx} = -\frac{1}{|x|\sqrt{x^2-1}}$.

Problem C. Computing some derivatives

(1) Find $\frac{dy}{dx}$ when $y = (2x+3)(5x+6)$.

(2) Find $\frac{dy}{dx}$ when $y = \left(x + \frac{1}{x}\right)\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)$.

(3) Find $\frac{dy}{dx}$ when $y = (2x-5)^2(3x-4)^3$.

(4) Find $\frac{dy}{dx}$ when $y = \left(ex^2 + \frac{\pi}{x^3} + x^{7/2}\right)$.

(5) Find $\frac{dy}{dx}$ when $y = \left(\frac{x-3}{x-4}\right)^2$.

(6) Find $\frac{dy}{dx}$ when $y = \frac{3x+5}{4-x^2}$.

(7) Find $\frac{dy}{dx}$ when $y = \frac{x}{\sqrt{1-2x}}$.

(8) Find $\frac{dy}{dx}$ when $y = \frac{1+\sqrt{x}}{1-\sqrt{x}}$.

(9) Find $\frac{dy}{dx}$ when $y = \frac{2(x+1)}{x^2+2x-3}$.

(10) Find $\frac{dy}{dx}$ when $y = \frac{\sqrt{a+x}-\sqrt{a-x}}{\sqrt{a+x}+\sqrt{a-x}}$.

(11) Find $\frac{dy}{dx}$ when $y = \frac{x^2-2}{x+1}$.

(12) Find $\frac{dy}{dx}$ when $y = \frac{\sqrt{x}}{\sqrt{x-3}}$.

(13) Find $\frac{dy}{dx}$ when $y = \frac{x^n+1}{x^n-1}$.

(14) Find $\frac{dy}{dx}$ when $y = \frac{\sqrt{1+x^2}}{\sqrt{1-x^2}}$.

(15) Find $\frac{dy}{dx}$ when $y = \frac{2x^2-1}{x\sqrt{1+x^2}}$

(16) Find $\frac{dy}{dx}$ when $y = u^n$.

(17) Find $\frac{dy}{dx}$ when $y = \sqrt{1-x^2}$.

Problem D. Correcting derivative identities

(1) Correct the identity $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} + u\frac{dv}{dx}}{v^2}$.

(2) Correct the identity $\frac{d}{dx}(u+v) = \frac{du}{dx} - \frac{dv}{dx}$.

(3) Correct the identity $\frac{d}{dx}(uv) = \frac{du}{dx} \cdot \frac{dv}{dx}$.

Problem E. Verifying derivative identities

(1) If $y = x^{7/2}$ show that $2x\frac{dy}{dx} - 7y = 0$.

(2) If $y = 3 - x^2$ prove that $\left(\frac{dy}{dx}\right)^2 - 4x^2 = 0$.

(3) If $y = \sqrt{x} + \frac{1}{\sqrt{x}}$ show that $2x\frac{dy}{dx} + y - 2\sqrt{x} = 0$.

(4) If $y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$ show that $\frac{dy}{dx} - y + \frac{x^n}{n!} = 0$.

(5) If $y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ show that $\frac{dy}{dx} = y$.

(6) If $z = \frac{3}{1+t}$ show that $3t\frac{dz}{dt} = z(z-3)$.

(7) If $y = \frac{1}{a-z}$ show that $\frac{dz}{dy} = (z-a)^2$.

(8) If $y = \dfrac{x}{x-p}$ prove that $x\dfrac{dy}{dx} = y(1-y)$.

(9) If $y = x - \sqrt{1+x^2}$ show that $(1+x^2)\left(\dfrac{dy}{dx}\right)^2 = y^2$.

(10) If $y = x^2$ show that $\left(\dfrac{dy}{dx}\right)^2 = 4y$.

(11) If $y = \sqrt{1+x^5}$ show that $\dfrac{dy}{dx} = \dfrac{5x^4}{2y}$.

Problem F. Derivatives at a point

(1) Find $\dfrac{dy}{dx}$ at $x = 3$ when $y = x^6 + 3x^2 + 5$.

(2) Find $\left.\dfrac{dy}{dx}\right|_{x=3}$ when $y = (x+1)(x+2)$.

Problem G. Derivatives with respect to functions

(1) Differentiate $t^2 - \dfrac{4}{t^2}$ with respect to t^5.

(2) Differentiate $\dfrac{x^2}{1+x^2}$ with respect to x^2.

(3) Differentiate $\dfrac{ax+b}{cx+d}$ with respect to $\dfrac{a_1x+b_1}{c_1x+d_1}$.

(4) Differentiate x^3 with respect to x^2.

(5) Differentiate $\dfrac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}$ with respect to $\sqrt{1-x^4}$.

(6) Differentiate $\dfrac{x}{1+x^2}$ with respect to x^3.

(7) Differentiate $x - \sqrt{1-x^2}$ with respect to $\sqrt{1-x^2}$.

(8) Differentiate $7x^5 - 11x^2$ with respect to $7x^2 - 15x$.

Problem H. Derivatives of parametric equations

(1) Find $\dfrac{dy}{dx}$ when $x = pt$ and $y = p/t$.

(2) Find $\dfrac{dy}{dx}$ when $x = at^2$ and $y = 2at$.

(3) Find $\dfrac{dy}{dx}$ when $y = \dfrac{2at^2}{1+t^2}$ and $x = \dfrac{2a}{1+t^2}$.

(4) Find $\frac{dy}{dx}$ when $x = a\frac{1-t^2}{1+t^2}$ and $y = b\frac{2t}{1+t^2}$.

(5) Find $\frac{dy}{dx}$ when $x = a\sqrt{\frac{t^2-1}{t^2+1}}$ and $y = at\sqrt{\frac{t^2-1}{t^2+1}}$.

(6) Find $\frac{dy}{dx}$ when $x = a\frac{1+t^2}{1-t^2}$ and $y = \frac{2bt}{1-t^2}$.

(7) Find $\frac{dy}{dx}$ when $x = \frac{3at}{1+t^3}$ and $y = \frac{3at^2}{1+t^3}$.

(8) Find $\frac{dy}{dx}$ when $x = \frac{1-t^2}{1+t^2}$ and $y = \frac{2t}{1+t^2}$.

Problem I. Implicit differentiation

(1) Find $\frac{dy}{dx}$ when $x^4 + y^4 = 4a^2x^2y^2$.

(2) Find $\frac{dy}{dx}$ when $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

(3) Find $\frac{dy}{dx}$ when $x^5 + y^5 - 5ax^2y^2 = 0$.

(4) If $ax^2 + by^2 + 2gx + 2fy + 2hxy + c = 0$ show that $\frac{dy}{dx} + \frac{ax+hy+g}{hx+by+f} = 0$.

(5) If $xy + px + q = 0$ prove that $x^2\frac{dy}{dx}$ is always constant.

(6) Find $\frac{dy}{dx}$ when $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$.

Problem J. Derivatives with trigonometric functions.

(1) Find $\frac{dy}{dx}$ when $y = \sin(3x+2)$.

(2) Find $\frac{dy}{dx}$ when $y = \sqrt{\sin x^4}$.

(3) Find $\frac{dy}{dx}$ when $y = x^2 \sin x$.

(4) Find $\frac{dy}{dx}$ when $y = \tan x \sin 2x$.

(5) Find $\frac{dy}{dx}$ when $y = \sin x^2 - \frac{\tan x}{1+x^2}$.

(6) Find $\frac{dy}{dx}$ when $y = \frac{2\cos x - x}{x+2}$.

(7) Find $\dfrac{dy}{dx}$ when $y = (1+x^2) + \dfrac{x}{\sin x}$.

(8) Find $\dfrac{dy}{dx}$ when $y = \dfrac{\sin 2x}{\cos x}$.

(9) Find $\dfrac{dy}{dx}$ when $y = \sin(x/3)\csc(2x/3)$.

(10) Find $\dfrac{dy}{dx}$ when $y = \sin(\sin x + \cos x)$.

(11) Find $\dfrac{dy}{dx}$ when $y = \sqrt{\sec^2 x + \csc^2 x}$.

(12) Find $\dfrac{dy}{dx}$ when $y = (x^2-1)\left(\cot x + \dfrac{\tan x}{1+x^2}\right)$.

(13) Find $\dfrac{dy}{dx}$ when $y = \sqrt{\dfrac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta}}$.

(14) Find $\dfrac{dy}{dx}$ when $y = \dfrac{\sec x + \tan x}{\sec x - \tan x}$.

(15) Find $\dfrac{dy}{dx}$ when $y = \sqrt{\dfrac{1-\cos x}{1+\cos x}}$.

(16) Find $\dfrac{dy}{dx}$ when $y = x^3 \tan^2(x/2)$.

(17) If $y = \tan(\cos(\sin\theta))$ find dy/dx.

Problem K. Derivatives with exponentials and logs.

(1) Find $\dfrac{dy}{dx}$ when $y = \left(ex^2 + \dfrac{\pi}{x^3} + x^{7/2}\right)$.

(2) Find $\dfrac{dy}{dx}$ when $y = a^{ax+b}$.

(3)$\dfrac{dy}{dx}$ when $y = a^{x^3}$.

(4) Find $\dfrac{dy}{dx}$ when $y = 6^{2x}$.

(5) Find $\dfrac{dy}{dx}$ when $y = \ln(ax^2+b)$.

(6) Find $\dfrac{dy}{dx}$ when $y = e^{3\ln x}$.

(7) Find $\dfrac{dy}{dx}$ when $y = e^{2x} - e^{-2x}$.

(8) Find $\frac{dy}{dx}$ when $y = e^{x^2+2x}$.

(9) Find $\frac{dy}{dx}$ when $y = a^x x^a$.

(10) Find $\frac{dy}{dx}$ when $y = xe^x$.

2.4 HW4 Fall 2006: October 2, 2006 (expansions, equations, derivatives)

Problem A. Expansions

For questions 1-9 suppose that

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \cdots.$$

(1) Show that $c_0 = f(a)$.

(2) Show that $c_1 = \frac{df}{dx}\Big|_{x=a}$.

(3) Show that $c_2 = \frac{1}{2}\left(\frac{d^2f}{dx^2}\Big|_{x=a}\right)$.

(4) Show that $c_3 = \frac{1}{3!}\left(\frac{d^3f}{dx^3}\Big|_{x=a}\right)$.

(5) Show that $c_4 = \frac{1}{4!}\left(\frac{d^4f}{dx^4}\Big|_{x=a}\right)$.

(6) Show that $c_5 = \frac{1}{5!}\left(\frac{d^5f}{dx^5}\Big|_{x=a}\right)$.

(7) Explain why $c_n = \frac{1}{n!}\left(\frac{d^nf}{dx^n}\Big|_{x=a}\right)$.

(8) Show that

$$f(a+\Delta x) = f(a) + \left(\frac{df}{dx}\Big|_{x=a}\right)\Delta x + \frac{1}{2}\left(\frac{d^2f}{dx^2}\Big|_{x=a}\right)(\Delta x)^2$$

$$+ \frac{1}{3!}\left(\frac{d^3f}{dx^3}\Big|_{x=a}\right)(\Delta x)^3 + \frac{1}{4!}\left(\frac{d^4f}{dx^4}\Big|_{x=a}\right)(\Delta x)^4 + \cdots.$$

(9) Show that $\lim_{\Delta x\to 0} \frac{f(a+\Delta x)-f(a)}{\Delta x} = \frac{df}{dx}\Big|_{x=a}$.

(10) Give a series expansion for e^x.

(11) Give a series expansion for $\sin x$.

(12) Give a series expansion for $\cos x$.

(13) Give a series expansion for $\frac{1}{1-x}$.

(14) Give a series expansion for $\frac{1}{1+x}$.

(15) Give a series expansion for $\dfrac{1}{1+x^2}$.

(16) Explain why $1+\dfrac{1}{3}+\dfrac{1}{3^2}+\dfrac{1}{3^3}+\dfrac{1}{3^4}+\cdots=\dfrac{3}{2}$.

(17) Explain why $1+\dfrac{1}{3}+\dfrac{1}{3^2}+\dfrac{1}{3^3}+\dfrac{1}{3^4}+\cdots+\dfrac{1}{3^{50}}=\dfrac{3}{2}-\dfrac{1}{2\cdot 3^{50}}$.

Problem B. Derivatives at a point.

(1) Let $y=\tan 2x-2\tan x+2$. Find $\dfrac{dy}{dx}$ at $x=\pi/4$.

(2) Let $y=\dfrac{\sin^2 x+\cos x}{1+x^2}$. Find $\dfrac{dy}{dx}\Big|_{x=0}$ and $\dfrac{dy}{dx}\Big|_{x=\pi/2}$.

(3) Let $y=\cos(\sin x^2)$. Find $\dfrac{dy}{dx}\Big|_{x=\pi/3}$.

(4) Let $y=(\cot\sqrt{x}+5\sin^2\sqrt{x})^2$. Find $\dfrac{dy}{dx}$ at $x=\pi^2/16$.

(5) Let $y=\dfrac{\sin x^2}{\sqrt{1+x^2}}$. Find $\dfrac{dy}{dx}\Big|_{x=0}$ and $\dfrac{dy}{dx}\Big|_{x=\sqrt{\pi/2}}$.

Problem C. Differential equations.

(1) If $y=x+\tan x$ show that $\cos^2 x\cdot\dfrac{d^2y}{dx^2}-2y+2x=0$.

(2) If $y=A\cos nx+B\sin nx$ show that $\dfrac{d^2y}{dx^2}+n^2y=0$.

(3) If $y=2\sin x+3\cos x$ show that $y+\dfrac{d^2y}{dx^2}=0$.

(4) If $y=a\sin x+b\cos x$ show that $\dfrac{d^2y}{dx^2}+y=0$.

(5) If $y=\sin(\sin x)$ show that $\dfrac{d^2y}{dx^2}+\tan x\dfrac{dy}{dx}+y\cos^2 x=0$.

(6) If $y=a\sin x+b\cos x$ prove that $y^2+\left(\dfrac{dy}{dx}\right)^2=a^2+b^2$.

(7) If $y=\sqrt{\sin x+\sqrt{\sin x+\sqrt{\sin x+\sqrt{\cdots}}}}$ show that $(2y-1)\dfrac{dy}{dx}=\cos x$.

Problem D. Parametric equations.

(1) Find $\dfrac{dy}{dx}$ when $x=a\cos\theta$ and $y=b\sin\theta$.

(2) Find $\frac{dy}{dx}$ when $x = a(\theta + \sin\theta)$ and $y = a(1 - \cos\theta)$.

(3) Find $\frac{dy}{dx}$ when $x = a\sec^2\theta$ and $y = b\tan^3\theta$.

(4) Find $\frac{dy}{dx}$ when $x = b\sin^3\phi$ and $y = b\cos^3\phi$.

(5) If $x = a(t - \sin t)$ and $y = a(1 - \cos t)$ find $\frac{d^2y}{dx^2}$.

(6) If $x = a(\theta + \sin\theta)$ and $y = a(1 - \cos\theta)$ find $\frac{d^2y}{dx^2}$ at $\theta = \pi/2$.

(7) If $x = 2\cos\theta - \cos 2\theta$ and $y = 2\sin\theta - \sin 2\theta$ find $\frac{d^2y}{dx^2}$ at $\theta = \pi/2$.

(8) If $x = \sin t$ and $y = \sin mt$ prove that $(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + m^2y = 0$.

Problem E. Implicit differentiation.

(1) Find $\frac{dy}{dx}$ when $y^2\sin x + y\tan x + (1 + x^2)\cos x = 0$.

(2) Find $\frac{dy}{dx}$ when $\sin(xy) + \frac{x}{y} = x^2 - y$.

(3) Find $\frac{dy}{dx}$ when $2y^2 + \frac{y}{1 + x^2} + \tan^2 x + \sin y = 0$.

(4) Find $\frac{dy}{dx}$ when $\tan(x + y) + \tan(x - y) = 1$.

(5) Find $\frac{dy}{dx}$ when $a\sin(xy) + b\cos(x/y) = 0$.

(6) If $x = \ln(\tan(y/x))$ find $\frac{dy}{dx}$.

Problem F. Derivatives with inverse trig functions.

(1) Find $\frac{dy}{dx}$ when $y = \sin^{-1} x^3$.

(2) Find $\frac{dy}{dx}$ when $y = \tan^{-1}\sqrt{x}$.

(3) Find $\frac{dy}{dx}$ when $y = \sin^{-1} 3x$.

(4) Find $\frac{dy}{dx}$ when $y = \csc^{-1} x^2$.

(5) Find $\frac{dy}{dx}$ when $y = \cos^{-1}\sqrt{x}$.

(6) Find $\frac{dy}{dx}$ when $y = \csc^{-1}(\sin x)$.

(7) Find $\frac{dy}{dx}$ when $y = \tan^{-1}\sqrt{x-1}$.

(8) Find $\frac{dy}{dx}$ when $y = \sin(\tan^{-1} x)$.

(9) Find $\frac{dy}{dx}$ when $y = x\cos^{-1} x$.

(10) Find $\frac{dy}{dx}$ when $y = x\sin^{-1} x$.

(11) Find $\frac{dy}{dx}$ when $y = \tan^{-1}\sqrt{x} - \tan^{-1} x$.

(12) Find $\frac{dy}{dx}$ when $y = (1+x^2)\tan^{-1} x$.

(13) Find $\frac{dy}{dx}$ when $y = \tan x \cos^{-1} x$.

(14) Find $\frac{dy}{dx}$ when $y = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) + \tan^{-1} x$.

(15) Find $\frac{dy}{dx}$ when $y = (1-x^2)\cos^{-1} x$.

(16) Find $\frac{dy}{dx}$ when $y = \tan x \cdot \tan^{-1} x$.

(17) Find $\frac{dy}{dx}$ when $y = \sec^{-1} x + \csc^{-1} x$.

(18) Find $\frac{dy}{dx}$ when $y = \tan^{-1}(a/x) \cdot \cot^{-1}(x/a)$.

(19) Find $\frac{dy}{dx}$ when $y = (\tan^{-1} 2x)^3$.

(20) Find $\frac{dy}{dx}$ when $y = \cos^{-1}(\tan x^2)$.

(21) Find $\frac{dy}{dx}$ when $y = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$.

(22) Find $\frac{dy}{dx}$ when $y = \sec^{-1}\left(\frac{1-x^2}{1+x^2}\right)$.

(23) Find $\frac{dy}{dx}$ when $y = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$.

(24) Find $\frac{dy}{dx}$ when $y = \tan^{-1}\left(\frac{1+x^2}{1-x^2}\right)$.

(25) Find $\frac{dy}{dx}$ when $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$.

(26) Find $\frac{dy}{dx}$ when $y = \cot^{-1}\left(\frac{1+\cos x}{1-\cos x}\right)^{1/2}$.

(27) Find $\frac{dy}{dx}$ when $y = \cot^{-1}\left(\frac{1+\cos 3x}{1-\cos 3x}\right)^{1/2}$.

(28) Find $\frac{dy}{dx}$ when $y = \sin^{-1}\left(\frac{a+b\cos x}{b+a\cos x}\right)$.

(29) Find $\frac{dy}{dx}$ when $y = \cos^{-1}\left(\frac{1+2\cos x}{2+\cos x}\right)$.

(30) Find $\frac{dy}{dx}$ when $y = \tan^{-1}\left(\frac{1-\cos x}{\sin x}\right)$.

(31) Differentiate $\sin^{-1}\left(\frac{x^2-1}{1+x^2}\right)$ with respect to $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$.

(32) If $y = \frac{\sin^{-1} x}{\sqrt{1-x^2}}$ prove that $(1-x^2)\frac{dy}{dx} - xy = 1$.

Problem G. Derivatives with trigonometric functions.

(1) Find $\frac{dy}{dx}$ when $y = x\cos x - \sin x$.

(2) Find $\frac{dy}{dx}$ when $y = \cos^3 3x$.

(3) Find $\frac{dy}{dx}$ when $y = (x^2 + \cos x)^4$.

(4) Find $\frac{dy}{dx}$ when $y = \sin x \sin 2x$.

(5) Find $\frac{dy}{dx}$ when $y = \frac{\sin 2x}{x^2}$.

Problem H. Derivatives with exponentials and logs.

(1) Find $\frac{dy}{dx}$ when $y = \ln\left(x + \sqrt{x^2 + a^2}\right)$.

(2) Find $\frac{dy}{dx}$ when $y = \frac{1+e^x}{1-e^x}$.

(3) Find $\dfrac{dy}{dx}$ when $y = \ln\left(\dfrac{x^2+x+1}{x^2-x-1}\right)$.

(4) Find $\dfrac{dy}{dx}$ if $y = \ln\left[e^x\left(\dfrac{x-2}{x+2}\right)^{3/4}\right]$.

(5) Find $\dfrac{dy}{dx}$ when $y = \ln\ln\ln x^4$.

Problem I. Derivatives with exponentials, logs and trig functions.

(1) Find $\dfrac{dy}{dx}$ when $y = a^{\cos 1}$.

(2) Find $\dfrac{dy}{dx}$ when $y = \ln\dfrac{\sin^m x}{\cos^n x}$.

(3) Find $\dfrac{dy}{dx}$ when $\quad y = e^{ax}\sin bx$.

(4) Find $\dfrac{dy}{dx}$ when $y = \ln\left(\dfrac{1-\cos x}{1+\cos x}\right)$.

(5) Find $\dfrac{dy}{dx}$ when $y = \ln\sqrt{\dfrac{1-\tan x}{1+\tan x}}$.

(6) Find $\dfrac{dy}{dx}$ when $y = e^{ax}\cos(bx+c)$.

(7) Find $\dfrac{dy}{dx}$ when $y = \dfrac{\sqrt{x+\ln\tan x}}{xe^{2x}}$.

(8) Find $\dfrac{dy}{dx}$ when $y = \ln\dfrac{1+x\sin x}{1-x\sin x}$.

(9) Find $\dfrac{dy}{dx}$ when $y = \ln\left(\dfrac{1-\cos x}{1+\cos x}\right)^{1/2}$.

(10) Find $\dfrac{dy}{dx}$ when $y = \ln\sqrt{\dfrac{1+\sin x}{1-\sin x}}$.

(11) If $y = \ln(\sin x)$ show that $\dfrac{d^3y}{dx^3} = 2\csc^2 x\cot x$.

(12) If $y = e^{ax}\cos bx$ show that $\dfrac{d^2y}{dx^2} - 2a\dfrac{dy}{dx} + (a^2+b^2)y = 0$.

(13) If $y = a\cos(\ln x) + b\sin(\ln x)$ show that $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = 0$.

(14) If $y = Ae^{-kt}\cos(pt+c)$ show that $\dfrac{d^2y}{dt^2} + 2k\dfrac{dy}{dt} + n^2y = 0$, where $n^2 = p^2+k^2$.

(15) If $y = e^{-x}\cos x$ prove that $\dfrac{d^4y}{dx^4} + 4y = 0$.

2.5 HW5 Fall 2006: Due October 9, 2006 (Limits)

Problem A. Evaluating limits when $x \to 0$.

(1) Evaluate $\lim_{x\to 0}(x^2-2)^2+6$.

(2) Evaluate $\lim_{x\to 0}\frac{5x}{x}$.

(3) Evaluate $\lim_{x\to 0}\frac{17x}{2x}$.

(4) Evaluate $\lim_{x\to 0}\frac{-317x}{422x}$.

(5) Evaluate $\lim_{x\to 0}\frac{-317x-3}{422x+5}$.

(6) Evaluate $\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}$.

(7) Evaluate $\lim_{x\to 0}\frac{\sqrt{1+x+x^2}-1}{x}$.

(8) Evaluate $\lim_{x\to 0}\frac{\sqrt{2+x}-\sqrt{2}}{x}$.

(9) Evaluate $\lim_{h\to 0}\frac{1}{h}\left(\frac{1}{\sqrt{x+h}}-\frac{1}{\sqrt{x}}\right)$.

(10) Evaluate $\lim_{x\to 0}\frac{2x}{\sqrt{a+x}-\sqrt{a-x}}$.

(11) Evaluate $\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x}$.

(12) Evaluate $\lim_{x\to 0}\frac{x}{\sqrt{1+x}-1}$.

(13) Evaluate $\lim_{x\to 0}\frac{e^x+e^{-x}-2}{x^2}$.

(14) Calculate $\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{(x+\Delta x)-x}$ when $f(x)=\sqrt{ax+b}$.

(15) Calculate $\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{(x+\Delta x)-x}$ when $f(x)=(mx+c)^n$.

Problem B. Evaluating limits when $x \to a$.

(1) Evaluate $\lim_{x\to 1}(6x^2-4x+3)$.

(2) Evaluate $\lim_{x\to 7} \frac{x^2-49}{x-7}$.

(3) Evaluate $\lim_{x\to 2} \frac{x^2-6x+8}{x-2}$.

(4) Evaluate $\lim_{x\to -5} \frac{2x^2+9x-5}{x+5}$.

(5) Evaluate $\lim_{x\to 1} \frac{x^3-1}{x-1}$.

(6) Evaluate $\lim_{x\to 3} \frac{x^2-4x+3}{x^2-2x-3}$.

(7) Evaluate $\lim_{x\to -2} \frac{x^3+8}{x+2}$.

(8) Evaluate $\lim_{x\to 3} \frac{x^4-81}{x-3}$.

(9) Evaluate $\lim_{x\to 5} \frac{x^5-3125}{x-5}$.

(10) Evaluate $\lim_{x\to a} \frac{x^{12}-a^{12}}{x-a}$.

(11) Evaluate $\lim_{x\to a} \frac{x^{5/2}-a^{5/2}}{x-a}$.

(12) Evaluate $\lim_{x\to a} \frac{(x+2)^{5/3}-(a+2)^{5/3}}{x-a}$.

(13) Evaluate $\lim_{x\to 4} \frac{x^3-64}{x^2-16}$.

(14) Evaluate $\lim_{x\to 2} \frac{x^5-32}{x^3-8}$.

(15) Evaluate $\lim_{x\to 1} \frac{x^n-1}{x-1}$.

(16) Evaluate $\lim_{x\to a} \frac{\sqrt{x}-\sqrt{a}}{x-a}$.

(17) Evaluate $\lim_{x\to 2} \frac{\sqrt{3-x}-1}{2-x}$.

(18) Evaluate $\lim_{x\to a} \frac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}$.

(19) Evaluate $\lim_{x\to a} \frac{x^n-a^n}{x-a}$.

Problem C. Evaluating limits as $x \to \infty$.

(1) Evaluate $\lim_{x\to\infty} \dfrac{x+2}{x-2}$.

(2) Evaluate $\lim_{x\to\infty} \dfrac{3x^2+2x-5}{5x^2+3x+1}$.

(3) Evaluate $\lim_{x\to\infty} \dfrac{x^2-7x+11}{3x^2+10}$.

(4) Evaluate $\lim_{x\to\infty} \dfrac{2x^3-5x+7}{7x^3+2x^2-6}$.

(5) Evaluate $\lim_{x\to\infty} \dfrac{(3x-1)(4x-5)}{(x+6)(x-3)}$.

(6) Evaluate $\lim_{n\to\infty} \dfrac{1}{3}+\dfrac{1}{3^2}+\dfrac{1}{3^3}+\cdots+\dfrac{1}{3^n}$.

(7) Evaluate $\lim_{x\to\infty} \dfrac{x}{\sqrt{4x^2+1}-1}$.

(8) Evaluate $\lim_{x\to-\infty} 2^x$.

(9) Evaluate $\lim_{n\to\infty} \left(1+\dfrac{1}{n}\right)^n$.

(10) Evaluate $\lim_{t\to\infty} \dfrac{t+1}{t^2+1}$.

(11) Evaluate $\lim_{n\to\infty} \sqrt{n^2+1}-n$.

(12) Evaluate $\lim_{n\to\infty} \sqrt{n^2+n}-n$.

Problem D. Limits with exponential and log functions.

(1) Evaluate $\lim_{x\to 0} \dfrac{e^x-1}{x}$.

(2) Evaluate $\lim_{x\to 0} \dfrac{a^x-1}{x}$.

(3) Evaluate $\lim_{x\to 0} \dfrac{\ln(1+x)}{x}$.

(4) Evaluate $\lim_{x\to 0}(1+x)^{1/x}$.

(5) Evaluate $\lim_{x\to 0} \dfrac{a^x-b^x}{x}$.

(6) Evaluate $\lim_{x\to 0} \frac{e^x + e^{-x} - 2}{x^2}$.

(7) Evaluate $\lim_{x\to -\infty} 2^x$.

(8) Explain why $\lim_{x\to -1} \ln x$ does not exist.

(9) Explain why $\lim_{x\to 0} 2^{1/x}$ does not exist.

(10) Explain why $\lim_{x\to 1} 2^{1/(x-1)}$ does not exist.

(11) Calculate $\lim_{\Delta x\to 0} \frac{f(x+\Delta x) - f(x)}{(x+\Delta x) - x}$ when $f(x) = e^{\sqrt{x}}$.

(12) Calculate $\lim_{\Delta x\to 0} \frac{f(x+\Delta x) - f(x)}{(x+\Delta x) - x}$ when $f(x) = \ln(ax+b)$.

(13) Calculate $\lim_{\Delta x\to 0} \frac{f(x+\Delta x) - f(x)}{(x+\Delta x) - x}$ when $f(x) = x^x$.

Problem E. Limits with trigonometric functions.

(1) Evaluate $\lim_{x\to 0} \frac{\sin 3x}{4x}$.

(2) Evaluate $\lim_{x\to 0} \frac{\sin x \cos x}{3x}$.

(3) Evaluate $\lim_{x\to 0} \frac{\tan x}{x}$.

(4) Evaluate $\lim_{x\to 0} \frac{1 - \cos x}{\sin^2 x}$.

(5) Evaluate $\lim_{x\to 0} \frac{\tan ax}{\tan bx}$.

(6) Evaluate $\lim_{x\to 0} \frac{\sin(x/4)}{x}$.

(7) Evaluate $\lim_{x\to 0} \frac{\sin mx}{\tan nx}$.

(8) Evaluate $\lim_{\theta\to 0} \frac{1 - \cos 6\theta}{\theta}$.

(9) Evaluate $\lim_{x\to 0} \frac{1 - \cos 2x}{3\tan^2 x}$.

(10) Evaluate $\lim_{x\to 0} \frac{\cos^2 x}{1 - \sin x}$.

(11) Evaluate $\displaystyle\lim_{x\to 0} \frac{\tan 2x - x}{3x - \sin x}$.

(12) Evaluate $\displaystyle\lim_{x\to a} \frac{\sin x - \sin a}{x - a}$.

(13) Evaluate $\displaystyle\lim_{x\to 0} \frac{\sin 5x - \sin 3x}{\sin x}$.

(14) Evaluate $\displaystyle\lim_{x\to 0} \frac{\tan 3x - 2x}{3x - \sin^2 x}$.

(15) Evaluate $\displaystyle\lim_{x\to 0} \frac{x^2 - \tan 2x}{\tan x}$.

(16) Evaluate $\displaystyle\lim_{x\to \pi/4} \frac{1 - \tan x}{x - \pi/4}$.

(17) Evaluate $\displaystyle\lim_{x\to 0} \frac{\tan(x/2)}{3x}$.

(18) Evaluate $\displaystyle\lim_{x\to 0} \frac{1 - \cos 2x + \tan^2 x}{x \sin x}$.

(19) If $\displaystyle\lim_{x\to 0} kx \csc x = \lim_{x\to 0} x \csc kx$, explain why $k = \pm 1$.

(20) Evaluate $\displaystyle\lim_{h\to 0} \frac{\sin(a+h) - \sin a}{h}$.

(21) Evaluate $\displaystyle\lim_{h\to \infty} \frac{\cos(\pi/h)}{h - 2}$.

Problem F. Calculating $\displaystyle\lim_{\Delta x\to 0} \frac{\Delta f}{\Delta x}$.

(1) Calculate $\displaystyle\lim_{\Delta x\to 0} \frac{f(x+\Delta x) - f(x)}{(x+\Delta x) - x}$ when $f(x) = \sin 2x$.

(2) Calculate $\displaystyle\lim_{\Delta x\to 0} \frac{f(x+\Delta x) - f(x)}{(x+\Delta x) - x}$ when $f(x) = \cos 2x$.

(3) Calculate $\displaystyle\lim_{\Delta x\to 0} \frac{f(x+\Delta x) - f(x)}{(x+\Delta x) - x}$ when $f(x) = \sin x^2$.

(4) Calculate $\displaystyle\lim_{\Delta x\to 0} \frac{f(x+\Delta x) - f(x)}{(x+\Delta x) - x}$ when $f(x) = \cos x^2$.

(5) Calculate $\displaystyle\lim_{\Delta x\to 0} \frac{f(x+\Delta x) - f(x)}{(x+\Delta x) - x}$ when $f(x) = \sqrt{\sin x}$.

(6) Calculate $\displaystyle\lim_{\Delta x\to 0} \frac{f(x+\Delta x) - f(x)}{(x+\Delta x) - x}$ when $f(x) = x \sin x$.

(7) Calculate $\lim_{\Delta x\to 0} \dfrac{f(x+\Delta x)-f(x)}{(x+\Delta x)-x}$ when $f(x) = e^{\sin x}$.

(8) Calculate $\lim_{\Delta x\to 0} \dfrac{f(x+\Delta x)-f(x)}{(x+\Delta x)-x}$ when $f(x) = e^{\cos x}$.

(9) Calculate $\lim_{\Delta x\to 0} \dfrac{f(x+\Delta x)-f(x)}{(x+\Delta x)-x}$ when $f(x) = \dfrac{\sin x}{x}$.

Problem G. Limits with inverse trig functions.

(1) Evaluate $\lim_{x\to 1} \dfrac{1-x}{(\cos^{-1} x)^2}$.

(2) Evaluate $\lim_{x\to 1/\sqrt{2}} \dfrac{x-\cos(\sin^{-1} x)}{1-\tan(\sin^{-1} x)}$.

(3) Evaluate $\lim_{x\to 0} \dfrac{x(1-\sqrt{1-x^2})}{(\sin^{-1} x)^3\sqrt{1-x^2}}$.

(4) Evaluate $\lim_{x\to 1} \dfrac{1-x}{\pi - 2\sin^{-1} x}$.

(5) Evaluate $\lim_{x\to 0} \dfrac{\tan^{-1} 2x}{\sin 3x}$.

2.6 HW6 Fall 2006: Due October 16, 2006 (Graphing, continuity, concavity)

Problem A. Graphs of the basic functions.

(1) Graph $f(x) = |x|$.

(2) Graph $f(x) = \lfloor x \rfloor$.

(3) Graph $f(x) = 2$.

(4) Graph $f(x) = x$.

(5) Graph $f(x) = x^2$.

(6) Graph $f(x) = x^3$.

(7) Graph $f(x) = x^4$.

(8) Graph $f(x) = x^5$.

(9) Graph $f(x) = x^6$.

(10) Graph $f(x) = x^{100}$.

(11) Graph $f(x) = x^{-1}$.

(12) Graph $f(x) = x^{-2}$.

(13) Graph $f(x) = x^{-3}$.

(14) Graph $f(x) = x^{-4}$.

(15) Graph $f(x) = x^{-100}$.

(16) Graph $f(x) = e^x$.

(17) Graph $f(x) = \sin x$.

(18) Graph $f(x) = \cos x$.

(19) Graph $f(x) = \tan x$.

(20) Graph $f(x) = \cot x$.

(21) Graph $f(x) = \sec x$.

(22) Graph $f(x) = \csc x$.

(23) Graph $f(x) = \sqrt{x}$.

(24) Graph $f(x) = x^{1/3}$.

(25) Graph $f(x) = x^{1/4}$.

(26) Graph $f(x) = x^{1/5}$.

(27) Graph $f(x) = x^{1/6}$.

(28) Graph $f(x) = \dfrac{1}{\sqrt{x}}$.

(29) Graph $f(x) = x^{-1/3}$.

(30) Graph $f(x) = x^{-1/4}$.

(31) Graph $f(x) = \ln x$.

(32) Graph $f(x) = \sin^{-1} x$.

(33) Graph $f(x) = \cos^{-1} x$.

(34) Graph $f(x) = \tan^{-1} x$.

(35) Graph $f(x) = \cot^{-1} x$.

(36) Graph $f(x) = \sec^{-1} x$.

(37) Graph $f(x) = \csc^{-1} x$.

Problem B. Where is a function continuous?

(1) For which values of x is the function $f(x) = x^2 + 3x + 4$ continuous? Justify your answer with limits if necessary and draw a graph of the function to illustrate your answer.

(2) For which values of x is the function $f(x) = \begin{cases} \dfrac{x^2 - x - 6}{x - 3}, & \text{if } x \neq 3, \\ 5, & \text{if } x = 3, \end{cases}$ continuous? Justify your answer with limits if necessary and draw a graph of the function to illustrate your answer.

(3) For which values of x is the function $f(x) = \begin{cases} \dfrac{\sin 3x}{x}, & \text{if } x \neq 0, \\ 1, & \text{if } x = 0, \end{cases}$ continuous? Justify your answer with limits if necessary and draw a graph of the function to illustrate your answer.

(4) For which values of x is the function $f(x) = \begin{cases} \dfrac{1 - \cos x}{x^2}, & \text{if } x \neq 0, \\ 1, & \text{if } x = 0, \end{cases}$ continuous? Justify your answer with limits if necessary and draw a graph of the function to illustrate your answer.

(5) Determine the value of k for which the function $f(x) = \begin{cases} \dfrac{\sin 2x}{5x}, & \text{if } x \neq 0, \\ k, & \text{if } x = 0, \end{cases}$ is continuous at $x = 0$. Justify your answer with limits if necessary and draw a graph of the function to illustrate your answer.

(6) For which values of x is the function $f(x) = \begin{cases} x-1, & \text{if } 1 \le x < 2, \\ 2x-3, & \text{if } 2 \le x \le 3, \end{cases}$ continuous? Justify your answer with limits if necessary and draw a graph of the function to illustrate your answer.

(7) For which values of x is the function $f(x) = \begin{cases} \cos x, & \text{if } x \ge 0, \\ -\cos x, & \text{if } x < 0, \end{cases}$ continuous? Justify your answer with limits if necessary and draw a graph of the function to illustrate your answer.

(8) For which values of x is the function $f(x) = \begin{cases} \sin(1/x), & \text{if } x \ne 0, \\ 0, & \text{if } x = 0, \end{cases}$ continuous? Justify your answer with limits if necessary and draw a graph of the function to illustrate your answer.

(9) Find the value of a for which the function $f(x) = \begin{cases} ax+5, & \text{if } x \le 2, \\ x-1, & \text{if } x > 2, \end{cases}$ is continuous at $x = 2$. Justify your answer with limits if necessary and draw a graph of the function to illustrate your answer.

(10) For which values of x is the function $f(x) = \begin{cases} 1+x^2, & \text{if } 0 \le x \le 1, \\ 2-x, & \text{if } x > 1, \end{cases}$ continuous? Justify your answer with limits if necessary and draw a graph of the function to illustrate your answer.

(11) For which values of x is the function $f(x) = 2x - |x|$ continuous? Justify your answer with limits if necessary and draw a graph of the function to illustrate your answer.

(12) Find the value of a for which the function $f(x) = \begin{cases} 2x-1, & \text{if } x < 2, \\ a, & \text{if } x = 2, \\ x+1, & \text{if } x > 2, \end{cases}$ is continuous at $x = 2$. Justify your answer with limits if necessary and draw a graph of the function to illustrate your answer.

(13) For which values of x is the function $f(x) = \begin{cases} \dfrac{|x-a|}{x-a}, & \text{if } x \ne a, \\ 1, & \text{if } x = a, \end{cases}$ continuous? Justify your answer with limits if necessary and draw a graph of the function to illustrate your answer.

(14) For which values of x is the function $f(x) = \begin{cases} \dfrac{x-|x|}{2}, & \text{if } x \ne 0, \\ 2, & \text{if } x = 0, \end{cases}$ continuous? Justify your answer with limits if necessary and draw a graph of the function to illustrate your answer.

(15) For which values of x is the function $f(x) = \begin{cases} \sin x, & \text{if } x < 0, \\ x, & \text{if } x \ge 0, \end{cases}$ continuous? Justify your answer with limits if necessary and draw a graph of the function to illustrate your answer.

(16) For which values of x is the function $f(x) = \begin{cases} \dfrac{x^n-1}{x-1}, & \text{if } x \ne 1, \\ n, & \text{if } x = 1, \end{cases}$ continuous? Justify your answer with limits if necessary and draw a graph of the function to illustrate your answer.

(17) Explain how you know that $f(x) = \sec x$ is continuous for all values of x. Justify your answer with limits if necessary and draw a graph of the function to illustrate your answer.

(18) For which values of x is the function $f(x) = \cos|x|$ continuous? Justify your answer with limits if necessary and draw a graph of the function to illustrate your answer.

(19) For which values of x is the function $f(x) = \lfloor x \rfloor$ continuous? Justify your answer with limits if necessary and draw a graph of the function to illustrate your answer.

(20) For which values of x is the function $f(x) = \begin{cases} x^3 - x^2 + 2x - 2, & \text{if } x \neq 1, \\ 4, & \text{if } x = 1, \end{cases}$ continuous? Justify your answer with limits if necessary and draw a graph of the function to illustrate your answer.

(21) For which values of x is the function $f(x) = |x| + |x-1|$, $-1 \le x \le 2$, continuous? Justify your answer with limits if necessary and draw a graph of the function to illustrate your answer.

Problem C. Existence of limits.

(1) Explain why $\lim\limits_{x\to 0} 1/x$ does not exist.

(2) Explain why $\lim\limits_{x\to \pi/2} \tan x$ does not exist.

(3) Explain why $\lim\limits_{x\to \pi/2} \sec x$ does not exist.

(4) Explain why $\lim\limits_{x\to 0} \csc x$ does not exist.

(5) Explain why $\lim\limits_{x\to -1} \ln x$ does not exist.

(6) Explain why $\lim\limits_{x\to 0} \sin(1/x)$ does not exist.

(7) Explain why $\lim\limits_{x\to \infty} \cos x$ does not exist.

(8) Let $\mathrm{sgn}(x)$ be the sign function. This function is given by $\mathrm{sgn}(x) = \begin{cases} 1, & \text{if } x > 0, \\ 0, & \text{if } x = 0, \\ -1, & \text{if } x < 0. \end{cases}$ Explain why $\lim\limits_{x\to \infty} \mathrm{sgn}(x)$ does not exist.

(9) Explain why $\lim\limits_{x\to 0} 2^{1/x}$ does not exist.

(10) Explain why $\lim\limits_{x\to 1} 2^{1/(x-1)}$ does not exist.

Problem D. Increasing, decreasing, and concavity.

(1) What does it mean for a function $f(x)$ to be continuous at $x = a$? Explain how to test if a function is continuous at $x = a$.

(2) What does it mean for a function $f(x)$ to be differentiable at $x = a$? Explain how to test if a function is differentiable at $x = a$.

(3) What does $\left.\dfrac{df}{dx}\right|_{x=a}$ indicate you about the graph of $y = f(x)$? Explain why this is true.

(4) What does it mean for a function to be increasing? Explain how to use calculus to tell if a function is increasing. Explain why this works.

(5) What does it mean for a function to be concave up? Explain how to use calculus to tell if a function is concave up. Explain why this works.

(6) What is a critical point? Explain how to find critical points of a function $f(x)$?

(7) What is a point of inflection? Explain how to find points of inflection of a function $f(x)$?

(8) What is an asymptote of a function $f(x)$? Explain how to justify that a given line is an asymptote of $f(x)$?

(9) If $f(x) = |x|$ what is $\left.\dfrac{df}{dx}\right|_{x=2}$?

(10) Find the values of a and b so that the function $f(x) = \begin{cases} x^2 + 3x + a, & \text{if } x \le 1, \\ bx + 2, & \text{if } x > 1, \end{cases}$ is differentiable for all values of x.

Problem E. Graphing polynomials.

For each of the following graphing problems also determine

(a) where $f(x)$ is defined,
(b) where $f(x)$ is continuous,
(c) where $f(x)$ is differentiable,
(d) where $f(x)$ is increasing and where it is decreasing,
(e) where $f(x)$ is concave up and where it is concave down,
(f) what the critical points of $f(x)$ are,
(g) where the points of inflection are, and
(h) what the asymptotes to $f(x)$ are (if $f(x)$ has asymptotes).

(1) Graph $f(x) = a$, where a is a constant.

(2) Graph $f(x) = ax + b$, where a and b are constants.

(3) Graph $f(x) = a(x - c) + b$, where a, b and c are constants.

(4) Graph $f(x) = \begin{cases} 2 - x, & \text{if } x \ge 1, \\ x, & \text{if } 0 \le x \le 1. \end{cases}$

(5) Graph $f(x) = \begin{cases} 2 + x, & \text{if } x \ge 0, \\ 2 - x, & \text{if } x < 0. \end{cases}$

(6) Graph $f(x) = \begin{cases} 1-x, & \text{if } x < 1, \\ x^2-1, & \text{if } x \geq 1. \end{cases}$

(7) Graph $f(x) = 2x - x^2$.

(8) Graph $f(x) = x - x^2 - 27$.

(9) Graph $f(x) = 3x^2 - 2x - 1$.

(10) Graph $f(x) = x^3$.

(11) Graph $f(x) = x^3 - x + 1$.

(12) Graph $f(x) = x^3 - x - 1$.

(13) Graph $f(x) = (x-2)^2(x-1)$.

(14) Graph $f(x) = 2x^3 - 21x^2 + 36x - 20$.

(15) Graph $f(x) = 2x^3 + x^2 - 20x$.

(16) Graph $f(x) = 1 - x^4$.

(17) Graph $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$.

(18) Graph $f(x) = 3x^4 - 16x^3 + 18x^2$.

(19) Graph $f(x) = x^5 - 4x^4 + 4x^3$.

(20) Graph $f(x) = x^3(x-2)^2$.

(21) Graph $f(x) = (x-2)^4(x+1)^3(x-1)$.

2.7 HW 7 Fall 2006: Due October 23, 2006 (Graphing, tangents, max/mins)

For each of the following graphing problems also determine

(a) where $f(x)$ is defined,
(b) where $f(x)$ is continuous,
(c) where $f(x)$ is differentiable,
(d) where $f(x)$ is increasing and where it is decreasing,
(e) where $f(x)$ is concave up and where it is concave down,
(f) what the critical points of $f(x)$ are,
(g) where the points of inflection are, and
(h) what the asymptotes to $f(x)$ are (if $f(x)$ has asymptotes).

Problem A. Graphing rational functions.

(1) Graph $f(x) = 1/x$.

(2) Graph the function $f(x)$ such that $\dfrac{df}{dx} = 1/x$ and $f(-1) = 2$ and $f(1) = 1$.

(3) Graph $f(x) = x + 1/x$.

(4) Graph $f(x) = \dfrac{x^2 + 2x - 20}{x - 4}$.

(5) Graph $f(x) = \dfrac{1}{x^2 + 1}$.

(6) Graph $f(x) = \dfrac{1}{x^2 + 2x + c}$, where c is a constant.

(7) Graph $f(x) = \dfrac{x^3}{x^2 + 1}$.

(8) Graph $f(x) = \dfrac{x^2 - 1}{x^2 + 1}$.

(9) Graph $f(x) = \dfrac{2x^2}{x^2 - 1}$.

(10) Graph $f(x) = \dfrac{x^2 + 7x + 3}{x^2}$.

(11) Graph $f(x) = \dfrac{x^2(x + 1)^3}{(x - 2)^2(x - 4)^4}$.

(12) Graph $f(x) = \dfrac{x^2 - 1}{x^3 - 4x}$.

Problem B. Graphing functions with square roots.

(1) Graph $y = f(x)$ where $x^2 + y^2 = 1$.

(2) Graph $f(x) = \sqrt{1 - x^2}$.

(3) Graph $f(x) = \sqrt{a^2 - x^2}$, where a is a constant.

(4) Graph $y = f(x)$ when $(x - h)^2 + (y - k)^2 = r^2$, where h, k, and r are constants.

(5) Graph $y = f(x)$ when $x^2 + y^2 - 2hx - 2ky + h^2 + k^2 = r^2$, where h, k, and r are constants.

(6) Graph $y = f(x)$ when $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, where a and b are constants.

(7) Graph $y = f(x)$ when $x = a\cos\theta$ and $y = b\sin\theta$, where a and b are constants.

(8) Graph $f(x) = (b/a)\sqrt{a^2 - x^2}$, where a and b are constants.

(9) Graph $y = f(x)$ when $x^2 - y^2 = 1$.

(10) Graph $y = f(x)$ when $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, where a and b are constants.

(11) Graph $y = f(x)$ when $y = ax^2 - b$, where a and b are constants.

(12) Graph $y = f(x)$ when $x = 2y^2 - 1$.

(13) Graph $y = f(x)$ when $x = \cos 2\theta$ and $y = \cos\theta$.

(14) Graph $f(x) = b\sqrt{x - a}$, where a and b are constants.

(15) Graph $f(x) = \sqrt{x + 2}$.

(16) Graph $f(x) = -\sqrt{x + 2}$.

(17) Graph $y = f(x)$ when $y^2(x^2 - x) = x^2 - 1$.

(18) Graph $y = f(x)$ when $x = \dfrac{y^2 - 1}{y^2 + 1}$.

(19) Graph $y = f(x)$ when $y = \dfrac{\sqrt{1 + x}}{\sqrt{1 - x}}$.

(20) Graph $f(x) = \dfrac{x^2}{\sqrt{x + 1}}$.

(21) Graph $f(x) = x\sqrt{32 - x^2}$.

(22) Graph $f(x) = x\sqrt{1 - x^2}$.

Problem C. Graphing other functions.

(1) Graph $f(x) = \lfloor x \rfloor$.

(2) Graph $f(x) = |x|$.

(3) Graph $f(x) = |x - 5|$.

(4) Graph $f(x) = |x^2 - 1|$.

(5) Graph $f(x) = \begin{cases} 1, & \text{if } x > 0, \\ 0, & \text{if } x = 0, \\ -1, & \text{if } x < 0. \end{cases}$

(6) Graph $f(x) = (x-1)^{1/3}$.

(7) Graph $f(x) = x^{2/3}$.

(8) Graph $f(x) = \dfrac{1}{(x-1)^{2/3}}$.

(9) Graph $f(x) = x(1-x)^{2/5}$.

(10) Graph $f(x) = x^{2/3}(6-x)^{1/3}$.

(11) Graph $y = f(x)$ when $\sqrt{x} + \sqrt{y} = 1$.

(12) Graph $y = f(x)$ when $x^{2/3} + y^{2/3} = a^{2/3}$.

(13) Graph $y = f(x)$ when $x = a\cos^3\theta$ and $y = a\sin^3\theta$.

(14) Graph $f(x) = \sin x$.

(15) Graph $f(x) = \sin 2x - x$.

(16) Graph $f(x) = \sin x - \cos x$ for $-\pi/3 < x < 0$.

(17) Graph $f(x) = 2\cos x + \sin 2x$.

(18) Graph $f(x) = \dfrac{\sin x}{x}$.

(19) Graph $f(x) = \sin(1/x)$.

(20) Graph $f(x) = \sin(x + \sin 2x)$.

(21) Graph $f(x) = e^{-x}$.

(22) Graph $f(x) = e^{1/x}$.

(23) Graph $f(x) = e^{-x^2}$.

(24) Graph $f(x) = \ln(4 - x^2)$.

Problem D. Rolle's theorem and the mean value theorem.

(1) State Rolle's theorem and draw a picture which illustrates the statement of the theorem.

(2) State the mean value theorem and draw a picture which illustrates the statement of the theorem.

(3) Explain why Rolle's theorem is a *special case* of the mean value theorem.

(4) Verify Rolle's theorem for the function $f(x) = (x-1)(x-2)(x-3)$ on the interval $[1, 3]$.

(5) Verify Rolle's theorem for the function $f(x) = (x-2)^2(x-3)^6$ on the interval $[2, 3]$.

(6) Verify Rolle's theorem for the function $f(x) = \sin x - 1$ on the interval $[\pi/2, 5\pi/2]$.

(7) Verify Rolle's theorem for the function $f(x) = e^{-x}\sin x$ on the interval $[0, \pi]$.

(8) Verify Rolle's theorem for the function $f(x) = x^3 - 6x^2 + 11x - 6$.

(9) Let $f(x) = 1 - x^{2/3}$. Show that $f(-1) = f(1)$ but that there is no number c in the interval $(-1, 1)$ such that $\left.\frac{df}{dx}\right|_{x=c} = 0$. Why does this not contradict Rolle's theorem?

(10) Let $f(x) = (x-1)^{-2}$. Show that $f(0) = f(2)$ but that there is no number c in the interval $(0, 2)$ such that $\left.\frac{df}{dx}\right|_{x=c} = 0$. Why does this not contradict Rolle's theorem?

(11) Discuss the applicability of Rolle's theorem when $f(x) = (x-1)(2x-3)$ on the interval $1 \le x \le 3$.

(12) Discuss the applicability of Rolle's theorem when $f(x) = 2 + (x-1)^{2/3}$ on the interval $0 \le x \le 2$.

(13) Discuss the applicability of Rolle's theorem when $f(x) = \lfloor x \rfloor$ on the interval $-1 \le x \le 1$.

(14) At what point on the curve $y = 6 - (x-3)^2$ on the interval $[0, 6]$ is the tangent to the curve parallel to the x-axis?

(15) Show that the equation $x^5 + 10x + 3 = 0$ has exactly one real root.

(16) Show that a polynomial of degree three has at most three real roots.

(17) Verify the mean value theorem for the function $f(x) = x^{2/3}$ in the interval $[0, 1]$.

(18) Verify the mean value theorem for the function $f(x) = \ln x$ in the interval $[1, e]$.

(19) Verify the mean value theorem for the function $f(x) = x$ in the interval $[a, b]$.

(20) Verify the mean value theorem for the function $f(x) = \ell x^2 + mx + n$ in the interval $[a, b]$, where ℓ, m and n are constants.

(21) Show that the mean value theorem is not applicable to the function $f(x) = |x|$ in the interval $[-1, 1]$.

(22) Show that the mean value theorem is not applicable to the function $f(x) = 1/x$ in the interval $[-1, 1]$.

(23) Find the points on the curve $y = x^3 - 3x$ where the tangent is parallel to the chord joining $(1, -2)$ and $(2, 2)$.

(24) If $f(x) = x(1 - \ln x)$, $x > 0$, show that $(a - b)\ln c = b(1 - \ln b) - a(1 - \ln a)$, where $0 < a < b$.

Problem E. Tangents and normals.

(1) Find the slope of the tangent to the curve $y = x^3 - x$ at $x = 2$.

(2) Find the slope of the tangent to the curve $y = (\sin 2x + \cot x + 2)^2$ at $x = \pi/2$.

(3) Find the equations of the tangent and normal to the curve $y = x^3 - 2x + 7$ at the point $(1, 6)$.

(4) Find the equations of the tangent and normal to the curve $3xy^2 - 2x^2y = 1$ at the point $(1, 1)$.

(5) Find the equations of the tangent and normal to the curve $y = x^3 + 2x + 6$ at the point $(2, 18)$.

(6) Find the equations of the tangent and normal to the curve $y^2 = 4ax$ at the point $(a/m^2, 2a/m)$.

(7) Find the equations of the tangent and normal to the curve $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ at the point $(a\cos\theta, b\sin\theta)$.

(8) Find the equations of the tangent and normal to the curve $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ at the point $(a\sec\theta, b\tan\theta)$.

(9) Find the equations of the tangent and normal to the curve $c^2(x^2 + y^2) = x^2y^2$ at the point $(c/\cos\theta, c/\sin\theta)$.

(10) Find the equations of the tangent and normal to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$.

(11) Find the equation of the normal to the curve $ay^2 = x^3$ at the point $(am^2, 2m^3)$.

(12) Show that the equation of the tangent to the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ at the point (p, q) is $\dfrac{xp}{a^2} - \dfrac{yq}{b^2} = 1$

2.8 HW8 Fall 2006: Due October 30, 2006 (Optimization, rates)

Problem A. Tangents and normals.

(1) Find the equations of the tangent and normal to the curve $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at the point where $x = 1$.

(2) Find the equations of the tangent and normal to the curve $y = \cot^2 x - 2\cot x + 2$ at $x = \pi/4$.

(3) Find the equation of the tangent to the curve given by the equations $x = \theta + \sin\theta$ and $y = 1 + \cos\theta$ at $\theta = \pi/4$.

(4) Find the equation of the tangent to the curve given by the equations $x = a\cos\theta$ and $y = b\sin\theta$ at $\theta = \pi/4$.

(5) For a general t find the equation of the tangent and normal to the curve given by the equations $x = a\cos t$ and $y = b\sin t$.

(6) For a general t find the equation of the tangent and normal to the curve $x = a\sec t$, $y = b\tan t$.

(7) For a general t find the equation of the tangent and normal to the curve given by the equations $x = a(t + \sin t)$ and $y = b(1 - \cos t)$.

(8) Find the equations of the tangent and normal to the curve $16x^2 + 9y^2 = 144$ at (x_1, y_1) where $x_1 = 2$ and $y_1 > 0$.

(9) Find the equation of the tangent to the curve $y = \sec^4 x - \tan^4 x$ at $x = \pi/3$.

(10) Find the equation of the normal to the curve $y = (\sin 2x + \cot x + 2)^2$ at $x = \pi/2$.

(11) Find the equation of the normal to the curve $y = \dfrac{1 + \sin x}{\cos x}$ at $x = \pi/4$.

(12) Show that the tangents to the curve $y = 2x^3 - 3$ at the points where $x = 2$ and $x = -2$ are parallel.

(13) Show that the tangents to the curve $y = x^2 - 5x + 6$ at the points $(2, 0)$ and $(3, 0)$ are at right angles.

(14) Find the points on the curve $2a^2y = x^3 - 3ax^2$ where the tangent is parallel to the x-axis.

(15) For the curve $y(x-2)(x-3) = x - 7$ show that the tangent is parallel to the x-axis at the points for which $x = 7 \pm 2\sqrt{5}$.

(16) Find the points on the curve $y = 4x^3 - 2x^5$ for which the tangent passes through the origin.

(17) Find the points on the circle $x^2 + y^2 = 13$ where the tangent is parallel to the line $2x + 3y = 7$.

(18) Find the point on the curve $y = 3x^2 + 4$ at which the tangent is perpendicular to the line whose slope is $-1/6$.

(19) Find the equations of the normal to the curve $2x^2 - y^2 = 14$ parallel to the line $x + 3y = 4$.

(20) Find the equation of the tangent to the curve $x^2 + 2y = 8$ which is perpendicular to the line $x - 2y + 1 = 0$.

(21) (Bonus problem) If the straight line $x\cos\alpha + y\sin\alpha = p$ touches the curve $\left(\dfrac{x}{a}\right)^{n/(n-1)} + \left(\dfrac{y}{b}\right)^{n/(n-1)} = 1$ show that $(a\cos\alpha)^n + (b\sin\alpha)^n = p^n$.

Problem B. Optimization.

(1) Find the local maxima and minima of $f(x) = (5x-1)^2 + 4$ without using derivatives.

(2) Find the local maxima and minima of $f(x) = -(x-3)^2 + 9$ without using derivatives.

(3) Find the local maxima and minima of $f(x) = -|x+4| + 6$ without using derivatives.

(4) Find the local maxima and minima of $f(x) = \sin 2x + 5$ without using derivatives.

(5) Find the local maxima and minima of $f(x) = |\sin 4x + 3|$ without using derivatives.

(6) Find the local maxima and minima of $f(x) = x^4 - 62x^2 + 120x + 9$.

(7) Find the local maxima and minima of $f(x) = (x-1)(x+2)^2$.

(8) Find the local maxima and minima of $f(x) = -(x-1)^3(x+1)^2$.

(9) Find the local maxima and minima of $f(x) = x/2 + 2/x$ for $x > 0$.

(10) Find the local maxima and minima of $f(x) = 2x^3 - 24x + 107$ in the interval $[1, 3]$.

(11) Find the local maxima and minima of $f(x) = \sin x + (1/2)\cos x$ in $0 \le x \le \pi/2$.

(12) Show that the maximum value of $\left(\dfrac{1}{x}\right)^x$ is $e^{1/e}$.

(13) Show that $f(x) = x + 1/x$ has a local maximum and a local minimum, but the maximum value is less than the minimum value.

(14) Find the maximum profit that a company can make if the profit function is given by $p(x) = 41 + 24x - 18x^2$.

(15) An enemy jet is flying along the curve $y = x^2 + 2$. A soldier is placed at the point $(3, 2)$. At what point will the jet be at when the soldier and the jet are closest?

(16) Find the local maxima and minima of $f(x) = -x + 2\sin x$ in $[0, 2\pi]$.

(17) Divide 15 into two parts such that the square of one times the cube of the other is maximum.

(18) Suppose the sum of two numbers is fixed. Show that their product is maximum exactly when each one of them is half of the total sum.

(19) Divide a into two parts such that the pth power of one times the qth power of the other is maximum.

(20) Which fraction exceeds its pth power by the maximum amount?

(21) Find the dimensions of the rectangle of area 96 cm^2 which has minimum perimeter. What is this minimum perimeter?

(22) Show that the right circular cone with a given volume and minimum curved surface area has altitude equal to $\sqrt{2}$ times the radius of the base.

(23) Show that the altitude of the right circular cone with maximum volume that can be inscribed in a sphere of radius R is $4R/3$.

(24) Show that the height of a right circular cylinder with maximum volume that can be inscribed in a given right circular cone of height h is $h/3$.

(25) A cylindrical can is to be made to hold 1 liter of oil. Find the dimensions of the can which will minimize the cost of the metal to make the can.

(26) An open box is to be made out of a given quantity of cardboard of area p^2. Find the maximum volume of the box if its base is square.

(27) Show that $f(x) = \sin x(1 + \cos x)$ is maximum when $x = \pi/3$.

(28) An 8 inch piece of wire is to be cut into two pieces. Figure out where to cut the wire in order to make the sum of the squares of the lengths of the two pieces as small as possible.

(29) Find the dimensions of the maximum rectangular area that can be fenced with a fence 300 yards long.

(30) Given the perimeter of a rectangle show that its diagonal is minimum when it is a square. Make up a word problem for which this gives the solution.

(31) Prove that the rectangle of maximum area that can be inscribed in a circle is a square. Make up a word problem for which this gives the solution.

(32) Show that the triangle of the greatest area with given base and vertical angle is isosceles.

(33) Show that a right triangle with a given perimeter has greatest area when it is isosceles.

(34) Show that the angle of the cone with a given slant height and with maximum volume is $\tan^{-1}(\sqrt{2})$.

Problem C. Related rates.

(1) Find the rate of change of the volume of a sphere of radius r with respect to a change in the radius.

(2) Find the rate of change of the volume of a cylinder of radius r and height h with respect to a change in the radius.

(3) Find the rate of change of the curved surface of a cone of radius r and height h with respect to a change in the radius.

(4) The side of a square is increasing at the rate of 0.2 cm/s. Find the rate of increase of the perimeter of the square.

(5) A balloon which always remains spherical is being inflated by pumping in 900 cubic centimeters of gas per second. Find the rate at which the radius of the balloon is increasing when the radius is 15cm.

(6) The surface area of a spherical bubble is increasing at 2 cm^2/s. When the radius of the bubble is 6 cm at what rate is the volume of the bubble increasing?

(7) The bottom of a rectangular swimming pool is 25×40 meters. Water is pumped into the tank at the rate of 500 cubic meters per minute. Find the rate at which the level of the water in the tank is rising.

(8) A runner runs around a circular track of radius 100 m at a constant speed of 7 m/s. The runner's friend is standing 200 m from the center of the track. How fast is the distance between the friends changing when the distance between them is 200 m.

(9) A streetlight is at the top of a 15 foot tall pole. A man 6 ft tall walks away from the pole with a speed of 5 ft/s along a straight path.

(a) How fast is the tip of his shadow moving when he is 40 feet from the pole?
(b) How fast is his shadow lengthening at that point?

(10) A lighthouse is on a small island 3 km away from the nearest point P on a straight shoreline and its light turns four revolutions per minute. How fast is the beam of light moving along the shoreline when it is 1 km from P?

(11) A boat is pulled into a dock by a rope attached to the bow of the boat a passing through a pulley on the dock that is 1 m higher than the bow of the boat. If the rope is pulled in at a rate of 1 m/s how fast is the boat approaching the dock when it is 8 m from the dock?

(12) Gravel is being dumped from a conveyor belt at a rate of 30 ft^3/min and its coarseness is such that it forms a pile in the shape of a cone whose base diameter and height are always equal. How fast is the height of the pile increasing when the pile is 10 ft high?

(13) Water is dripping from a tiny hole in the vertex in the bottom of a conical funnel at a uniform rate of 4 cm^3/s. When the slant height of the water is 3 cm, find the rate of decrease of the slant height of the water, given that the vertical angle of the funnel is 120°.

(14) Water is leaking out of an inverted conical tank at a rate of $10,000$ cm^3/min at the same time that water is being pumped into the tank at a constant rate. The tank has height 6 m and the diameter at the top is 4 m. If the water height is rising at a rate of 20 cm/min when the height of the water is 2 m, find the rate at which water is being pumped into the tank.

(15) Oil is leaking from a cylindrical drum at a rate of 16 milliliters per second. If the radius of the drum is 7 cm and its height is 60 cm find the rate at which the level of oil is changing when the oil level is 18 cm.

(16) A ladder 10 feet long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a speed of 2 ft/s, how fast is the angle between the top of the ladder and the wall changing when the angle is $\pi/4$ radians?

(17) A ladder 13 meters long is leaning against a wall. The bottom of the ladder is pulled along the ground away from from the wall at the rate of 2 m/s. How fast is its height on the wall decreasing when the foot of the ladder is 5 m away from the wall?

(18) A man is moving away from a 40 meter tower at a speed of 2 m/s. Find the rate at which the angle of elevation of the top of the tower is changing when he is at a distance of 30 meters from the foot of the tower. Assume that the eye level of the man is 1.6 meters from the ground.

(19) Find the angle which increases twice as fast as its sine.

(20) A television camera is positioned 4000 ft from the base of a rocket launching pad. A rocket rises vertically and its speed is 600 ft/s when it has risen 3000 feet.

(a) How fast is the distance from the television camera to the rocket changing at that moment?

(b) How fast is the camera's angle of elevation changing at that same moment?

2.9 HW9 Fall 2006: November 6, 2006 (Integrals)

Problem A. Indefinite integrals.

(1) $\displaystyle\int x^7 \, dx$

(2) $\displaystyle\int x^{-7} \, dx$

(3) $\displaystyle\int x^{-1} \, dx$

(4) $\displaystyle\int x^{5/3} \, dx$

(5) $\displaystyle\int x^{-5/4} \, dx$

(6) $\displaystyle\int \sqrt[3]{x^2} \, dx$

(7) $\displaystyle\int \frac{1}{\sqrt[4]{x^3}} \, dx$

(8) $\displaystyle\int \frac{2}{x^2} \, dx$

(9) $\displaystyle\int (8 - x + 2x^3 - 6/x^3 + 2x^{-5} + 5x^{-1}) \, dx$

(10) $\displaystyle\int (2-5x)(3+2x)(1-x) \, dx$

(11) $\displaystyle\int \sqrt{x}(ax^2 + bx + c) \, dx$

(12) $\displaystyle\int (x^2 - 1/x^2)^3 \, dx$

(13) $\displaystyle\int (\sqrt{x} - 1/\sqrt{x}) \, dx$

(14) $\displaystyle\int \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 \, dx$

(15) $\displaystyle\int \frac{(1+2x)^3}{x^4} \, dx$

(16) $\displaystyle\int \frac{(1+x)^3}{\sqrt{x}} \, dx$

(17) $\displaystyle\int \frac{2x^2+x-2}{x-2}\ dx$

(18) If $\dfrac{df}{dx} = x - 1/x^2$ and $f(1) = 1/2$ find $f(x)$.

Problem B. Indefinite integrals with trigonometric functions.

(1) $\displaystyle\int \left(9\sin x - 7\cos x - \frac{6}{\cos^2 x} + \frac{2}{\sin^2 x} + \cot^2 x\right)\ dx$

(2) $\displaystyle\int \left(\frac{\cot x}{\sin x} - \tan^2 x - \frac{\tan x}{\cos x} + \frac{2}{\cos^2 x}\right)\ dx$

(3) $\displaystyle\int \sec x(\sec x + \tan x)\ dx$

(4) $\displaystyle\int \csc x(\csc x - \cot x)\ dx$

(5) $\displaystyle\int (\tan x + \cot x)^2\ dx$

(6) $\displaystyle\int \frac{1+2\sin x}{\cos^2 x}\ dx$

(7) $\displaystyle\int \frac{3\cos x+4}{\sin^2 x}\ dx$

(8) $\displaystyle\int \frac{1}{1-\cos x}\ dx$

(9) $\displaystyle\int \frac{1}{1+\cos x}\ dx$

(10) $\displaystyle\int \frac{\tan x}{\sec x+\tan x}\ dx$

(11) $\displaystyle\int \frac{\csc x}{\csc x-\cot x}\ dx$

(12) $\displaystyle\int \frac{\cos x}{1+\cos x}\ dx$

(13) $\displaystyle\int \frac{\sin x}{1-\sin x}\ dx$

(14) $\displaystyle\int \sqrt{1+\cos 2x}\ dx$

(15) $\displaystyle\int \sqrt{1-\cos 2x}\ dx$

(16) $\displaystyle\int \frac{1}{1+\cos 2x}\ dx$

(17) $\displaystyle\int \frac{1}{1-\cos 2x}\ dx$

(18) $\displaystyle\int \sqrt{1+\sin 2x}\ dx$

(19) $\displaystyle\int \frac{\sin^3 x+\cos^3 x}{\sin^2 x\cos^2 x}\ dx$

Problem C. Integrals with exponential functions and inverse functions.

(1) $\displaystyle\int 2^x\ dx$

(2) $\displaystyle\int (6x^5-2x^{-4}-7x+3/x-5+4e^x+7^x)\ dx$

(3) $\displaystyle\int (x/a+a/x+x^a+a^x+ax)\ dx$

(4) $\displaystyle\int \left(\sqrt{x}-\sqrt[3]{x^4}+\frac{7}{\sqrt[3]{x^2}}-6e^x+1\right)\ dx$

(5) $\displaystyle\int \frac{x^2-1}{x^2+1}\ dx$

(6) $\displaystyle\int \frac{x^6-1}{x^2+1}\ dx$

(7) $\displaystyle\int \frac{x^4}{1+x^2}\ dx$

(8) $\displaystyle\int \frac{x^2}{1+x^2}\ dx$

(9) $\displaystyle\int \left(1+\frac{1}{1+x^2}-\frac{2}{\sqrt{1-x^2}}+\frac{5}{x\sqrt{x^2-1}}+a^x\right)\ dx$

(10) $\displaystyle\int \tan^{-1}\left(\frac{\sin 2x}{1+\cos 2x}\right)\ dx$

(11) $\displaystyle\int \cos^{-1}\left(\frac{1-\tan^2 x}{1+\tan^2 x}\right)\ dx$

(12) $\displaystyle\int \cos^{-1}(\sin x)\ dx$

(13) $\displaystyle\int \cot^{-1}\left(\frac{\sin 2x}{1-\cos 2x}\right)\ dx$

Problem D. Integration by substitution.

(1) $\displaystyle\int (2x+9)^5\ dx$

(2) $\displaystyle\int (7-3x)^4\ dx$

(3) $\displaystyle\int \sqrt{3x-5}\ dx$

(4) $\displaystyle\int \frac{1}{\sqrt{4x+3}}\ dx$

(5) $\displaystyle\int \frac{1}{\sqrt{3-4x}}\ dx$

(6) $\displaystyle\int \frac{1}{(2x-3)^{3/2}}\ dx$

(7) $\displaystyle\int \frac{4x}{2x^2+3}\ dx$

(8) $\displaystyle\int \frac{x+1}{x^2+2x-3}\ dx$

(9) $\displaystyle\int \frac{4x-5}{2x^2-5x+1}\ dx$

(10) $\displaystyle\int \frac{9x^2-4x+5}{3x^3-2x^2+5x+1}\ dx$

(11) $\displaystyle\int \frac{2x+3}{\sqrt{x^2+3x-2}}\ dx$

(12) $\displaystyle\int \frac{2x-1}{\sqrt{x^2-x-1}}\ dx$

(13) $\displaystyle\int \frac{dx}{\sqrt{x+a}+\sqrt{x+b}}$

(14) $\displaystyle\int \frac{dx}{\sqrt{1-3x}-\sqrt{5-3x}}$

(15) $\displaystyle\int \frac{x^2}{1+x^6}\ dx$

(16) $\displaystyle\int \frac{x^3}{1+x^8}\ dx$

(17) $\displaystyle\int \frac{x}{1+x^4}\ dx$

(18) $\displaystyle\int \frac{x^5}{\sqrt{1+x^3}}\ dx$

(19) $\displaystyle\int \frac{x}{\sqrt{1+x}}\ dx$

(20) $\displaystyle\int \frac{1}{x\sqrt{x^4-1}}\ dx$

(21) $\displaystyle\int x\sqrt{x-1}\ dx$

(22) $\displaystyle\int (1-x)\sqrt{1+x}\ dx$

(23) $\displaystyle\int x\sqrt{x^2-1}\ dx$

(24) $\displaystyle\int x\sqrt{3x-2}\ dx$

(25) $\displaystyle\int (2x-3)\sqrt{x^2-3x+5}\ dx$

(26) $\displaystyle\int \frac{dx}{3-5x}$

(27) $\displaystyle\int \sqrt{1+x}\ dx$

Problem E. Integrals with trigonometric functions.

(1) $\displaystyle\int \sin 3x\ dx$

(2) $\displaystyle\int \cos(5+6x)\ dx$

(3) $\displaystyle\int \sin(5-3x)\ dx$

(4) $\displaystyle\int \csc^2(2x+5)\ dx$

(5) $\displaystyle\int \sin x\cos x\ dx$

(6) $\displaystyle\int \sin^3 x\cos x\ dx$

(7) $\displaystyle\int \sqrt{\cos x}\sin x\ dx$

(8) $\displaystyle\int \frac{\cos\sqrt{x}}{\sqrt{x}}\ dx$

(9) $\displaystyle\int \sin(ax+b)\cos(ax+b)\ dx$

(10) $\displaystyle\int \cos^3 x\ dx$

(11) $\displaystyle\int (1/x^2)\cos(1/x)\ dx$

(12) $\displaystyle\int 2x\sin(x^2+1)\ dx$

(13) $\displaystyle\int \frac{\tan\sqrt{x}\sec^2\sqrt{x}}{\sqrt{x}}\ dx$

(14) $\displaystyle\int \frac{\sec^2 x}{1+\tan x}\ dx$

(15) $\displaystyle\int \frac{\sin x}{1+\cos x}\ dx$

(16) $\displaystyle\int \frac{\sin x}{2+3\cos x}\ dx$

(17) $\displaystyle\int \frac{\sin 2x}{a^2+b^2\sin^2 x}\ dx$

(18) $\displaystyle\int \frac{\sin 2x}{a^2\cos^2 x+b^2\sin^2 x}\ dx$

(19) $\displaystyle\int \frac{2\cos x-3\sin x}{3\cos x+2\sin x}\ dx$

(20) $\displaystyle\int \frac{1+\cos x}{(x+\sin x)^3}\ dx$

(21) $\displaystyle\int \frac{\sin x}{(1+\cos x)^2}\ dx$

(22) $\displaystyle\int x^2\sin x^3\ dx$

(23) $\displaystyle\int \frac{\sin x}{\sin x-\cos x}\ dx$

(24) $\displaystyle\int \frac{dx}{1-\tan x}$

(25) $\displaystyle\int \frac{dx}{1-\cot x}$

(26) $\displaystyle\int \frac{\cos 2x}{(\sin x + \cos x)^2}$

(27) $\displaystyle\int \frac{\cos x - \sin x}{(1 + \sin 2x)}$

(28) $\displaystyle\int x \sin^3 x^2 \cos x^2 \ dx$

(29) $\displaystyle\int \frac{\sec^2 x}{\sqrt{1 - \tan^2 x}} \ dx$

(30) $\displaystyle\int 2x \sec^3(x^2 + 3) \tan(x^2 + 3) \ dx$

(31) $\displaystyle\int \frac{\sin 2x}{(a + b \cos 2x)^2} \ dx$

2.10 HW10 Fall 2006: Due November 13, 2006 (Definite integrals, FTC, areas)

Problem A. Integrals with exponential functions and logarithms.

(1) $\displaystyle\int e^{2x-1}\ dx$

(2) $\displaystyle\int e^{1-3x}\ dx$

(3) $\displaystyle\int 3^{2-3x}\ dx$

(4) $\displaystyle\int \frac{1}{x\ln x}\ dx$

(5) $\displaystyle\int \frac{\ln(x^2)}{x}\ dx$

(6) $\displaystyle\int \frac{(\ln x)^2}{x}\ dx$

(7) $\displaystyle\int \frac{1}{x^2}e^{-1/x}\ dx$

(8) $\displaystyle\int \frac{e^x}{1+e^{2x}}\ dx$

(9) $\displaystyle\int \frac{e^{2x}}{e^{2x}-2}\ dx$

(10) $\displaystyle\int \frac{\sqrt{2+\ln x}}{x}\ dx$

(11) $\displaystyle\int \frac{(x+1)(x+\ln x)^2}{x}\ dx$

(12) $\displaystyle\int \sqrt{e^x-1}\ dx$

Problem B. Definite integrals.

(1) $\displaystyle\int_{-2}^{4} (3x-5)\ dx$

(2) $\displaystyle\int_{1}^{2} x^{-2}\ dx$

(3) $\displaystyle\int_{0}^{1} (1-2x-3x^2)\ dx$

(4) $\displaystyle\int_1^2 (5x^2 - 4x + 3)\ dx$

(5) $\displaystyle\int_{-3}^0 (5y^4 - 6y^2 + 14)\ dy$

(6) $\displaystyle\int_0^1 (y^9 - 2y^5 + 3y)\ dy$

(7) $\displaystyle\int_0^4 \sqrt{x}\ dx$

(8) $\displaystyle\int_0^1 x^{3/7}\ dx$

(9) $\displaystyle\int_1^3 \left(\frac{1}{t^2} - \frac{1}{t^4}\right)\ dt$

(10) $\displaystyle\int_1^2 \frac{t^6 - t^2}{t^4}\ dt$

(11) $\displaystyle\int_1^2 \frac{x^2 + 1}{\sqrt{x}}\ dx$

(12) $\displaystyle\int_0^2 (x^3 - 1)^2\ dx$

(13) $\displaystyle\int_0^1 u(\sqrt{u} + \sqrt[3]{u})\ du$

(14) $\displaystyle\int_{-1}^1 \frac{3}{t^4}\ dt$

(15) $\displaystyle\int_1^2 (x + 1/x)^2\ dx$

(16) $\displaystyle\int_3^3 \sqrt{x^5 + 2}\ dx$

(17) $\displaystyle\int_{-4}^2 \frac{2}{x^6}\ dx$

(18) $\displaystyle\int_1^{-1} (x - 1)(3x + 2)\ dx$

(29) $\displaystyle\int_1^4 (\sqrt{t} - 2/\sqrt{t})\ dt$

(20) $\displaystyle\int_1^8 \left(\sqrt[3]{r} + \frac{1}{\sqrt[3]{r}}\right)\ dr$

(21) $\displaystyle\int_{-1}^{0} (x+1)^3 \ dx$

(22) $\displaystyle\int_{-5}^{-2} \frac{x^4-1}{x^2+1} \ dx$

(23) $\displaystyle\int_{1}^{e} \frac{x^2+x+1}{x} \ dx$

(24) $\displaystyle\int_{4}^{9} \left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2 \ dx$

(25) $\displaystyle\int_{0}^{1} \left(\sqrt[4]{x^5}+\sqrt[5]{x^4}\right) \ dx$

(26) $\displaystyle\int_{1}^{8} \frac{x-1}{\sqrt[3]{x^2}} \ dx$

Problem C. Definite integrals with trigonometric functions.

(1) $\displaystyle\int_{\pi/4}^{\pi/3} \sin t \ dt$

(2) $\displaystyle\int_{0}^{\pi/2} (\cos\theta + 2\sin\theta) \ d\theta$

(3) $\displaystyle\int_{\pi/2}^{\pi} \sec x \tan x \ dx$

(4) $\displaystyle\int_{\pi/3}^{\pi/2} \csc x \cot x \ dx$

(5) $\displaystyle\int_{\pi/6}^{\pi/3} \csc^2\theta \ d\theta$

(6) $\displaystyle\int_{\pi/4}^{\pi} \sec^2\theta \ d\theta$

(7) $\displaystyle\int_{1}^{\sqrt{3}} \frac{6}{1+x^2} \ dx$

(8) $\displaystyle\int_{0}^{0.5} \frac{dx}{\sqrt{1-x^2}} \ dx$

Problem D. Definite integrals with other functions.

(1) $\displaystyle\int_{4}^{8} 1/x \ dx$

(2) $\displaystyle\int_{\ln 3}^{\ln 6} 8e^x \, dx$

(3) $\displaystyle\int_{8}^{9} 2^t \, dt$

(4) $\displaystyle\int_{-e^2}^{-e} \frac{3}{x} \, dx$

(5) $\displaystyle\int_{-2}^{3} |x^2 - 1| \, dx$

(6) $\displaystyle\int_{-1}^{2} |x - x^2| \, dx$

(7) $\displaystyle\int_{-1}^{2} (x - 2|x|) \, dx$

(8) $\displaystyle\int_{0}^{2} (x^2 - |x - 1|) \, dx$

(9) $\displaystyle\int_{0}^{2} f(x) \, dx$ where $f(x) = \begin{cases} x^4, & \text{if } 0 \le x < 1, \\ x^5, & \text{if } 1 \le x \le 2. \end{cases}$

(10) $\displaystyle\int_{-\pi}^{\pi} f(x) \, dx$ where $f(x) = \begin{cases} x, & \text{if } -\pi \le x \le 0, \\ \sin x, & \text{if } 0 < x \le \pi. \end{cases}$

Problem E. The Fundamental Theorem of Calculus.

(1) What does $\displaystyle\int_a^b f(x)dx$ mean?

(2) How does one usually calculate $\displaystyle\int_a^b f(x)dx$? Give an example which shows that this method doesn't always work. Why doesn't it?

(3) Give an example which shows that $\displaystyle\int_a^b f(x)dx$ is not always the true area under $f(x)$ between a and b even if $f(x)$ is continuous between a and b.

(4) What is the Fundamental Theorem of Calculus?

(5) Let $f(x)$ be a function which is continuous and let $A(x)$ be the area under $f(x)$ from a to x. Compute the derivative of $A(x)$ by using limits.

(6) Why is the Fundamental Theorem of Calculus true? Explain carefully and thoroughly.

(7) Give an example which illustrates the Fundamental Theorem of Calculus. In order to do this compute an area by summing up the areas of tiny boxes and then show that applying the Fundamental Theorem of Calculus gives the same answer.

Problem F. Finding areas bounded by lines and a curve.

(1) Find the area of the region bounded by the curve $xy - 3x - 2y - 10 = 0$, the x-axis, and the lines $x = 3$ and $x = 4$.

(2) Find the area lying below the x-axis and above the parabola $y = 4x + x^2$.

(3) Graph the curve $y = 2\sqrt{9 - x^2}$ and determine the area enclosed between the curve and the x-axis.

(4) Find the area bounded by the curve $y = x(x-3)(x-5)$, the x-axis and the lines $x = 0$ and $x = 5$.

(5) Find the area enclosed between the curve $y = \sin 2x$, $0 \le x \le \pi/4$ and the axes.

(6) Find the area enclosed between the curve $y = \cos 2x$, $0 \le x \le \pi/4$ and the axes.

(7) Find the area enclosed between the curve $y = 3\cos x$, $0 \le x \le \pi/2$ and the axes.

(8) Show that the ratio of the areas under the curves $y = \sin x$ and $y = \sin 2x$ between the lines $x = 0$ and $x = \pi/3$ is $2/3$.

(9) Find the area enclosed between the curve $y = \cos 3x$, $0 \le x \le \pi/6$ and the axes.

(10) Find the area enclosed between the curve $y = \tan^2 x$, $0 \le x \le \pi/4$ and the axes.

(11) Find the area enclosed between the curve $y = \csc^2 x$, $0 \le x \le \pi/4$ and the axes.

(12) Compare the areas under the curves $y = \cos^2 x$ and $y = \sin^2 x$ between $x = 0$ and $x = \pi$.

(13) Graph the curve $y = x/\pi + 2\sin^2 x$ and find the area between the x-axis, the curve and the lines $x = 0$ and $x = \pi$.

(14) Find the area bounded by $y = \sin x$ and the x-axis between $x = 0$ and $x = 2\pi$.

Problem G. Areas between curves.

(1) Find the area of the region bounded by the parabola $y^2 = 4x$ and the line $y = 2x$.

(2) Find the area bounded by the curve $y = x(2-x)$ and the line $x = 2y$.

(3) Find the area bounded by the curve $x^2 = 4y$ and the line $x = 4y - 2$.

(4) Calculate the area of the region bounded by the parabolas $y = x^2$ and $x = y^2$.

(5) Find the area of the region included between the parabola $y^2 = x$ and the line $x + y = 2$.

(6) Find the area of the region bounded by the curves $y = \sqrt{x}$ and $y = x$.

(7) Find the area of the part of the first quadrant which is between the parabola $y^2 = 3x$ and the circle $x^2 + y^2 - 6x = 0$.

(8) Find the area of the region between the curves $y^2 = 4x$ and $x = 3$.

(9) Use integration to find the area of the triangular region bounded by the lines $y = 2x+1$, $y = 3x+1$ and $x = 4$.

(10) Find the area bounded by the parabola $x^2 - 2 = y$ and the line $x + y = 0$.

(11) Find the area bounded by the curves $y = 3x - x^2$ and $y = x^2 - x$.

(12) Graph the curve $y = (1/2)x^2 + 1$ and the straight line $y = x + 1$ and find the area between the curve and the line.

(13) Find the area of the region between the parabolas $4y^2 = 9x$ and $3x^2 = 16y$.

(14) Find the area of the region between the curves $x^2 + y^2 = 2$ and $x = y^2$.

(15) Find the area of the region between the curves $y = x^2$ and $x^2 + 4(y - 1) = 0$.

(16) Find the area of the region between the circles $x^2 + y^2 = 4$ and $(x - 2)^2 + y^2 = 4$.

(17) Find the area of the region enclosed by the parabola $y^2 = 4ax$ and the line $y = mx$.

(18) Find the area between the parabolas $y = 4ax$ and $y^2 = 4ay$.

(19) Find the area of the region between the two circles $x^2 + y^2 = 1$ and $(x - 1)^2 + y^2 = 1$.

(20) Find the area bounded by the curves $y = x$ and $y = x^3$.

(21) Graph $y = \sin x$ and $y = \cos x$ for $0 \leq x \leq \pi/2$ and find the area enclosed by them and the x-axis.

2.11 HW 11 Fall 2006: Due November 20, 2006 (Volumes)

Problem A. Volumes by washers.

(1) Show that the volume of a right circular cylinder of radius r and height h is $\pi r^2 h$ by using the washer method.

(2) Show that the volume of a right circular cone of radius r and height h is $\frac{1}{3}\pi r^2 h$ by using the washer method.

(3) Show that the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$ by using the washer method.

(4) Find the volume generated when the area bounded by the lines $x + y = 2$, $x = 0$, $y = 0$ is rotated about the x-axis.

(5) Find the volume generated when the area bounded by $y = \sin x$, $0 \leq x \leq \pi$, and $y = 0$ is rotated about the x-axis.

(6) Find the volume generated when the area bounded by $y = x - x^2$ and $y = 0$ is rotated about the x-axis.

(7) Using integration find the volume generated by rotating the triangle with vertices at $(0, 0)$, $(h, 0)$, and (h, r) about the x-axis.

(8) Using integration find the volume generated by rotating the triangle with vertices at $(0, 0)$, $(h, 0)$, and (h, r) about the y-axis.

(9) A hemispherical bowl of radius a contains water to a depth h.

(a) Find the volume of water in the bowl.

(b) Water runs into a hemispherical bowl of radius 5 ft at the rate of 0.2 ft^3/sec. How fast is the water level in the bowl rising when the water is 4 ft deep?

(10) Find the volume generated when the area bounded by $y = -3x - x^2$ and $y = 0$ is rotated about the x-axis.

(11) Find the volume generated when the area bounded by $y = x^2 - 2x$ and $y = 0$ is rotated about the x-axis.

(12) Find the volume generated when the area bounded by $y = x^3$, $x = 2$ and $y = 0$ is rotated about the x-axis.

(13) A football has a volume that is approximately the same as the volume generated by rotating the area inside the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ (where a and b are constants) about the x-axis. Find the volume generated.

(14) The cross sections of a certain solid by planes perpendicular to the x-axis are circles with diameters extending from the curve $y = x^2$ to the curve $y = 8 - x^2$. The solid lies between the points of intersection of these two curves. Find its volume.

(15) The base of a certain solid is the circle $x^2 + y^2 = a^2$. Each plane section of the solid cut out by a plane perrpendicular to the x-axis is a square with one edge of the square in the base of the solid. Find the volume of the solid.

(16) Find the volume generated when the area bounded by $y = x^4$, $x = 1$ and $y = 0$ is rotated about the x-axis.

(17) Find the volume generated when the area bounded by $y = \sqrt{\cos x}$, $0 \le x \le \pi/2$; $x = 0$ and $y = 0$ is rotated about the x-axis.

(18) Find the volume generated when the area bounded by $y = \sqrt{x}$, $y = 2$ and $x = 0$ is rotated about the y-axis.

(19) Two great circles, lying in planes that are perpendicular to each other are marked on a sphere of radius a. A portion of the sphere is shaved off so that any plane section of the remaining solid, perpendicular to the common diameter of the two great circles, is a square with vertices on these circles. Find the volume of the solid that remains.

(20) The base of a solid is the circle $x^2 + y^2 = a^2$. Each plane section of the solid cut out by a plane perpendicular to the y-axis is an isosceles right triangle with one leg in the base of the solid. Find the volume.

(21) The base of a solid is the region between the x-axis and the curve $y = \sin x$ between $x = 0$ and $x = \pi/2$. Each plane section of the solid perpendicular to the x-axis is an equilateral triangle with one side in the base of the solid. Find the volume.

(22) Find the volume generated when the area bounded by $y = \sqrt{x}$, $y = 2$ and $x = 0$ is rotated about the line $y = 2$.

Problem B. Finding volumes by cylindrical shells.

(1) Show that the volume of a right circular cylinder of radius r and height h is $\pi r^2 h$ by using cylindrical shells.

(2) Show that the volume of a right circular cone of radius r and height h is $\frac{1}{3}\pi r^2 h$ by using cylindrical shells.

(3) Show that the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$ by using cylindrical shells.

(4) A hole of diameter a is bored through the center of a sphere of radius a. Find the remaining volume.

(5) Find the volume of the bagel produced by rotating the circle $x^2 + y^2 = a^2$ about the line $x = b$, $(b \ge a)$.

(6) Find the volume generated by rotating the area bounded by the curves $x + y = 2$, $x = 0$ and $y = 0$ about the x-axis, by using cylindrical shells.

(7) Find the volume generated by rotating the area bounded by the curves $x = 2y - y^2$ and $x = 0$ about the x-axis.

(8) Find the volume generated by rotating the area bounded by the curves $y = 3x - x^2$ and $y = x$ about the x-axis.

(9) Find the volume generated by rotating the area bounded by the curves $y = x$, $y = 1$ and $x = 0$ about the x-axis.

(10) Find the volume generated by rotating the area bounded by the curves $y = x^2$ and $y = 4$ about the x-axis.

(11) Find the volume generated by rotating the area bounded by the curves $y = 3 + x^2$ and $y = 4$ about the x-axis.

(12) Find the volume generated by rotating the area bounded by the curves $y = x^2 + 1$ and $y = x + 3$ about the x-axis.

(13) Find the volume generated by rotating the area bounded by the curves $y = 4 - x^2$ and $y = 2 - x$ about the x-axis.

(14) Find the volume generated by rotating the area bounded by the curves $y = x^4$, $x = 1$ and $y = 0$ about the y-axis.

(15) Find the volume generated by rotating the area bounded by the curves $y = x^3$, $x = 2$ and $y = 0$ about the y-axis.

(16) Find the volume generated by revolving the triangle with vertices $(1, 1)$, $(1, 2)$ and $(2, 2)$ about the x-axis.

(17) Find the volume generated by revolving the triangle with vertices $(1, 1)$, $(1, 2)$ and $(2, 2)$ about the y-axis.

(18) Find the volume generated by revolving the area bounded by the curves $x = y - y^3$, $x = 0$, $y = 0$ and $y = 1$ about the x-axis.

(19) Find the volume generated by revolving the area bounded by $y = \sqrt{x}$, $x = 0$ and $y = 2$ about the x-axis.

(20) Find the volume generated by revolving the area bounded by $y = \sqrt{x}$, $x = 0$ and $y = 2$ about the line $x = 4$.

Problem C. Practical volumes.

(1) The cross section of a solid in any plane perpendicular to the x-axis is a circle having diameter AB with A on the curve $y^2 = 4x$ and B on the curve $x^2 = 4y$. Find the volume of the solid lying between the points of intersection of the curves.

(2) The base of a solid is the area bounded by $y^2 = 4ax$ and $x = a$. Each cross section perpendicular to the x-axis is an equilateral triangle. Find the volume of the solid.

(3) Find the volume of the slice obtained by cutting a slice off a sphere of radius r, if the slice has thickness h at its thickest point.

(4) Find the volume left after slicing off the top of a right circular cone, if the cone has radius r and height h and, after slicing the top off, what's left has height b.

(5) Find the volume of a tetrahedron, where each side of the tetrahedron is an equilateral triangle with side length a.

(6) The base of a solid is a circle of radius r and the cross sections perpendicular to the base are squares. Find the volume.

(7) The base of a solid is the ellipse $9x^2 + 4y^2 = 36$. Cross sections perpendicular to the x-axis are isosceles right triangles with hypotenuse in the base. Find the volume.

(8) The base of a solid is the parabolic region $\{(x, y) \mid x^2 \le y \le 1\}$. Cross sections perpendicular to the y-axis are equilateral triangles. Find the volume.

(9) Find the volume common to two spheres, each with radius r, where the center of each sphere is on the surface of the other.

(10) Find the volume common to two circular cylinders of radius r, such that the axes of the cylinders intersect a right angles.

(11) In 1715 Kepler published a book, *Stereometria doliorum*, which explained how to find the volumes of barrels. A barrel of height h and maximum radius R is constructed by rotating the parabola $y = R - cx^2$, $-h/2 \le x \le h/2$, where c is a positive constant.

(a) Show that the radius of each end of the barrel is $r = R - ch^2/4$.
(b) Show that the volume of the barrel is $(1/3)\pi h(2R^2 + r^2 - (1/40)c^2h^4)$.

(12) Suppose that you are given two spherical balls of wood, one of radius r and a second one of radius R. A circular hole is bored through the center of each ball and the resulting napkin rings have the same height h. Which napkin ring contains more wood? How much more?

(13) A right circular cone with height 1 meter and base radius r is to be separated into three pieces of equal volume by cutting twice parallel to the base. At what heights should the cuts be made?

(14) A drinking cup filled with water has the shape of a right circular cone with height h and semivertical angle θ. A ball is placed in the cup displacing some of the water. What is the radius of the ball that causes the greatest volume of water to spill out of the cup?

2.12 HW12 Fall 2006: Due November 27, 2006 (Lengths, surface areas, averages)

Problem A. Length of a plane curve.

(1) Use integration to show that the circumference of a circle of radius r is $2\pi r$.

(2) Find the length of the curve $y = x^{2/3}$ between $x = -1$ and $x = 8$.

(3) Find the total length of the curve determined by the equations $x = a\cos^3\theta$ and $y = a\sin^3\theta$.

(4) Find the length of the curve $y = (1/3)(x^2+2)^{3/2}$ from $x = 0$ to $x = 3$.

(5) Find the length of the curve $y = x^{3/2}$ from $(0,0)$ to $(4,8)$.

(6) Find the length of the curve $9x^2 = 4y^3$ from $(0,0)$ to $(2\sqrt{3},3)$.

(7) Find the length of the curve $y = (1/3)x^3 + 1/4x$ from $x = 1$ to $x = 3$.

(8) Find the length of the curve $x = y^4/4 + 1/8y^2$ from $y = 1$ to $y = 2$.

(9) Find the length of the curve $(y+1)^2 = 4x^3$ from $x = 0$ to $x = 1$.

(10) Find the distance traveled between $t = 0$ and $t = \pi/2$ by a particle $P(x,y)$ whose position at time t is given by $x = a\cos t + at\sin t$, $y = a\sin t - at\cos t$, where a is a positive constant.

(11) Find the length of the curve $x = t - \sin t$, $y = 1 - \cos t$, $0 \le t \le 2\pi$.

(12) Find the distance traveled by the particle $P(x,y)$ between $t = 0$ and $t = 4$ if the position at time t is given by $x = t^2/2$, $y = (1/3)(2t+1)^{3/2}$.

(13) The position of the particle $P(x,y)$ at time t is given by $x = (1/3)(2t+3)^{3/2}$, $y = t^2/2 + t$. Find the distance it travels between $t = 0$ and $t = 3$.

(14) Find the length of the curve $x = (3/5)y^{5/3} - (3/4)y^{1/3}$ from $y = 0$ to $y = 1$.

(15) Find the length of the curve $y = (2/3)x^{3/2} - (1/2)x^{1/2}$ from $x = 0$ to $x = 4$.

(16) Consider the curve $y = f(x)$, $x \ge 0$, such that $f(0) = a$. Let $s(x)$ denote the arc length along the curve from $(0,a)$ to $(x, f(x))$. Find $f(x)$ if $s(x) = Ax$. What are the permissible values of A?

(17) Consider the curve $y = f(x)$, $x \ge 0$, such that $f(0) = a$. Let $s(x)$ denote the arc length along the curve from $(0,a)$ to $(x, f(x))$. Is it possible for $s(x)$ to equal x^n with $n > 1$? Give a reason for your answer.

Problem B. Surface area.

(1) Use integration to show that the surface area of a sphere of radius r is $4\pi r^2$.

(2) Find the surface area of the bagel obtained by rotating the circle $x^2 + y^2 = r^2$ about the line $y = -r$.

(3) Find the surface area of the solid generated by rotating the portion of the curve $y = (1/3)(x^2+2)^{3/2}$ between $x = 0$ and $x = 3$ about the x-axis.

(4) Find the area of the surface generated by rotating the arc of the curve $y = x^3$ between $x = 0$ and $x = 1$ about the x-axis.

(5) Find the area of the surface generated by rotating the arc of the curve $y = x^2$ between $(0, 0)$ and $(2, 4)$ about the y-axis.

(6) The arc of the curve $y = x^3/3 + 1/4x$ from $x = 1$ to $x = 3$ is rotated about the line $y = -1$. Find the surface area generated.

(7) The arc of the curve $x = y^4/4 + 1/8y^2$ from $y = 1$ to $y = 2$ is rotated about the x-axis. Find the surface area generated.

(8) Find the area of the surface obtained by rotating about the y-axis the curve $y = x^2/2 + 1/2$, $0 \le x \le 1$.

(9) Find the area of the surface obtained by rotating the curve determined by $x = a\cos^3\theta$, $y = a\sin^3\theta$ about the x-axis.

(10) The curve described by the particle $P(x, y)$ with position given by $x = t + 1$, $y = t^2/2 + t$, from $t = 0$ to $t = 4$ is rotated about the y-axis. Find the surface area that is generated.

(11) The loop of the curve $9x^2 = y(3-y)^2$ is rotated about the x-axis. Find the surface area generated.

(12) Find the surface area generated when the curve $y = (2/3)x^{3/2} - (1/2)x^{1/2}$ from $x = 0$ to $x = 4$ is rotated about the y-axis.

(13) Find the surface area generated when the curve $x = (3/5)y^{5/3} - (3/4)y^{1/3}$ from $y = 0$ to $y = 1$ is rotated about the line $y = -1$.

Problem C. Center of mass.

(1) Find the center of mass of a thin homogeneous triangular plate of base b and height h.

(2) A thin homogeneous wire is bent to form a semicircle of radius r. Find its center of mass.

(3) Find the center of mass of a solid hemisphere of radius r if its density at any point P is proportional to the distance between P and the base of the hemisphere.

(4) Find the center of mass of a thin homogeneous plate covering the area in the first quadrant of the circle $x^2 + y^2 = a^2$.

(5) Find the center of mass of a thin homogeneous plate covering the area bounded by the parabola $y = h^2 - x^2$ and the x-axis.

(6) Find the center of mass of a thin homogeneous plate covering the “triangular” area in the first quadrant between the circle $x^2 + y^2 = a^2$ and the lines $x = a$, $y = a$.

(7) Find the center of mass of a thin homogeneous plate covering the area between the x-axis and the curve $y = \sin x$ between $x = 0$ and $x = \pi$.

(8) Find the center of mass of a thin homogeneous plate covering the area between the y-axis and the curve $x = 2y - y^2$.

(9) Find the distance, from the base, of the center of mass of a thin triangular plate of base b and height h if its density varies as the square root of the distance from the base.

(10) Find the distance, from the base, of the center of mass of a thin triangular plate of base b and height h if its density varies as the square of the distance from the base.

(11) Find the center of mass of a homogeneous right circular cone.

(12) Find the center of mass of a solid right circular cone if the density varies as the distance from the base.

(13) A thin homogeneous wire is bent to form a semicircle of radius r. Suppose that the density is $d = k \sin\theta$, where k is a constant. Find the center of mass.

(14) Find the center of gravity of a solid hemisphere of radius r.

(15) Find the center of gravity of a thin hemispherical shell of inner radius r and thickness t.

(16) Find the center of gravity of the area bounded by the x-axis and the curve $y = c^2 - x^2$.

(17) Find the center of gravity of the area bounded by the y-axis and the curve $x = y - y^3$, $0 \le y \le 1$.

(18) Find the center of gravity of the area bounded by the curve $y = x^2$ and the line $y = 4$.

(19) Find the center of gravity of the area bounded by the curve $y = x - x^2$ and the line $x + y = 0$.

(20) Find the center of gravity of the area bounded by the curve $x = y^2 - y$ and the line $y = x$.

(21) Find the center of gravity of a solid right circular cone of altitude h and base radius r.

(22) Find the center of gravity of the solid generated by rotating, about the y axis, the area bounded by the curve $y = x^2$ and the line $y = 4$.

(23) The area bounded by the curve $x = y^2 - y$ and the line $y = x$ is rotated about the x axis. Find the center of gravity of the solid thus generated.

(24) Find the center of gravity of a very thin right circular conical shell of base radius r and height h.

(25) Find the center of gravity of the surface area generated by rotating about the line $x = -r$, the arc of the circle $x^2 + y^2 = r^2$ that lies in the first quadrant.

(26) Find the moment, about the x-axis of the arc of the parabola $y = \sqrt{x}$ lying between $(0,0)$ and $(4,2)$.

(27) Find the center of gravity of the arc length of one quadrant of a circle.

Problem D. Average value of a function.

(1) Explain how to derive a formula for the average value of a function $f(x)$ as x ranges from a to b.

(2) Compute the average of the numbers $1, 2, 3, \ldots, 100$.

(3) Compute the average of the numbers $9, 10, 11, \ldots, 243$.

(4) Compute the average of the numbers $-9, -6, -3, 0, 3, 6, 9, \ldots, 243$.

(5) Compute the average of the numbers $3^0, 3^1, 3^2, \ldots, 3^{50}$.

(6) Explain why the average of the numbers $1, 1/2, 1/3, \ldots, 1/100$ is more than .04615 but less than .04705.

(7) Explain why the average of the numbers $1, e^{-1}, e^{-2}, \ldots, e^{-50}$ is more than .02 but less than .04.

(8) Show that the average of the numbers $1, e^{-1}, e^{-2}, \ldots, e^{-50}$ is equal to .031639534 (up to 7 decimal places).

(9) Explain why the average of the numbers $1, 1/4, 1/9, 1/16, 1/25, \ldots, 1/10000$ is more than .00333433 but less than .01333333.

(10) Graph $f(x) = \sin x$, $0 \le x \le \pi/2$, and find its average value. Indicate the average value on the graph. Draw a rectangle with base $0 \le x \le \pi/2$ and with area equal to the area under the graph of $f(x)$.

(11) Graph $f(x) = \sin x$, $0 \le x \le 2\pi$, and find its average value. Indicate the average value on the graph. Draw a rectangle with base $0 \le x \le 2\pi$ and with area equal to the area under the graph of $f(x)$.

(12) Graph $f(x) = \sin^2 x$, $0 \le x \le \pi/2$, and find its average value. Indicate the average value on the graph. Draw a rectangle with base $0 \le x \le \pi/2$ and with area equal to the area under the graph of $f(x)$.

(13) Graph $f(x) = \sin^2 x$, $\pi \le x \le 2\pi$, and find its average value. Indicate the average value on the graph. Draw a rectangle with base $\pi \le x \le 2\pi$ and with area equal to the area under the graph of $f(x)$.

(14) Graph $f(x) = \sqrt{2x+1}$, $4 \le x \le 12$, and find its average value. Indicate the average value on the graph. Draw a rectangle with base $4 \le x \le 12$ and with area equal to the area under the graph of $f(x)$.

(15) Graph $f(x) = 1/2 + (1/2)\cos 2x$, $0 \le x \le \pi$, and find its average value. Indicate the average value on the graph. Draw a rectangle with base $0 \le x \le \pi$ and with area equal to the area under the graph of $f(x)$.

(16) Graph $f(x) = \alpha x + \beta$, $a \le x \le b$, where α, β, a and b are constants, and find its average value. Draw a rectangle with base $a \le x \le b$ and with area equal to the area under the graph of $f(x)$.

(17) A mailorder company receives 600 cases of athletic socks every 60 days. The number of cases on hand t days after the shipment arrives is $I(t) = 600 - 20\sqrt{15t}$. Find the average daily inventory. If the holding cost for one case is 1/2 cent per day, find the total daily holding cost.

(18) Find the average value of y with respect to x for that part of the curve $y = \sqrt{ax}$ between $x = a$ and $x = 3a$.

(19) Find the average value of y^2 with respect to x for the curve $ay = b\sqrt{a^2 - x^2}$ between $x = 0$ and $x = a$. Also find the average value of y with respect to x^2 for $0 \le x \le a$.

(20) A point moves in a straight line during the time from $t = 0$ to $t = 3$ according to the law $s = 120t - 16t^2$.

(a) Find the average value of the velocity, with respect to time, for these three seconds.

(b) Find the average value of the velocity, with respect to the distance s, for these three seconds.

(21) The temperature in a certain city t hours after 9 am was approximated by the function $T(t) = 50 + 14\sin(\pi t/12)$. Find the average temperature during the period from 9 am to 9 pm.

2.13 HW13 Fall 2006: Due December 8, 2006 (Motion, exp applications)

Problem A. Motion.

(1) What do distance, speed and acceleration have to do with calculus? Explain thoroughly.

(2) A particle, starting from a fixed point P, moves in a straight line. Its distance from P after t seconds is $s = 11 + 5t + t^3$ meters. Find the distance, velocity and acceleration of the particle after 4 seconds, and find the distance it travels during the 4th second.

(3) The displacement of a particle at time t is given by $x = 2t^3 - 5t^2 + 4t + 3$. Find

(i) the time when the acceleration is $8\mathrm{cm/s}^2$, and

(ii) the velocity and displacement at that instant.

(4) A particle moves along a straight line according to the law $s = t^3 - 6t^2 + 19t - 4$. Find

(i) its displacement and acceleration when its velocity is 7m/s, and

(ii) its displacement and velocity when its acceleration is $6\mathrm{m/s}^2$.

(5) A particle moves along a straight line so that after t seconds its distance from a fixed point P on the line is s meters, where $s = t^3 - 4t^2 + 3t$. Find

(i) when the particle is at P, and

(ii) its velocity and acceleration at these instants.

(6) A particle moves along a straight line according to the law $s = at^2 - 2bt + c$, where a, b, c are constants. Prove that the acceleration of the particle is constant.

(7) If a particle moves along a straight line so that the distance described is proportional to the square of the time elapsed prove that

(i) the velocity is proportional to the time, and

(ii) the rate of increase of the velocity is constant.

(8) A car starts from rest and moves a distance s meters in t seconds, where $s = a\cos t + b\sin t$. Show that the acceleration at time t is the negative of the distance traveled in t seconds.

(9) The distance s in meters traveled by a particle in t seconds is given by $s = ae^t + be^{-t}$. Show that the acceleration of the particle at time t is equal to the distance the particle travels in t seconds.

(10) A particle moves in a line according to the law $s = at^2 + bt + c$ where a, b, c are constants and s is the distance of the particle from a fixed point P after t seconds. Initially the particle is 10 cm away from P and its initial velocity is 12 cm/s. If the particle moves with a uniform acceleration of $4\mathrm{cm/s}^2$ find the distance travelled by it during the 7th second.

(11) The displacement of a particle moving in a straight line is $x = 2t^3 - 9t^2 + 12t + 1$ meters at time t. Find

(i) the velocity and acceleration at $t = 1$ second,

(ii) the time when the particle stops momentarily, and

(iii) the distance between two stops.

(12) A particle moves in a straight line according to the law $s = 6t^3 + 20t^2 + 9t$ where s is in centimeters and t is in seconds. Find the initial velocity and acceleration of the particle.

(13) A particle is moving on a line according to the law $s = \tan^{-1} t + at^2 + bt + c$ where a, b, c are constants. Given that, at $t = 1$, s is 3.5 m, the velocity is 3 m/s, and the acceleration is 1.5 m/s^2, find the values of a, b and c.

(14) The height of a stone thrown vertically upwards is given by $s = 49t - 4.9t^2$ where s is in meters and t is in seconds. Find the maximum height reached by the stone.

(15) A particle is moving in a vertical line according to the equation $x = 100t - 4.9t^2$ where x is in meters and t is in seconds. Find its velocity at $t = 1$. A what time is its velocity 0? What is the maximum value of s?

(16) An arrow shot vertically upwards moves according to the formula $s = 49t - 4.9t^2$ where s is in meters and t is in seconds. Find the time that it takes to reach a height of 117.6 meters. What is its velocity after 8 seconds? How long before it hits the ground?

(17) A ball projected vertically upwards has equation of motion $s = ut - 4.9t^2$ where s is in meters and t is in seconds and u is the initial velocity. If the maximum height reached by the ball is 44.1 meters find the value of u.

(18) A shot fired vertically upwards is known to be at a point A at the end of 2 seconds and also again there after 3 more seconds. The equation of motion of the bullet is $s = ut - 4.9t^2$ where s is in meters, t is in seconds and u is the initial velocity of the bullet. Find the height of the point A above the point where the shot is fired.

(19) A particle falls from the top of a tower and in the last second before it hits the ground it falls 9/25 of the total height of the tower. Find the height of the tower.

Problem B. Applications of the exponential function.

(1) Find all functions $y(t)$ such that $\dfrac{dy}{dt} = ky$ and $y(a) = b$, where k, a and b are constants.

(2) What is the idea behind computing radioactive decay? Explain why exponential functions arise.

(3) What is the idea for computing population growth? Explain why exponential functions are used.

(4) Explain why exponential functions are used to compute money owed on loans. Explain why the limit $\lim_{n\to\infty} \left(1 + \dfrac{a}{n}\right)^n$ is used for computing interest.

(5) What is the idea for computing temperatures of objects during cooling? Explain why exponential functions appear.

(6) If you borrow \$500 on your credit card at 14% interest find the amounts due at the end of 2 years if the interest is compounded (a) annually, (b) quarterly, (c) monthly, (d) daily, (e) hourly, (f) continuously.

(7) If you buy a \$24,000 car and put 15% down and take out a 3 year loan at 7% per year compute how much your monthly payments are if the interest is compounded continuously.

(8) If you buy a \$24,000 car and put 15% down and take out a 3 year loan at 7% per year compute how much your payment would be if you paid it all off in one big payment at the end of 3 years.

(9) If you buy a \$24,000 car and put 15% down and take out a 3 year loan at 7% per year compute how much interest you pay during the first month.

(10) If you buy a \$200,000 home and put 10% down and take out a 30 year fixed rate mortgage at 8% per year compute how much your monthly payments are if the interest is compounded continuously.

(11) If you buy a \$200,000 home and put 10% down and take out a 30 year fixed rate mortgage at 8% per year compute how much your payment would be if you paid it all off in one big payment at the end of 30 years.

(12) If you buy a \$200,000 home and put 10% down and take out a 30 year fixed rate mortgage at 8% per year compute how much interest you pay during the first month.

(13) A roast turkey is taken from an oven when its temperature has reached 185° F and is placed on a table in a room where the temperature is 75° F. Assume that it cools at a rate proportional to the difference between its current temperature and the room temperature.

(a) If the temperature of the turkey is 150° F after half an hour, what is the temperature after 45 min?
(b) When will the turkey have cooled to 100° F?

(14) Radiocarbon dating works on the principle that ${}^{14}C$ decays according to radioactive decay with a half life of 5730 years. A parchment fragment was discovered that had about 74% as much ${}^{14}C$ as does plant material on earth today. Estimate the age of the parchment.

(15) After 3 days a sample of radon-222 decayed to 58% of its original amount.

(a) What is the half life of radon-222?
(b) How long would it take the sample to decay to 10% of its original amount?

(16) Polonium-210 has a half life of 140 days.

(a) If a sample has a mass of 200 mg find a formula for the mass that remains after t days.
(b) Find the mass after 100 days.
(c) When will the mass be reduced to 10 mg?
(d) Sketch the graph of the mass function.

(17) If the bacteria in a culture increase continuously at a rate proportional to the number present, and the initial number is N_0, find the number at time t.

(18) If a radioactive substance disintegrates at a rate proportional to the amount present how much of the substance remains at time t if the initial amount is Q_0?

(19) If an object cools at a rate proportional to the the difference between its temperature and the temperature of its surroundings, the initial temperature of the object is T_0 and the temperature of the surroundings are a constant temperature S what is the temperature of the object at time t?

(20) Current agricultural experts believe that the worlds farms can feed about 10 billion people. The 1950 world population was 2517 million and the 1992 world population was 5.4 billion. When can we expect to run out of food?

(21) Suppose that the GNP in a country is increasing at an annual rate of 4 percent. How many years, at that rate of growth, are required to double the present GNP?

(22) What percent of a sample of ${}^{226}_{88}\mathrm{Ra}$ remains after 100 years? The half life of ${}^{226}_{88}\mathrm{Ra}$ is 1620 years.

(23) A sample contains 4.6 mg of ${}^{131}_{53}\mathrm{I}$. How many mg will remain after 3.0 days? The half life of ${}^{131}_{53}\mathrm{I}$ is 8.0 days.

(24) The majority of naturally occuring rhenium is ${}^{187}_{75}\mathrm{Re}$, which is radioactive and has a half life of 7×10^{10} years. In how many years will 5% of the earth's ${}^{187}_{75}\mathrm{Re}$ decompose?

(25) A piece of paper from the Dead Sea scrolls was found to have a ${}^{14}_{6}\mathrm{C}/{}^{12}_{6}\mathrm{C}$ ratio 79.5% of that in a plant living today. Estimate the age of the paper, given that the half life of ${}^{14}_{6}\mathrm{C}$ is 5720 years.

(26) The charcoal from ashes found in a cave gave a ${}^{14}\mathrm{C}$ activity of 8.6 counts per gram per minute. Calculate the age of the charcoal (wood from a growing tree gives a comparable count of 15.3). For ${}^{14}\mathrm{C}$ the half life is 5720 years.

(27) In a certain activity meter, a pure sample of ${}^{90}_{38}\mathrm{Sr}$ has an activity (rate of decay) of 1000.0 disintegrations per minute. If the activity of this sample after 2.00 years is 953.2 disintegrations per minute, what is the half life of ${}^{90}_{38}\mathrm{Sr}$.

(28) A sample of a wooden artifact from an Egyptian tomb has a ${}^{14}\mathrm{C}/{}^{12}\mathrm{C}$ ratio which is 54.2 % of that of freshly cut wood. In approximately what year was the old wood cut? The half life of ${}^{14}\mathrm{C}$ is 5720 years.

Problem C. Logarithmic differentiation.

(1) Find $\dfrac{dy}{dx}$ when $y = \dfrac{(x+2)^{5/2}}{(x+6)^{1/2}(x+3)^{7/2}}$.

(2) Find $\dfrac{dy}{dx}$ when $y = (x+1)^2(x-2)^3(x+4)\ln x$.

(3) Find $\dfrac{dy}{dx}$ when $y = \sqrt{\dfrac{(x-a)(x-b)}{(x-p)(x-q)}}$.

(4) Find $\dfrac{dy}{dx}$ when $y = (\sin x)^{\ln x}$.

(5) Find $\dfrac{dy}{dx}$ when $y = (\sin x)^{\cos x}$.

(6) Find $\dfrac{dy}{dx}$ when $y = (\sin x)^{\tan x} + (\tan x)^{\sin x}$.

(7) Find $\dfrac{dy}{dx}$ when $x^y + y^x = a$.

(8) Find $\dfrac{dy}{dx}$ when $x + y = x^y$.

(9) Find $\dfrac{dy}{dx}$ when $(\cos x)^y = (\sin y)^x$.

(10) Find $\dfrac{dy}{dx}$ when $y = a^x + e^{\tan x} + (\cot x)^{\cos x}$.

(11) Find $\dfrac{dy}{dx}$ when $y = (\tan x)^{\cot x}$.

(12) Find $\dfrac{dy}{dx}$ when $y = x^x + x^{1/x}$.

(13) Find $\dfrac{dy}{dx}$ when $y = (\sec x)^{\csc x} + (\csc x)^{\sec x}$.

(14) Find $\dfrac{dy}{dx}$ when $y = \log_y x$.

(15) Find $\dfrac{dy}{dx}$ when $y = (\cos x)^{(\cos x)^{(\cos x)^{\cdot^{\cdot^{\cdot}}}}}$.

(16) Find $\dfrac{dy}{dx}$ when $y = x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}$.

(17) Find $\dfrac{dy}{dx}$ when $y = x^{y^x}$.

(18) Find $\dfrac{dy}{dx}$ when $y = x^{1/x}$.

(19) Find $\dfrac{dy}{dx}$ when $y = \left(\dfrac{x^x + x^{-x}}{x^x - x^{-x}}\right)^{1/2}$.

(20) If $x^m y^n = (x+y)^{m+n}$ show that $\dfrac{dy}{dx} = \dfrac{y}{x}$.

Problem D. L'Hopital's rule.

(1) State L'Hôpital's rule and give an example which illustrates how it is used.

(2) Explain why L'Hôpital's rule works. Hint: Expand the numerator and the denominator in terms of Δx.

(3) Give three examples which illustrate that a limit problem that looks like it is coming out to $0/0$ could be really getting closer and closer to almost anything and must be looked at a different way.

(4) Give three examples which illustrate that a limit problem that looks like it is coming out to 1^∞ could be really getting closer and closer to almost anything and must be looked at a different way.

(5) Give three examples which illustrate that a limit problem that looks like it is coming out to 0^0 could be really getting closer and closer to almost anything and must be looked at a different way.

(6) Evaluate $\lim_{x\to 1} \dfrac{x^2+3x-4}{x-1}$. You may use L'Hôpital's rule.

(7) Evaluate $\lim_{x\to 1} \dfrac{x^a-1}{x^b-1}$. You may use L'Hôpital's rule.

(8) Evaluate $\lim_{x\to 1} \dfrac{\ln x}{x-1}$. You may use L'Hôpital's rule.

(9) Evaluate $\lim_{x\to \pi} \dfrac{\tan x}{x}$. You may use L'Hôpital's rule.

(10) Evaluate $\lim_{x\to 3\pi/2} \dfrac{\cos x}{x-(3\pi/2)}$. You may use L'Hôpital's rule.

(11) Evaluate $\lim_{x\to 0^+} \dfrac{\ln x}{\sqrt{x}}$. You may use L'Hôpital's rule.

(12) Evaluate $\lim_{x\to \infty} \dfrac{(\ln x)^3}{x^2}$. You may use L'Hôpital's rule.

(13) Evaluate $\lim_{x\to 0} \dfrac{6^x-2^x}{x}$. You may use L'Hôpital's rule.

(14) Evaluate $\lim_{x\to 0} \dfrac{e^x-1-x-(x^2/2)}{x^3}$. You may use L'Hôpital's rule.

(15) Evaluate $\lim_{x\to 0} \dfrac{\sin x - x}{x^3}$. You may use L'Hôpital's rule.

(16) Evaluate $\lim_{x\to \infty} \dfrac{\ln(1+e^x)}{5x}$. You may use L'Hôpital's rule.

(17) Evaluate $\lim_{x\to 0} \dfrac{\tan \alpha x}{x}$. You may use L'Hôpital's rule.

(18) Evaluate $\lim_{x\to 0} \dfrac{2x-\sin^{-1} x}{2x+\cos^{-1} x}$. You may use L'Hôpital's rule.

(19) Evaluate $\lim_{x\to 0^+} \sqrt{x}\ln x$. You may use L'Hôpital's rule.

(20) Evaluate $\lim_{x\to\infty} e^{-x}\ln x$. You may use L'Hôpital's rule.

(21) Evaluate $\lim_{x\to\infty} x^3e^{-x^2}$. You may use L'Hôpital's rule.

(22) Evaluate $\lim_{x\to\pi} (x-\pi)\cot x$. You may use L'Hôpital's rule.

(23) Evaluate $\lim_{x\to 0} x^{-4} - x^{-2}$. You may use L'Hôpital's rule.

(24) Evaluate $\lim_{x\to 0} x^{-1} - \csc x$. You may use L'Hôpital's rule.

(25) Evaluate $\lim_{x\to\infty} x - \sqrt{x^2-1}$. You may use L'Hôpital's rule.

(26) Evaluate $\lim_{x\to\infty} \left(\frac{x^3}{x^2-1} - \frac{x^3}{x^2+1}\right)$. You may use L'Hôpital's rule.

(27) Evaluate $\lim_{x\to 0^+} x^{\sin x}$. You may use L'Hôpital's rule.

(28) Evaluate $\lim_{x\to 0}(1-2x)^{1/x}$. You may use L'Hôpital's rule.

(29) Evaluate $\lim_{x\to\infty} (1+3/x+5/x^2)^x$. You may use L'Hôpital's rule.

(30) Evaluate $\lim_{x\to\infty} x^{1/x}$. You may use L'Hôpital's rule.

(31) Evaluate $\lim_{x\to 0^+} (\cot x)^{\sin x}$. You may use L'Hôpital's rule.

(32) Evaluate $\lim_{x\to\infty} \left(\frac{x}{x+1}\right)^x$. You may use L'Hôpital's rule.

(33) Evaluate $\lim_{x\to 0^+} (-\ln x)^x$. You may use L'Hôpital's rule.

2.14 HW14 Fall 2006: December 15, 2006 (Review)

Problem A. Derivatives with all functions mixed together.

(1) Find $\frac{dy}{dx}$ when $y = \frac{2\tan x}{\tan x + \cos x}$.

(2) Find $\frac{dy}{dx}$ when $y = \sqrt{x \sin x}$.

(3) Find $\frac{dy}{dx}$ when $y = \frac{x + \sin 2x}{\cos 3x}$.

(4) Find $\frac{dy}{dx}$ when $y = e^{5x} \ln(\sec x)$.

(5) Find $\frac{dy}{dx}$ when $y = \frac{x^5}{\sin^{-1} 2x}$.

(6) Find $\frac{dy}{dx}$ when $y = \sin x^2 - \frac{\tan x}{1+x^2}$.

(7) Find $\frac{dy}{dx}$ when $y = (\tan\sqrt{x} + x^2 - \sin x)^3$.

(8) Find $\frac{dy}{dx}$ when $y = \frac{\sin^3 x \cos^3 x}{\cos 3x}$.

(9) Find $\frac{dy}{dx}$ when $y = e^x \tan x + \frac{\ln x}{\sin x}$.

(10) Find $\frac{dy}{dx}$ when $y = 2a^x \ln x$.

(11) Find $\frac{dy}{dx}$ when $y = \frac{x^{3/2} + 1}{3\sqrt{x}}$.

(12) Find $\frac{dy}{dx}$ when $y = \frac{1 - x^2}{x^2 - x + 1}$.

(13) Find $\frac{dy}{dx}$ when $y = \sqrt{1 + \ln x \ln \sin x}$.

(14) Find $\frac{dy}{dx}$ when $y = 7x^{1/2} + 5x^{-7/2} + \sin^{-1}(x^4) - \ln\cot x$.

(15) Find $\frac{dy}{dx}$ when $y = \sin^2 x \cos^3 x$.

(16) Find $\frac{dy}{dx}$ when $y = \sin mx \cos nx$.

(17) Find $\frac{dy}{dx}$ when $y = \sin^m x \cos^n x$.

(18) Find $\dfrac{dy}{dx}$ when $y = \cos^{-1}(1-2x^2)$.

(19) Find $\dfrac{dy}{dx}$ when $y = \sin^{-1}(3x-4x^3)$.

(20) Find $\dfrac{dy}{dx}$ when $y = \dfrac{\sqrt{a+x}-\sqrt{a-x}}{\sqrt{a+x}+\sqrt{a-x}}$.

(21) Find $\dfrac{dy}{dx}$ when $y = (1+x)(1+2x)(1+3x)(1+4x)$.

(22) Find $\dfrac{dy}{dx}$ when $y = \tan^2\sqrt{1-x^2}$.

(23) Find $\dfrac{dy}{dx}$ when $y = \dfrac{\tan x}{x^2}$.

(24) Find $\dfrac{dy}{dx}$ when $y = \dfrac{e^{2x}}{\ln x}$.

(25) Find $\dfrac{dy}{dx}$ when $y = \dfrac{e^{x^2}\tan^{-1}x}{\sqrt{1+x^2}}$.

(26) Find $\dfrac{dy}{dx}$ when $y = e^{\sqrt{x}+2} - e^{\sqrt{x+2}}$.

(27) Find $\dfrac{dy}{dx}$ when $y = 7^{x^2+2x}$.

(28) Find $\dfrac{dy}{dx}$ when $y = \cot^2(e^{3x}x^x)$.

(29) Find $\dfrac{dy}{dx}$ when $y = \tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)$.

(30) Find $\dfrac{dy}{dx}$ when $y = \ln(\tan^{-1}x)$.

(31) Find $\dfrac{dy}{dx}$ when $y = \csc^{-1}\left(\dfrac{1+x^2}{2x}\right)$.

(32) Find $\dfrac{dy}{dx}$ when $y = \tan^{-1}\left(\dfrac{x}{\sqrt{1-x^2}}\right)$.

(33) Find $\dfrac{dy}{dx}$ when $y = \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$.

(34) Find $\dfrac{dy}{dx}$ when $y = \sin^{-1}x + \sin^{-1}\sqrt{1-x^2}$.

(35) Find $\dfrac{dy}{dx}$ when $y = \sec^{-1}\left(\dfrac{x^2+1}{x^2-1}\right)$.

(36) Find $\dfrac{dy}{dx}$ when $y = x\sin^{-1}x + \sqrt{1-x^2}$.

(37) Find $\dfrac{dy}{dx}$ when $y = x\cos^{-1}2x - \frac{1}{2}\sqrt{1-4x^2}$.

(38) Find $\dfrac{dy}{dx}$ when $y = \frac{1}{2}\tan^{-1}\left(\frac{1}{2}\tan(x/2)\right)$.

(39) Find $\dfrac{dy}{dx}$ when $y = \tan^{-1}(\sec x + \tan x)$.

(40) Find $\dfrac{dy}{dx}$ when $y = \dfrac{x\cos^{-1}x}{\sqrt{1-x^2}}$.

(41) Find $\dfrac{dy}{dx}$ when $y = \dfrac{1}{2}x\sqrt{a^2-x^2} + \dfrac{a^2}{2}\sin^{-1}(x/a)$.

(42) Find $\dfrac{dy}{dx}$ when $y = \sin^{-1}(2x\sqrt{1-x^2})$.

(43) Find $\dfrac{dy}{dx}$ when $y = \tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$.

(44) Find $\dfrac{dy}{dx}$ when $y = x^3\sin 2x + \dfrac{\cos x}{x+1}$.

(45) Find $\dfrac{dy}{dx}$ when $y = x^4\sin 2x + \dfrac{x^2}{x^3+1}$.

(46) Find $\dfrac{dy}{dx}$ when $y = x^{\ln x}$.

(47) Find $\dfrac{dy}{dx}$ when $y = (\tan x)^{\cot x}$.

(48) Find $\dfrac{dy}{dx}$ when $y = (\tan x)^{\cot x} + (\cot x)^{\tan x} + x^{\cos^{-1}x}$.

(49) Find $\dfrac{dy}{dx}$ when $y = (\sin x)(e^x)(\ln x)(x^x)(x^{\cos^{-1}x})$.

(50) Find $\dfrac{dy}{dx}$ when $x^y = y^x$.

(51) Find $\dfrac{dy}{dx}$ when $x^{2/3} + y^{2/3} = a^{2/3}$.

(52) Find $\dfrac{dy}{dx}$ when $e^{xy} - 4xy = 0$.

(53) Find $\dfrac{dy}{dx}$ when $xy = \sin(x+y)$.

(54) Find $\dfrac{dy}{dx}$ when $\dfrac{x^m}{a^m} + \dfrac{y^m}{b^m} = 1$.

(55) Find $\dfrac{dy}{dx}$ when $x^m y^n = (x+y)^{m+n}$.

(56) Find $\dfrac{dy}{dx}$ where $y \ln x = x - y$.

Problem B. Integrals with mixed functions.

(1) $\displaystyle\int \frac{\sin^{-1} x}{\sqrt{1-x^2}}\ dx$

(2) $\displaystyle\int \frac{\sin(2\tan^{-1} x)}{1+x^2}\ dx$

(3) $\displaystyle\int \frac{\cos(\ln x)}{x}\ dx$

(4) $\displaystyle\int \frac{\csc^2(\ln x)}{x}\ dx$

(5) $\displaystyle\int e^{\tan x}\sec^2 x\ dx$

(6) $\displaystyle\int e^{\cos^2 x}\sin 2x\ dx$

(7) $\displaystyle\int \cot x \ln(\sin x)\ dx$

(8) $\displaystyle\int \frac{\cot x}{\ln(\sin x)}\ dx$

(9) $\displaystyle\int \sec x \ln(\sec x + \tan x)\ dx$

(10) $\displaystyle\int \frac{x\tan^{-1} x^2}{1+x^4}\ dx$

(11) $\displaystyle\int \frac{x\sin^{-1} x^2}{\sqrt{1-x^4}}\ dx$

(12) $\displaystyle\int \frac{1}{\sqrt{1-x^2}\sin^{-1} x}\ dx$

(13) $\displaystyle\int \frac{1+\tan x}{x+\ln(\sec x)}\ dx$

(14) $\displaystyle\int \frac{\sec x \csc x}{\ln(\tan x)}\ dx$

(15) $\displaystyle\int \frac{dx}{x\cos^2(1+\ln x)}$

(16) $\displaystyle\int e^{-x}\csc^2(2e^{-x}+5)\ dx$

(17) $\displaystyle\int x^2 e^{x^3}\cos e^{x^3}\ dx$

(18) $\displaystyle\int \frac{e^{m\tan^{-1}x}}{1+x^2}\ dx$

(19) $\displaystyle\int \frac{(x+1)e^x}{\cos^2(xe^x)}\ dx$

(20) $\displaystyle\int \frac{e^{\sqrt{x}}\cos e^{\sqrt{x}}}{\sqrt{x}}\ dx$

Problem C. Areas of regions.

(1) Find the area inside the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

(2) Using integration find the area of the triangle with vertices $(-1,1)$, $(0,5)$ and $(3,2)$.

(3) Graph the region $\{(x,y)\mid 4x^2+9y^2\le 36\}$ and find its area.

(4) Find the area of the region $\{(x,y)\mid y^2\le 8x, x^2+y^2\le 9\}$.

(5) Find the area of the region $\{(x,y)\mid y^2\le x, x^2+y^2\le 2\}$.

(6) Find the area of the region $\{(x,y)\mid x^2+y^2\le 2ax, y^2\ge ax, x\ge 0, y\ge 0\}$.

(7) Find the area of the region $\{(x,y)\mid y^2\le 4x, 4x^2+4y^2\le 9\}$.

(8) Find the area of the region $\{(x,y)\mid x^2+y^2\le 1\le x+y\}$.

(9) Find the area of the region $\{(x,y)\mid 0\le y\le x^2+1, 0\le y\le x+1, 0\le x\le 2\}$.

(10) Find the area of the region $\{(x,y)\mid x^2\le y\le |x|\}$.

Problem D. Different types of volume problems.

(1) A solid is generated by rotating, about the x-axis, the area bounded by the curve $y=f(x)$, the x-axis, and the lines $x=a$, $x=b$. Its volume, for all $b>a$, is b^2-ab. Find $f(x)$.

(2) A solid is generated by rotating the curve $y=f(x)$, $0\le x\le a$, about the x-axis. Its volume, for all a, is a^2+a. Find $f(x)$.

(3) The area bounded by the curve $y^2=4x$ and the straight line $y=x$ is rotated about the x-axis. Find the volume generated.

(4) Sketch the area bounded by the curve $y^2=4ax$, the line $x=a$, and the x-axis. Find the volumes generated by rotating this area in each of the following ways:

(a) about the x-axis,

(b) about the line $x = a$.

(c) about the y-axis,

(5) The area bounded by the curve $y = x/\sqrt{x^3+8}$, the x-axis, and the line $x = 2$ is rotated about the y-axis. Compute the volume.

(6) Find the volume of the solid produced by rotating the larger area bounded by $y^2 = x - 1$, $x = 3$ and $y = 1$ about the y-axis.

(7) The area bounded by the curve $y^2 = 4ax$ and the line $x = a$ is rotated about the line $x = 2a$. Find the volume generated.

(8) A twisted solid is generated as follows: We are given a fixed line L in space, and a square of side length s in a plane perpendicular to L. One vertex of the square is on L. As this vertex moves a distance h along L, the square turns through a full revolution, with L as the axis. Find the volume generated.

(9) A twisted solid is generated as follows: We are given a fixed line L in space, and a square of side length s in a plane perpendicular to L. One vertex of the square is on L. As this vertex moves a distance h along L, the square turns through *two* full revolutions, with L as the axis. Find the volume generated.

(10) Two circles have a common diameter and lie in perpendicular planes. A square moves so that its plane is perpendicular to this diameter and its diagonals are chords of the circles. Find the volume generated.

(11) Find the volume generated by rotating the area bounded by the x-axis and one arch of the curve $y = \sin 2x$ about the x-axis.

(12) A round hole of radius $\sqrt{3}$ ft is bored through the center of a solid sphere of radius 2 ft. How much volume is cut out?

2.15 HW 1 Fall 2006: Selected answers

Problem B. Computing with complex numbers

(1) $z = 0.$ (2) $x = 1.$ (3) $5 - 2i$

(4) $-19 + 8i$ (5) $32 + 4i$ (6) $-2.9 + 1.7i$

(7) $-9 - 46i$ (8) $\pm(1 + i)$ (9) $\dfrac{a}{a^2 + b^2} - \dfrac{b}{a^2 + b^2}i.$

(10) $10 - 3i$ (11) $2 + 4i$ (12) $14 - 8i$

(13) $\dfrac{11}{10} - \dfrac{4}{5}i$ (14) $1, i, -1, -i$ (15) $2, 2i, -2, -2i$

(16) $81\left(\dfrac{1+\sqrt{3}i}{2}\right)$, $531141\left(\dfrac{1-\sqrt{3}}{2}\right)$ (17) 2

(18) $2,\ 6,\ 24,\ 120,\ 720$ (19) $2.71828182845904523536...$

2.16 HW 2 Fall 2006: Selected answers

Problem A. Angles

(1) The number π is the circumference of a circle divided by its diameter.

(2) $360^\circ = 2\pi$ radians.

(4) $2\pi r$ (5) $r\theta$ (6) πr^2 (7) $(1/2)\theta r^2$

Problem B. Computing trigonometric functions

(1) $\sin\frac{\pi}{6} = 1/2,\quad \cos\frac{\pi}{6} = \sqrt{3}/2,\quad \tan\frac{\pi}{6} = \sqrt{3}/3,$
$\cot\frac{\pi}{6} = \sqrt{3},\quad \sec\frac{\pi}{6} = 2\sqrt{3}/3,\quad \csc\frac{\pi}{6} = 2.$

(2) $\sin\frac{\pi}{3} = \sqrt{3}/2,\quad \cos\frac{\pi}{3} = 1/2,\quad \tan\frac{\pi}{3} = \sqrt{3},$
$\cot\frac{\pi}{3} = \sqrt{3}/3,\quad \sec\frac{\pi}{3} = 2,\quad \csc\frac{\pi}{3} = 2\sqrt{3}/3.$

(3) $\sin\frac{\pi}{4} = \sqrt{2}/2,\quad \cos\frac{\pi}{4} = \sqrt{2}/2,\quad \tan\frac{\pi}{4} = 1,$
$\cot\frac{\pi}{4} = 1,\quad \sec\frac{\pi}{4} = \sqrt{2},\quad \csc\frac{\pi}{4} = \sqrt{2}.$

(4) $\sin\frac{\pi}{2} = 1,\quad \cos\frac{\pi}{2} = 0,\quad \tan\frac{\pi}{2}$ is undefined,
$\cot\frac{\pi}{2} = 0,\quad \sec\frac{\pi}{2}$ is undefined, $\quad \csc\frac{\pi}{2} = 1.$

(5) $\sin 0 = 0,\quad \cos 0 = 1,\quad \tan 0 = 0,$

$\cot 0$ is undefined, $\sec 0 = 1$, $\csc 0$ is undefined.

(6) $\sin \frac{3\pi}{4} = \sqrt{2}/2$, $\cos \frac{3\pi}{4} = -\sqrt{2}/2$, $\tan \frac{3\pi}{4} = -1$,
$\cot \frac{3\pi}{4} = -1$, $\sec \frac{3\pi}{4} = -\sqrt{2}$, $\csc \frac{3\pi}{4} = \sqrt{2}$.

(7) $\sin \frac{-2\pi}{3} = -\sqrt{3}/2$, $\cos \frac{-2\pi}{3} = -1/2$, $\tan \frac{-2\pi}{3} = \sqrt{3}$,
$\cot \frac{-2\pi}{3} = \sqrt{3}/3$, $\sec \frac{-2\pi}{3} = -2$, $\csc \frac{-2\pi}{3} = -2\sqrt{3}/3$.

(8) $\dfrac{1+\sqrt{3}}{2}$ (9) $\frac{\sqrt{3}}{4}$ (10) 1

2.17 HW 3 Fall 2006: Selected answers

Problem C. Computing some derivatives

(1) $\dfrac{dy}{dx} = 20x + 27.$

(2) $\dfrac{dy}{dx} = \dfrac{3}{2}x^{1/2} + \dfrac{1}{2}x^{-1/2} - \dfrac{1}{2}x^{-3/2} - \dfrac{3}{2}x^{-5/2}.$

(3) $\dfrac{dy}{dx} = (2x-5)(3x-4)^2(30x-61).$

(4) $\dfrac{dy}{dx} = 2ex - \dfrac{3\pi}{x^4} + \dfrac{7}{2}x^{5/2}.$

(5) $\dfrac{dy}{dx} = -\dfrac{2(x-3)}{(x-4)^3}.$

(6) $\dfrac{dy}{dx} = \dfrac{3x^2+10x+12}{(4-x^2)^2}.$

(7) $\dfrac{dy}{dx} = \dfrac{1-x}{(1-2x)^{3/2}}.$

(8) $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{x}(1-\sqrt{x})^2}.$

(9) $\dfrac{dy}{dx} = -\dfrac{1}{(x-1)^2} - \dfrac{1}{(x+3)^2}.$

(10) $\dfrac{dy}{dx} = \dfrac{a}{\sqrt{a^2-x^2}(a+\sqrt{a^2-x^2})}.$

(11) $\dfrac{dy}{dx} = \dfrac{x^2+2x+2}{(x+1)^2}.$

(12) $\dfrac{dy}{dx} = \dfrac{-3}{2\sqrt{x}(x-3)^{3/2}}.$

(13) $\dfrac{dy}{dx} = \dfrac{-2nx^{n-1}}{(x^n-1)^2}.$

(14) $\dfrac{dy}{dx} = \dfrac{2x}{(1-x^2)(1-x^4)^{1/2}}.$

(15) $\dfrac{dy}{dx} = \dfrac{4x^2+1}{x^2(x^2+1)^{3/2}}$.

(16) $\dfrac{dy}{dx} = nu^{n-1}\dfrac{du}{dx}$.

(17) $\dfrac{dy}{dx} = \dfrac{-x}{\sqrt{1-x^2}}$.

Problem D. Correcting derivative identities

(1) $\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$.

(2) $\dfrac{d}{dx}(u+v) = \dfrac{du}{dx} + \dfrac{dv}{dx}$.

(3) $\dfrac{d}{dx}(u \cdot v) = u\dfrac{dv}{dx} + \dfrac{du}{dx}v$.

Problem F. Derivatives at a point

(1) $\left.\dfrac{dy}{dx}\right|_{x=3} = 1476$.

(2) $\left.\dfrac{dy}{dx}\right|_{x=3} = 9$.

Problem G. Derivatives with respect to functions

(1) $\dfrac{dy}{dx} = \dfrac{2t^4+8}{5t^7}$.

(2) $\dfrac{dy}{dx} = \dfrac{1}{(1+x^2)^2}$.

(3) $\dfrac{dy}{dx} = \dfrac{(ad-bc)(c_1x+d_1)^2}{(a_1d_1-b_1c_1)(cx+d)^2}$.

(4) $\dfrac{dy}{dx} = \dfrac{3}{2}x$.

(5) $\dfrac{dy}{dx} = \dfrac{-1}{x^2(1+\sqrt{1-x^4})}$.

(6) $\dfrac{dy}{dx} = \dfrac{1-x^2}{3x^2(1+x^2)^2}$.

(7) $\dfrac{dy}{dx} = \dfrac{x+\sqrt{1-x^2}}{-x}$.

(8) $\dfrac{dy}{dx} = \dfrac{35x^4-22x}{14x-15}$.

Problem H. Derivatives of parametric equations

(1) $\dfrac{dy}{dx} = \dfrac{-1}{t^2}$.

(2) $\dfrac{dy}{dx} = 1/t$.

(3) $\dfrac{dy}{dx} = -1$.

(4) $\dfrac{dy}{dx} = \dfrac{b(t^2-1)}{2at}$.

(5) $\dfrac{dy}{dx} = \dfrac{t^3+2t-t^{-1}}{2}$.

(6) $\dfrac{dy}{dx} = \dfrac{b(1+t^2)}{2at}$.

(7) $\dfrac{dy}{dx} = \dfrac{t(2-t^3)}{1-2t^3}$.

(8) $\dfrac{dy}{dx} = -x/y$.

Problem I. Implicit differentiation

(1) $\dfrac{dy}{dx} = \dfrac{x(2a^2y^2-x^2)}{y(y^2-2a^2x^2)}$.

(2) $\dfrac{dy}{dx} = \dfrac{-b^2x}{a^2y}$.

(3) $\dfrac{dy}{dx} = \dfrac{2axy^2-x^4}{y^4-2ax^2y}$.

(6) $\dfrac{dy}{dx} = -\dfrac{ax+hy+g}{hx+by+f}$.

Problem J. Derivatives with trigonometric functions.

(1) $\dfrac{dy}{dx} = 3\cos(3x+2)$.

(2) $\dfrac{dy}{dx} = 2x^3(\sin x^4)^{-1/2}\cos x^4$.

(3) $\dfrac{dy}{dx} = x^2\cos x + 2x\sin x$.

(4) $\dfrac{dy}{dx} = 2\sin 2x$.

(5) $\dfrac{dy}{dx} = 2x\cos x^2 - \left(\dfrac{(1+x^2)\sec^2 x - 2x\tan x}{(1+x^2)^2}\right)$.

(6) $\dfrac{dy}{dx} = -\left(\dfrac{2x\sin x + 4\sin x + 2\cos x + 2}{(x+2)^2}\right)$.

(7) $\dfrac{dy}{dx} = 2x + \dfrac{\sin x - x\cos x}{\sin^2 x}$.

(8) $\dfrac{dy}{dx} = 2\cos x$.

(9) $\dfrac{dy}{dx} = \dfrac{1}{6}\sec(x/3)\tan(x/3)$.

(10) $\dfrac{dy}{dx} = (\cos x - \sin x)\cos(\sin x + \cos x)$.

(11) $\dfrac{dy}{dx} = -2\csc 2x \cos 2x$.

(12) $\dfrac{dy}{dx} = 2x\left(\cot x + \dfrac{\tan x}{1+x^2}\right) + \dfrac{x^2-1}{(1+x^2)^2}((1+x^2)\sec^2 x - (1+x^2)^2\csc^2 x - 2x\tan x)$.

(13) $\dfrac{dy}{dx} = \dfrac{-\dfrac{d\theta}{dx}}{\sqrt{\cos 2\theta}(\cos\theta + \sin\theta)}$.

(14) $\dfrac{dy}{dx} = \dfrac{2\cos x}{(1-\sin x)^2}$.

(15) $\dfrac{dy}{dx} = \dfrac{1}{2}\sec^2(x/2)$.

(16) $\dfrac{dy}{dx} = x^3\tan(x/2)\sec^2(x/2) + 3x^2\tan^2(x/2)$.

(17) $\dfrac{dy}{dx} = -\sec^2(\cos(\sin\theta))\sin(\sin\theta)\cos\theta \cdot \dfrac{d\theta}{dx}$.

Problem K. Derivatives with exponentials and logs.

(1) $\dfrac{dy}{dx} = 2ex - \dfrac{3\pi}{x^4} + \dfrac{7}{2}x^{5/2}$.

(2) $\dfrac{dy}{dx} = a^{ax+b+1}\ln a$.

(3) $\dfrac{dy}{dx} = 3x^2 a^{x^3}\ln a$.

(4) $\dfrac{dy}{dx} = 2\cdot 6^{2x}\ln 6$.

(5) $\dfrac{dy}{dx} = \dfrac{2ax}{ax^2+b}$.

(6) $\dfrac{dy}{dx} = 3x^2$.

(7) $\dfrac{dy}{dx} = 2(e^{2x} + e^{-2x})$.

(8) $\dfrac{dy}{dx} = 2(x+1)e^{x^2+2x}$.

(9) $\dfrac{dy}{dx} = ax^{a-1}a^x + x^a a^x \ln a$.

(10) $\dfrac{dy}{dx} = (x+1)e^x$.

2.18 HW4 Fall 2006: Selected Answers

Problem A. Expansions.

(10) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$.

(11) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$.

(12) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$.

(13) $\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \cdots$.

(14) $\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots$.

(15) $\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - x^{10} + \cdots$.

Problem B. Derivatives at a point.

(1) $\left.\frac{dy}{dx}\right|_{x=\pi/4}$ is undefined.

(2) $\left.\frac{dy}{dx}\right|_{x=0} = 0$ and $\left.\frac{dy}{dx}\right|_{x=\pi/2} = -4\left(\frac{2+\pi}{4+\pi^2}\right)^2$.

(3) $\left.\frac{dy}{dx}\right|_{x=\pi/3} = -\frac{2\pi}{3}\sin(\sin(\pi^2/9))\cos(\pi^2/9)$.

(4) $\left.\frac{dy}{dx}\right|_{x=\pi^2/16} = \frac{27}{\pi}$.

(5) $\left.\frac{dy}{dx}\right|_{x=0} = 0$ and $\left.\frac{dy}{dx}\right|_{x=\sqrt{\pi/2}} = \frac{-2\sqrt{\pi}}{(2+\pi)^{3/2}}$.

Problem D. Parametric equations.

(1) $\frac{dy}{dx} = (-b/a)\cot\theta$.

(2) $\frac{dy}{dx} = \tan(\theta/2)$.

(3) $\frac{dy}{dx} = \frac{3b}{2a}\tan\theta$.

(4) $\frac{dy}{dx} = -\cot\phi$.

(5) $\frac{-1}{4a}\csc^4(t/2)$ (6) $-1/a$ (7) $-3/2$

Problem E. Implicit differentiation.

(1) $\dfrac{dy}{dx} = \dfrac{(1+x^2)\sin x - (y^2+2x)\cos x - y\sec^2 x}{2y\sin x + \tan x}$.

(2) $\dfrac{dy}{dx} = \dfrac{y(2xy - y^2\cos(xy) - 1)}{xy^2\cos(xy) + y^2 - x}$.

(3) $\dfrac{dy}{dx} = \dfrac{2xy - 2(1+x^2)^2\tan x\sec^2 x}{(1+x^2)^2(4y+\cos y) + (1+x^2)}$.

(4) $\dfrac{dy}{dx} = \dfrac{\sec^2(x+y) + \sec^2(x-y)}{\sec^2(x-y) - \sec^2(x+y)}$.

(5) $\dfrac{dy}{dx} = \dfrac{-y(ay^2\cos(xy) - b\sin(x/y))}{x(ay^2\cos(xy) + b\sin(x/y))}$.

(6) $\dfrac{dy}{dx} = \tan^{-1} e^x + \dfrac{xe^x}{1+e^{2x}}$.

Problem F. Derivatives with inverse trig functions.

(1) $\dfrac{dy}{dx} = \dfrac{3x^2}{\sqrt{1-x^6}}$.

(2) $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{x}(1+x)}$.

(3) $\dfrac{dy}{dx} = \dfrac{3}{\sqrt{1-9x^2}}$.

(4) $\dfrac{dy}{dx} = \dfrac{2}{x\sqrt{x^4-1}}$.

(5) $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{x(1-x)}}$.

(6) $\dfrac{dy}{dx} = \dfrac{-\cot x}{\sqrt{\sin^2 x - 1}}$.

(7) $\dfrac{dy}{dx} = \dfrac{1}{2x\sqrt{x-1}}$.

(8) $\dfrac{dy}{dx} = \dfrac{\cos(\tan^{-1} x)}{1+x^2}$.

(9) $\dfrac{dy}{dx} = \dfrac{-x}{\sqrt{1-x^2}} + \cos^{-1} x$.

(10) $\dfrac{dy}{dx} = \dfrac{x}{\sqrt{1-x^2}} + \sin^{-1} x$.

(11) $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{x}(1+x)} - \dfrac{1}{1+x^2}$.

(12) $\dfrac{dy}{dx} = 2x\tan^{-1}x + 1$.

(13) $\dfrac{dy}{dx} = \dfrac{-\tan x}{\sqrt{1-x^2}} + \cos^{-1}x\sec^2 x$.

(14) $\dfrac{dy}{dx} = \dfrac{2}{1-x^4}$.

(15) $\dfrac{dy}{dx} = -\sqrt{1-x^2} - 2x\cos^{-1}x$.

(16) $\dfrac{dy}{dx} = \dfrac{\tan x}{1+x^2} + \tan^{-1}x\sec^2 x$.

(17) $\dfrac{dy}{dx} = 0$.

(18) $\dfrac{dy}{dx} = \left(\dfrac{-a}{x^2+a^2}\right)(-\tan^{-1}(a/x) + \cot^{-1}(x/a))$.

(19) $\dfrac{dy}{dx} = \dfrac{6(\tan^{-1}2x)^2}{1+4x^2}$.

(20) $\dfrac{dy}{dx} = \dfrac{2x\sec^2 x}{\sqrt{1-\tan^2 x^2}}$.

(21) $\dfrac{dy}{dx} = \dfrac{2}{1+x^2}$.

(22) $\dfrac{dy}{dx} = \dfrac{-2}{1+x^2}$.

(23) $\dfrac{dy}{dx} = \dfrac{3}{(1+x^2)^2}$.

(24) $\dfrac{dy}{dx} = \dfrac{2x}{1+x^4}$.

(25) $\dfrac{dy}{dx} = \dfrac{2}{1+x^2}$.

(26) $\dfrac{dy}{dx} = 1/2$.

(27) $\dfrac{dy}{dx} = 3/2$.

(28) $\dfrac{dy}{dx} = \dfrac{-(b^2-a^2)\sin x}{b+a\cos x}$.

(29) $\dfrac{dy}{dx} = \dfrac{\sqrt{3}}{2+\cos x}$.

(30) $\dfrac{dy}{dx} = 1/2$.

(31) $\dfrac{dy}{dx} = 1$.

Problem G. Derivatives with trig functions.

(1) $\dfrac{dy}{dx} = -x\sin x$.

(2) $\dfrac{dy}{dx} = -9\cos^2 3x \sin 3x$.

(3) $\dfrac{dy}{dx} = 4(x^2+\cos x)^3(2x-\sin x)$.

(4) $\dfrac{dy}{dx} = 2\sin x(3\cos^2 x - 1)$.

(5) $\dfrac{dy}{dx} = \dfrac{2x\cos 2x - 2\sin 2x}{x^3}$.

Problem H. Derivatives with exponentials and logs.

(1) $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{x^2+a^2}}$.

(2) $\dfrac{dy}{dx} = \dfrac{2e^x}{(1-e^x)^2}$.

(3) $\dfrac{dy}{dx} = \dfrac{-2x(x+2)}{(x^2+x+1)(x^2-x-1)}$.

(4) $\dfrac{dy}{dx} = \dfrac{x^2-1}{x^2-4}$.

(5) $\dfrac{dy}{dx} = \dfrac{4}{\ln(\ln x^4)(\ln x^4)x}$.

Problem I. Derivatives with exponentials, logs and trig functions.

(1) $\dfrac{dy}{dx} = 0$.

(2) $\dfrac{dy}{dx} = m\cot x + n\tan x$.

(3) $\dfrac{dy}{dx} = e^{ax}(a\sin bx + b\cos bx)$.

(4) $\dfrac{dy}{dx} = 2\csc x$.

(5) $\dfrac{dy}{dx} = \sec 2x$.

(6) $\dfrac{dy}{dx} = e^{ax}(a\cos(bx+c) - b\sin(bx+c))$.

(7) $\dfrac{dy}{dx} = \dfrac{x(1+2\csc 2x) - 2(x+\ln\tan x)(1+2x)}{2(\sqrt{x+\ln\tan x})(x^2e^{2x})}$.

(8) $\dfrac{dy}{dx} = \dfrac{2(x\cos x+\sin x)}{1-x^2\sin^2 x}$.

(9) $\dfrac{dy}{dx} = \csc x$.

(10) $\dfrac{dy}{dx} = \sec x$.

2.19 HW5 Fall 2006: Selected Answers

Problem A. Evaluating limits when $x \to 0$.

(1) 10	(2) 5	(3) $17/2$	(4) $-317/422$
(5) $-3/5$	(6) $\dfrac{1}{2\sqrt{x}}$	(7) $1/2$	(8) $\sqrt{2}/4$
(9) $-(1/2)x^{-3/2}$	(10) $2\sqrt{a}$	(11) $1/2$	(12) 2
(13) 1	(14) $\dfrac{a}{2\sqrt{ax+b}}$	(15) $mn(mx+c)^{n-1}$	

Problem B. Evaluating limits when $x \to a$.

(1) 5	(2) 14	(3) -2	(4) -11
(5) 3	(6) $1/2$	(7) 4	(8) 108
(9) 3125	(10) $12a^11$	(11) $(5/2)a^{3/2}$	(12) $(5/3)(a+2)^{2/3}$
(13) 6	(14) $20/3$	(15) n	(16) $\dfrac{1}{2\sqrt{a}}$
(17) $1/2$	(18) $\dfrac{2\sqrt{3}}{9}$	(19) na^{n-1}	

Problem C. Evaluating limits as $x \to \infty$.

(1) 1	(2) $3/5$	(3) $1/3$	(4) $2/7$	(5) 12
(6) $1/2$	(7) $1/2$	(8) 0	(9) e	(10) 0
(11) 0	(12) $1/2$			

Problem D. Limits with exponential and log functions.

(1) 1	(2) $\ln a$	(3) 1	(4) e
(5) $\ln a - \ln b$	(6) 1	(7) 0	
(11) $\dfrac{1}{2\sqrt{x}}\, e^{\sqrt{x}}$	(12) $\dfrac{a}{ax+b}$	(13) $x^x(\ln x + 1)$	

Problem E. Limits with trigonometric functions.

(1) $3/4$	(2) $1/3$	(3) 1	(4) $1/2$	(5) a/b
(6) $1/4$	(7) m/n	(8) 0	(9) $2/3$	(10) 1
(11) $1/2$	(12) $\cos a$	(13) 2	(14) $1/3$	(15) -2
(16) -2	(17) $1/6$	(18) 2	(20) $\cos a$	(21) 0

Problem G. Limits with inverse trigonometric functions.

(1) $1/2$	(2) $-\sqrt{2}/2$	(3) $1/2$	(4) 0
(5) $2/3$			

2.20 HW6 Fall 2006: Selected Answers

Problem B. Where is a function continuous?

(1) all x	(2) all x	(3) $x \neq 0$
(4) $x \neq 0$	(5) $k = 2/5$	(6) $1 \le x \le 3$
(7) $x \neq 0$	(8) $x \neq 0$	(9) $a = -2$
(10) $x \ge 0, x \neq 1$	(11) all x	(12) $a = 3$
(13) $x \neq a$	(14) $x \neq 0$	(15) all x
(16) all x	(18) all x	(19) x not an integer
(20) $x \neq 1$	(21) $-1 \le x \le 2$	

Problem D. Increasing, decreasing, and concavity.

(9) 1 (10) $a = 3$ and $b = 5$

Problem E. Graphing polynomials.

(1) Defined for all x; continuous for all x; differentiable for all x; increasing nowhere; decreasing nowhere; concave up nowhere; concave down nowhere; all x are critical points; no points of inflection; $y = a$ is an asymptote.

(2) Defined for all x; continuous for all x; differentiable for all x; increasing for all x if $a > 0$; decreasing for all x if $a < 0$; concave up nowhere; concave down nowhere; all x are critical points if $a = 0$ and there are no critical points if $a \neq 0$; no points of inflection; $y = ax + b$ is an asymptote.

(3) Same as (2).

(4) Defined for $x \ge 0$; continuous for $x \ge 0$; differentiable for $x \neq 1$; increasing for $0 < x < 1$; decreasing for $x > 1$; concave up nowhere; concave down nowhere; critical points at $x = 1$ and $x = 0$; no points of inflection; $y = 2 - x$ is an asymptote as $x \to \infty$.

(5) Defined for all x; continuous for all x; differentiable for $x \neq 0$; increasing for $x > 0$; decreasing for $x < 0$; concave up nowhere; concave down nowhere; critical point at $x = 0$; no points of inflection; $y = 2 + x$ is an asymptote as $x \to \infty$, $y = 2 - x$ is an asymptote as $x \to -\infty$.

(6) Defined for all x; continuous for all x; differentiable for $x \neq 1$; increasing for $x > 1$; decreasing for $x < 1$; concave up for $x > 1$; concave down nowhere; critical point at $x = 1$; no points of inflection; $y = 1 - x$ is an asymptote as $x \to -\infty$.

(7) Defined for all x; continuous for all x; differentiable for all x; increasing for $x < 1$; decreasing for $x > 1$; concave up nowhere; concave down for all x; critical point at $x = 1$; no points of inflection; no asymptotes.

(8) Defined for all x; continuous for all x; differentiable for all x; increasing for $x < 1/2$; decreasing for $x > 1/2$; concave up nowhere; concave down for all x; critical point at $x = 1/2$; no points of inflection; no asymptotes.

(9) Defined for all x; continuous for all x; differentiable for all x; increasing for $x > 1/3$; decreasing for $x < 1/3$; concave up for all x; concave down nowhere; -critical point at $x = 1/3$; no points of inflection; no asymptotes.

(10) Defined for all x; continuous for all x; differentiable for all x; increasing for all x; decreasing nowhere; concave up for $x > 0$; concave down for $x < 0$; critical point at $x = 0$; point of inflection at $x = 0$; no asymptotes.

(11) Defined for all x; continuous for all x; differentiable for all x; increasing for $x < -1/\sqrt{3}, x > 1/\sqrt{3}$; decreasing for $-1/\sqrt{3} < x < 1/\sqrt{3}$; concave up for $x > 0$; concave down for $x < 0$; critical points at $x = \pm 1/\sqrt{3}$; point of inflection at $x = 0$; no asymptotes.

(12) Same as (11).

(13) Defined for all x; continuous for all x; differentiable for all x; increasing for $x < 4/3, x > 2$; decreasing for $4/3 < x < 2$; concave up for $x > 5/3$; concave down for $x < 5/3$; critical points at $x = 2$ and $x = 4/3$; point of inflection at $x = 5/3$; no asymptotes.

(14) Defined for all x; continuous for all x; differentiable for all x; increasing for $x < 1, x > 6$; decreasing for $1 < x < 6$; concave up for $x > 7/2$; concave down for $x < 7/2$; critical points at $x = 6$ and $x = 1$; point of inflection at $x = 7/2$; no asymptotes.

(15) Defined for all x; continuous for all x; differentiable for all x; increasing for $x < -5/3, x > 2$; decreasing for $-5/3 < x < 2$; concave up for $x > 1/6$; concave down for $x < 1/6$; critical points at $x = -5/3$ and $x = 2$; point of inflection at $x = 1/6$; no asymptotes.

(16) Defined for all x; continuous for all x; differentiable for all x; increasing for $x < 0$; decreasing for $x > 0$; concave up nowhere; concave down for all x; critical points at $x = 0$; no points of inflection; no asymptotes.

(17) Defined for all x; continuous for all x; differentiable for all x; increasing for $-1 < x < 0$ and $x > 2$; decreasing for $x < -1$ and $0 < x < 2$; concave up for x less than about $-1/2$, and for x greater than about 1.2; concave down for x between about $-1/2$ and 1.2; critical points at $x = -1$, $x = 0$ and $x = 2$; points of inflection at about $-1/2$ and about 1.2; no asymptotes.

(18) Defined for all x; continuous for all x; differentiable for all x; increasing for $0 < x < 1$ and $x > 3$; decreasing for $x < 0$ and $1 < x < 3$; concave up for x less than about $1/2$, and for x greater than about 2.2; concave down for x between about $1/2$ and 2.2; critical points at $x = 0$, $x = 1$ and $x = 3$; points of inflection at about $1/2$ and about 2.2; no asymptotes.

(20) Defined for all x; continuous for all x; differentiable for all x; increasing for $x < 1.2$ and $x > 2$; decreasing for $1.2 < x < 2$; concave up for x between 0 and about 1, and for x greater than about 1.3; concave down for $x < 0$ and x between about 1 and 1.3; critical points at $x = 0$, $x = 1.2$ and $x = 2$; points of inflection at $x = 0$ and x about 1 and about 1.3; no asymptotes.

(21) Defined for all x; continuous for all x; differentiable for all x; increasing for $x > 0$ and less than about 1.2 and for $x > 2$; decreasing for $x < 0$ and between about 1.2 and 2; concave up for x less

than about $-.9$ and between about $-.5$ and $.5$ and for x greater than about 1.5; concave down for x between about $-.9$ and $-.5$ and between about $.5$ and 1.5; critical points at $x = 0$, $x = 2$ and approximately $-.9$ and 1.2; points of inflection at approximately $-.9$, $-.5$, $.5$ and 1.5; no asymptotes.

2.21 HW 7 Fall 2006: Selected Answers

Problem A. Graphing rational functions.

(1) Defined for $x \neq 0$; continuous for $x \neq 0$; differentiable for all $x \neq 0$; increasing nowhere; decreasing for all $x \neq 0$; concave up for $x > 0$; concave down for $x < 0$; critical point at $x = 0$; no point of inflection; asymptote $y = 0$ as $x \to 0$; asymptote $x = 0$ as $x \to \pm\infty$.

(2) Defined for $x \neq 0$; continuous for $x \neq 0$; differentiable for all $x \neq 0$; increasing for $x > 0$; decreasing for all $x < 0$; concave up nowhere; concave down for $x \neq 0$; critical points at $x = 0$; no points of inflection; asymptote $y = 0$ as $x \to 0$;

(3) Defined for $x \neq 0$; continuous for $x \neq 0$; differentiable for all $x \neq 0$; increasing for $x < -1, x > 1$; decreasing for all $-1 < x < 0, 0 < x < 1$; concave up for $x > 0$; concave down for $x < 0$; critical points at $x = 0, \pm 1$; no points of inflection; asymptote $x = 0$ as $x \to 0$; asymptote $y = x$ as $x \to \pm\infty$;

(4) Defined for $x \neq 4$; continuous for $x \neq 4$; differentiable for all $x \neq 4$; increasing for $x < 2, x > 6$; decreasing for all $2 < x < 4, 4 < x < 6$; concave up for $x > 4$; concave down for $x < 4$; critical points at $x = 2, 4, 6$; no points of inflection; asymptote $x = 4$ as $x \to 4$; asymptote $y = x$ as $x \to \pm\infty$;

(5) Defined for all x; continuous for all x; differentiable for all x; increasing for $x < 0$; decreasing for all $x > 0$; concave up for $x < -1/\sqrt{3}, x > 1/\sqrt{3}$; concave down for $-1/\sqrt{3} < x < \sqrt{3}$; critical point at $x = 0$; points of inflection at $x = \pm 1/\sqrt{3}$; asymptote $y = 0$ as $x \to \pm\infty$;

Problem B. Graphing functions with square roots.

(1-5) Circles.

(6-8) Ellipses.

(9-10) Hyperbolas.

(11-16) Parabolas.

(18) This problem appeared before on this homework assignment (almost).

(19) Make this one into problem (B18).

Problem C. Graphing other functions.

(1) Defined for all x; continuous for all $x \neq 0, \pm 1, \pm 2, \ldots$; differentiable for all $x \neq 0, \pm 1, \pm 2, \ldots$; increasing nowhere; decreasing nowhere; concave up nowhere; concave down nowhere; all points are critical points; no points of inflection; no asymptotes;

(2) Defined for all x; continuous for all x; differentiable for $x \neq 0$; increasing for $x > 0$; decreasing for $x < 0$; concave up nowhere; concave down nowhere; critical point at $x = 0$; no points of inflection; asymptote $y = x$ as $x \to \infty$; asymptote $y = -x$ as $x \to -\infty$;

(3) Defined for all x; continuous for all x; differentiable for $x \neq 5$; increasing for $x > 5$; decreasing for $x < 5$; concave up nowhere; concave down nowhere; critical point at $x = 5$; no points of inflection; asymptote $y = x$ as $x \to \infty$; asymptote $y = -x$ as $x \to -\infty$;

(4) Defined for all x; continuous for all x; differentiable for $x \neq \pm 1$; increasing for $-1 < x < 0$, $x > 1$; decreasing for $x < -1$, $0 < x < 1$; concave up for $x < -1$, $x > 1$; concave down for $-1 < x < 1$; critical points at $x = \pm 1, 0$; no points of inflection; no asymptotes;

(5) Defined for all x; continuous for all $x \neq 0$; differentiable for $x \neq 0$; increasing nowhere; decreasing nowhere; concave up nowhere; concave down nowhere; critical point at $x = 0$; no points of inflection; asymptotes $y = 1$ as $x \to \infty$; asymptotes $y = -1$ as $x \to -\infty$;

(6) Defined for all x; continuous for all x; differentiable for $x \neq 1$; increasing for all x; decreasing nowhere; concave up for $x < 1$; concave down for $x > 1$; critical point at $x = 1$; point of inflection at $x = 1$; no asymptotes;

(7) Defined for all x; continuous for all x; differentiable for $x \neq 0$; increasing for $x > 0$; decreasing for $x < 0$; concave up nowhere; concave down everywhere; critical point at $x = 0$; no points of inflection; no asymptotes;

(8) Defined for $x \neq 1$; continuous for all $x \neq 1$; differentiable for $x \neq 1$; increasing for $x < 1$; decreasing for $x > 1$; concave up everywhere; concave down nowhere; critical point at $x = 1$; no points of inflection; no asymptotes;

(12) See page 153 in the text.

(13) Make this one into problem (C12).

(14) Defined for all x; continuous for all x; differentiable for x; increasing for $2k\pi + -\pi/2 < x < 2k\pi + \pi/2$, where k is an integer; decreasing for $2k\pi + \pi/2 < x < 2k\pi + 3\pi/2$, where k is an integer; concave up for $2k\pi - \pi < x < 2k\pi$, where k is an integer; concave down for $2k\pi < x < 2k\pi + \pi$, where k is an integer; critical points at $x = k\pi + \pi/2$, where k is an integer; points of inflection at $x = k\pi$, where k is an integer; no asymptotes;

(15) Compare the graphs of $y = \sin 2x$ and $y = x$.

Problem D. Rolle's theorem and the mean value theorem.

(4) $c = 2 \pm \sqrt{3}/3$ (5) $c = 9/4$ (6) $c = 3\pi/2$

(7) $c = \pi/4$ (8) $c = 2 \pm \sqrt{3}/3$ (11) $f(1) \neq f(3)$

(12) $f'(1)$ does not exist (13) $f(x)$ is discontinuous at $x = 0$

(14) $(3, 6)$ (17) $c = 8/27$ (18) $c = e - 1$ (20) $c = (a + b)/2$

(21) $f'(0)$ does not exist (22) $f(x)$ is discontinuous at $x = 0$

(23) $(\sqrt{7/3}, (-2/3)(\sqrt{7/3}))$, $(-\sqrt{7/3}, (2/3)(\sqrt{7/3}))$

Problem E. Tangents and Normals.

(1) 11 (2) -12

(3) $x - y + 5 = 0, \quad x + y - 7 = 0$

(4) $x - 4y + 3 = 0, \quad 4x + y - 5 = 0$

(5) $14x - y - 10 = 0, \quad x + 14y - 254 = 0$

(6) $m^2x - my + a = 0, \quad m^2x + m^3y - 2am^2 - a = 0$

(7) $bx\cos\theta + ay\sin\theta = ab, \quad ax\sec\theta - by\csc\theta = (a^2 - b^2)$

(8) $bx\sec\theta - ay\tan\theta = ab, \quad x\sin\theta - y\cos\theta = (a/4)\sin 4\theta$

(9) $x\cos^3\theta + y\sin^3\theta = c, \quad x\sin^3\theta - y\cos^3\theta + 2c\cot 2\theta = 0$

(10) $x - ty + at^2 = 0, \quad tx + y = at^3 + 2at$

(11) $2x + 3my - 18m^2 - 27m^4 = 0$

2.22 HW 8 Fall 2006: Selected Answers

Problem A. Tangents and normals.

(1) $10x + y - 5 = 0, \quad x - 10y + 50 = 0$

(2) $y = 1, \quad x = \pi/4$

(3) $y - 1 - (1/\sqrt{2}) = (1 - \sqrt{2})(x - \pi/4 - 1/\sqrt{2})$

(4) $x/a + y/b = \sqrt{2}$

(5) $bx \cos t + ay \sin t = ab, \quad ax \sec t - by \csc t = (a^2 - b^2)$

(6) $bx \sec t - ay \tan t = ab, \quad ax \cos t + by \cot t = (a^2 + b^2)$

(7) $y = (x - at)\tan(t/2), \quad (y - 2a)\tan(t/2) + x - at = 0$

(8) $8x + 3\sqrt{5}y - 36 = 0, \quad 9\sqrt{5}x - 24y + 14\sqrt{5}$

(9) $3y - 48\sqrt{3}x + 16\pi\sqrt{3} - 21 = 0$

(10) $24y - 2x + \pi - 96 = 0$

(11) $x + (2 + \sqrt{2})yi - 4 - 3\sqrt{2} - \pi/4 = 0$

(14)	$(0,0),\ (2a, -2a)$	(16)	$(0,0),\ (1,2),\ (-1,-2)$
(17)	$(2,3),\ (-2,-3)$	(18)	$(1,7)$
(19)	$x + 3y - 9 = 0$	(20)	$2x + y - 6 = 0$

Problem B. Optimization.

(1) $(1/5, 4)$ (2) $(3, 9)$ (3) $(-4, 6)$

(4) $(k\pi + \pi/4, 6)$ and $(k\pi + 3\pi/4, 4)$ where k is an integer

(5) $(k\pi/2 + \pi/8, 4)$ and $(k\pi/2 + 3\pi/8, 2)$ where k is an integer

(6)	$(1, 68)$, $(-6, -1647)$, $(5, -316)$	(7)	$(-2, 0)$, $(0, -4)$
(8)	$(1, 0)$, $(-1, 0)$, $(-1/5, -3456/3125)$	(9)	$(2, 2)$
(10)	$(-2, 139)$, $(3, 89)$	(11)	$(\pi/6, 3/4)$, $(\pi/2, 1/2)$
(14)	49	(15)	$(1, 3)$

(16) $(\pi/3, -\pi/3 + \sqrt{3})$, $(5\pi/3, -(5\pi/3 + \sqrt{3}))$

(17) 6, 9 (18) $\dfrac{ap}{p+q}, \dfrac{aq}{p+q}$ (20) $(1/p)^{1/(p-1)}$

(21) length $4\sqrt{6}$ cm, width $4\sqrt{6}$ cm, perimeter $16\sqrt{6}$ cm

(25) $r = (500/\pi)^{1/3}$ cm, $h = \dfrac{1000}{\pi^{1/3}500^{1/3}}$ cm

(26) $p^3/6\sqrt{3}$ cubic units (28) 4, 4 (29) 75×75 yards

Problem C. Related rates.

(1) $\dfrac{dV}{dr} = 4\pi r^2$

(2) $\dfrac{dV}{dr} = 2\pi rh$

(3) $\dfrac{dS}{dr} = \dfrac{\pi(h^2+2r^2)}{\sqrt{h^2+r^2}}$

(4) $\dfrac{dP}{dt} = 0.8$ cm/s

(5) $\dfrac{dr}{dt} \approx 0.32$ cm/s

(6) $\dfrac{dV}{dt} = 6$ cm^3/s

(7) $\dfrac{dh}{dt} = 0.5$ cm/min

(8) 6.78 m/s

(9) (a) 25/3 ft/s (b) 10/3 ft/s

(10) $80\pi/3$ km/min

(11) $\sqrt{65}/8$ m/sec

(12) $6/5\pi$ ft/min

(13) $32/27\pi$ cm/s

(14) $\approx 2.89 \times 10^5$ cm^3/min

(15) $16/49\pi$ cm/sec

(16) $\sqrt{2}/5$ rad/sec

(17) 5/6 m/s

(18) 0.032 rad/s

(19) $\pi/3$ radians

(20) (a) 360 ft/s (b) 0.096 rad/s

2.23 HW 9 Fall 2006: Selected Answers

Problem A. Indefinite integrals.

(1) $\frac{x^8}{8}+c$ (2) $-\frac{1}{6x^6}+c$

(3) $\ln|x|+c$ (4) $\frac{3}{8}x^{8/3}+c$

(5) $-4x^{-1/4}+c$ (6) $\frac{3}{5}x^{5/3}+c$

(7) $4x^{1/4}+c$ (8) $\frac{-2}{x}+c$

(9) $8x-\frac{x^2}{2}+\frac{x^4}{2}+\frac{3}{x^2}-\frac{1}{2x^4}+5\ln|x|+c$

(10) $6x-\frac{17x^2}{2}+\frac{x^3}{3}+\frac{5x^4}{2}+c$

(11) $\frac{2ax^{7/2}}{7}+\frac{2bx^{5/2}}{5}+\frac{2cx^{3/2}}{3}+c$

(12) $\frac{x^7}{7}+\frac{1}{5x^5}-x^3-\frac{3}{x}+c$

(13) $\frac{2x^{3/2}}{3}-2x^{1/2}+c$ (14) $\frac{x^2}{2}+\ln|x|+2x+c$

(15) $-\frac{1}{3x^3}+8\ln|x|-\frac{3}{x^2}-\frac{12}{x}+c$

(16) $2\sqrt{x}+\frac{2}{7}x^{7/2}+2x^{3/2}+\frac{6}{5}x^{5/2}+c$

(17) $x^2+5x+8\ln|x-2|+c$ (18) $f(x)=x^2/2+1/x-1$

Problem B. Indefinite integrals with trigonometric functions.

(1) $-9\cos x-7\sin x-6\tan x-3\cot x-x+c$

(2) $-\csc x + \tan x + x - \sec x + c$ (3) $\tan x + \sec x + c$

(4) $-\cot x + \csc x + c$ (5) $\tan x - \cot x + c$

(6) $\tan x + 2\sec x + c$ (7) $-3\csc x - 4\cot x + c$

(8) $-\cot x - \csc x + c$ (9) $-\cot x + \csc x + c$

(10) $\sec x - \tan x + x + c$ (11) $-\cot x - \csc x + c$

(12) $-\csc x + \cot x + x + c$ (13) $\sec x + \tan x - x + c$

(14) $\sqrt{2}\sin x + c$ (15) $-\sqrt{2}\cos x + c$

(16) $(1/2)\tan x + c$ (17) $(-1/2)\cot x + c$

(18) $\sin x - \cos x + c$ (19) $\sec x - \csc x + c$

Problem C. Integrals with exponential functions and inverse functions.

(1) $\dfrac{2^x}{\ln 2} + c$ (2) $x^6 + \dfrac{2}{3x^3} - \dfrac{7x^2}{2} + 3\ln|x| - 5x + 4e^x + \dfrac{7^x}{\ln 7} + c$

(3) $\dfrac{x^2}{2a} + a\ln|x| + \dfrac{x^{a+1}}{a+1} + \dfrac{a^x}{\ln a} + ax^2/2 + c$

(4) $\dfrac{2}{3}x^{3/2} - \dfrac{3}{7}x^{7/3} + 21x^{1/3} - 6e^x + x + c$

(5) $x - 2\tan^{-1} x + c$ (6) $x^5/5 - x^3/3 + x - 2\tan^{-1} x + c$

(7) $x^3/3 - x + \tan^{-1} x + c$ (8) $x - \tan^{-1} x + c$

(9) $x + \tan^{-1} x - 2\sin^{-1} x + 5\sec^{-1} x + \dfrac{a^x}{\ln a} + c$

(10) $x^2/2 + c$ (11) $x^2 + c$

(12) $\pi x/2 - x^2/2 + c$ (13) $x^2/2 + c$

Problem D. Integration by substitution.

(1) $\dfrac{(2x+9)^6}{12}+c$ (2) $-\dfrac{(7-3x)^5}{15}+c$

(3) $(2/9)(3x-5)^{3/2}+c$ (4) $(1/2)\sqrt{4x+3}+c$

(5) $-(1/2)\sqrt{3-4x}+c$ (6) $\dfrac{-1}{\sqrt{2x-3}}+c$

(7) $\ln|2x^2+3|+c$ (8) $(1/2)\ln|x^2+2x-3|+c$

(9) $\ln|2x^2-5x+1|+c$ (10) $\ln|3x^3-2x^2+5x+1|+c$

(11) $2\sqrt{x^2+3x-2}+c$ (12) $2\sqrt{x^2-x-1}+c$

(13) $\dfrac{2}{3(a-b)}\left((x+a)^{3/2}-(x+b)^{3/2}\right)+c$

(14) $\dfrac{1}{18}\left((1-3x)^{3/2}+(5-3x)^{3/2}\right)+c$ (15) $(1/3)\tan^{-1}x^3+c$

(16) $(1/4)\tan^{-1}x^4+c$ (17) $(1/2)\tan^{-1}x^2+c$

(18) $(2/9)(1+x^3)^{3/2}-(2/3)(1+x^3)^{1/2}+c$

(19) $(2/3)(1+x)^{3/2}-2\sqrt{1+x}+c$ (20) $(1/2)\sec^{-1}x^2+c$

(21) $(2/5)(x-1)^{5/2}+(2/3)(x-1)^{3/2}+c$

(22) $(4/3)(1+x)^{3/2}-(2/5)(1+x)^{5/2}+c$

(23) $(1/3)(x^2-1)^{3/2}+c$

(24) $(2/45)(3x-2)^{5/2}+(4/27)(3x-2)^{3/2}+c$

(25) $(2/3)(x^2-3x+5)^{3/2}+c$ (26) $\dfrac{-\ln|3-5x|}{5}+c$

(27) $(2/3)(1+x)^{3/2}+c$

Problem E. Integrals with trigonometric functions.

(1) $\frac{-\cos 3x}{3} + c$ (2) $(1/6)\sin(5+6x) + c$

(3) $(1/3)\cos(5-3x) + c$ (4) $(-1/2)\cot(2x+5) + c$

(5) $(1/2)\sin^2 x + c$ (6) $(1/4)\sin^4 x + c$

(7) $(-2/3)(\cos x)^{3/2} + c$ (8) $2\sin\sqrt{x} + c$

(9) $\frac{\sin^2(ax+b)}{2a} + c$ (10) $\frac{\sin 3x}{12} + (3/4)\sin x + c$

(11) $-\sin(1/x) + c$ (12) $-\cos(x^2+1) + c$

(13) $\tan^2\sqrt{x} + c$ (14) $\ln|1+\tan x| + c$

(15) $-\ln|1+\cos x| + c$ (16) $-(1/3)\ln|2+3\cos x| + c$

(17) $(1/b^2)\ln|a^2 + b^2\sin^2 x| + c$

(18) $\frac{1}{b^2-a^2}\ln|a^2\cos^2 x + b^2\sin^2 x| + c$

(19) $\ln|3\cos x + 2\sin x| + c$ (20) $\frac{-1}{2(x+\sin x)^2} + c$

(21) $\frac{1}{(1+\cos x)} + c$ (22) $(-1/3)\cos x^3 + c$

(23) $(1/2)x + (1/2)\ln|\sin x - \cos x| + c$

(24) $(1/2)x - (1/2)\ln|\sin x - \cos x| + c$

(25) $(1/2)x + (1/2)\ln|\sin x - \cos x| + c$

(26) $\ln|\sin x + \cos x| + c$ (27) $\frac{-1}{\sin x + \cos x} + c$

(28) $(1/8)\sin^4 x^2 + c$ (29) $\sin^{-1}(\tan x) + c$

(30) $\frac{\sec^3(x^2+3)}{3} + c$ (31) $\frac{1}{2b(a+b\cos 2x)} + c$

2.24 HW 10 Fall 2006: Selected Answers

Problem A. Integrals with exponential functions and logarithms.

(1) $(1/2)e^{(2x-1)} + c$ (2) $(-1/3)e^{(1-3x)} + c$

(3) $\dfrac{-3^{(2-3x)}}{3\ln 3} + c$ (4) $\ln|\ln x| + c$

(5) $(\ln x)^2 + c$ (6) $(1/3)(\ln x)^3 + c$

(7) $e^{-1/x} + c$ (8) $\tan^{-1}(e^x) + c$

(9) $(1/2)\ln|e^{2x} - 2| + c$ (10) $(2/3)(2 + \ln x)^{3/2} + c$

(11) $(1/3)(x + \ln x)^3 + c$ (12) $2\sqrt{e^x - 1} + 2\tan^{-1}\sqrt{e^x - 1} + c$

Problem B. Definite integrals.

(1) -12 (2) $1/2$ (3) -1 (4) $26/3$

(5) 231 (6) $19/15$ (7) $16/3$ (8) $7/10$

(9) $28/81$ (10) $11/6$ (11) $(6/5)(3\sqrt{2} - 2)$

(12) $86/7$ (13) $29/35$ (14) Does not exist

(15) $29/6$ (16) 0 (17) Does not exist

(18) 2 (19) $2/3$ (20) $63/4$ (21) $1/4$

(22) 36 (23) $(1/2)e^2 + e - 1/2$ (24) $85/2 + \ln(9/4)$

(25) 1 (26) $33/4$

Problem C. Definite integrals with trigonometric functions.

(1) $(\sqrt{2} - 1)/2$ (2) 3 (3) Does not exist

(4) $-1 + 2\sqrt{3}/3$ (5) $2\sqrt{3}/3$ (6) Does not exist

(7) $\pi/2$ (8) $\pi/6$

Problem D. Definite integrals with other functions.

(1) $\ln 2$ (2) 24 (3) $\dfrac{2^8}{\ln 2}$ (4) -3

(5) $28/3$ (6) $11/6$ (7) -3.5 (8) $5/3$

(9) 10.7 (10) $2 - \pi^2/2$

Problem F. Finding areas bounded by lines and curves.

(1) $3 + 16\ln 2$ sq. units
(2) $32/3$ sq. units
(3) 9π sq. units
(4) $253/12$ sq. units
(5) $1/2$ sq. units
(6) $1/2$ sq. units
(7) 3 sq. units
(9) $1/3$ sq. units
(10) $1 - \pi/4$ sq. units
(11) infinite
(12) both areas are $\pi/2$ sq. units
(13) $3\pi/2$ sq. units
(14) 4 sq. units

Problem G. Areas between curves.

(1) $1/3$ sq. units
(2) $9/16$ sq. units
(3) $9/8$ sq. units
(4) $1/3$ sq. units
(5) $41/6$ sq. units
(6) $1/6$ sq. units
(7) $(3/4)(3\pi - 8)$ sq. units
(8) $8\sqrt{3}$ sq. units
(9) 8 sq. units
(10) $9/2$ sq. units
(11) $8/3$ sq. units
(12) $2/3$ sq. units
(13) 4 sq. units
(14) $\pi/2 + 1/3$ sq. units
(15) $8\sqrt{5}/15$ sq. units
(16) $8\pi/3 - 2\sqrt{3}$ sq. units
(17) $8a^2/3m^3$ sq. units
(18) $16a^2/3$ sq. units
(19) $2\pi/3 - \sqrt{3}/2$ sq. units
(20) $1/2$ sq. units
(21) $2 - \sqrt{2}$ sq. units

2.25 HW 11 Fall 2006: Selected Answers

Problem A. Volumes by washers.

(4) $8\pi/3$ cubic units

(5) $\pi^2/2$ cubic units

(6) $\pi/30$ cubic units

(7) $\pi r^2 h/3$ cubic units

(8) $2\pi h^2 r/3$ cubic units

(9a) $\pi h^2(a - h/3)$ cubic units

(9b) $6/\pi$ inches per minute

(10) $81\pi/10$ cubic units

(11) $16\pi/15$ cubic units

(12) $128\pi/7$ cubic units

(13) $(4/3)\pi ab^2$ cubic units

(14) $512\pi/15$ cubic units

(15) $16a^3/3$ cubic units

(16) $\pi/9$ cubic units

(17) π cubic units

(18) $32\pi/5$ cubic units

(19) $8a^3/3$ cubic units

(20) $8a^3/3$ cubic units

(21) $\pi\sqrt{3}/16$ cubic units

(22) $8\pi/3$ cubic units

Problem B. Finding volumes by cylindrical shells.

(4) $(\sqrt{3}/2)\pi a^3$ cubic units

(5) $2\pi^2 a^2 b$ cubic units

(6) $8\pi/3$ cubic units

(7) $8\pi/3$ cubic units

(8) $56\pi/15$ cubic units

(9) $2\pi/3$ cubic units

(10) $256\pi/5$ cubic units

(11) $48\pi/5$ cubic units

(12) $117\pi/5$ cubic units

(13) $108\pi/5$ cubic units

(14) $\pi/3$ cubic units

(15) $64\pi/5$ cubic units

(16) $5\pi/3$ cubic units

(17) $4\pi/3$ cubic units

(18) $4\pi/15$ cubic units

(19) 8π cubic units

(20) $224\pi/15$ cubic units

Problem C. Practical volumes.

(1) $72\pi/35$ cubic units

(2) $2\sqrt{3}\,a^3$ cubic units

(3) $(\pi/3)(2r^3 - 3r^2h + h^3)$ cubic units

(4) $(1/3)\pi b(r^2 + rR + R^2)$ cubic units, where $R = r(h-b)/h$.

(5) $\sqrt{3}\,a^3/12$ cubic units (6) $16r^3/3$ cubic units

(7) 24 cubic units (8) $\sqrt{3}/2$ cubic units

(9) $5\pi r^3/12$ cubic units (10) $16r^3/3$ cubic units

(12) They both contain the *same* amount of wood.

(13) At $1-(1/3)^{1/3}$ and $1-(2/3)^{1/3}$. (14) $r = \dfrac{h\sin\theta}{\sin\theta + \cos 2\theta}$.

2.26 HW 12 Fall 2006: Selected Answers

Problem A. Length of a plane curve.

(2) 10.5 (3) $6a$ (4) 12

(5) $(8/27)(10\sqrt{10}-1)$ (6) $14/3$

(7) $53/6$ (8) $123/32$ (9) $(4/27)(10\sqrt{10}-1)$

(10) $a\pi^2/8$ (11) 8 (12) 12 (13) $21/2$

(14) $27/20$ (15) $19/3$ (16) $f(x)=a\pm x\sqrt{A^2-1}$, $|A|\geq 1$

(17) No

Problem B. Surface area.

(2) $4\pi^2r^2$ (3) $99\pi/2$ (4) $(\pi/27)(10\sqrt{10}-1)$

(5) $(\pi/6)(17\sqrt{17}-1)$ (6) $1823\pi/18$ (7) $253\pi/20$

(8) $(2\pi/3)(2\sqrt{2}-1)$ (9) $12\pi a^2/5$ (10) $(2\pi/3)(26\sqrt{26}-2\sqrt{2})$

(11) $56\pi\sqrt{3}/5$ (12) $424\pi/15$ (13) $153\pi/40$

Problem C. Center of mass.

(1) At the intersection of the lines through each vertex which are perpendicular to the opposite side.

(2) At $(0,(2/\pi)r,0)$ if the center is at $(0,0)$ and the y-axis cuts the semicircle in half.

(3) At $(0,(8/15)r,0)$ if the hemisphere is sitting on the x-z plane with its apex at $(0,r,0)$.

(4) $(4a/3\pi, 4a/3\pi)$ (5) $(0,(2/5)h^2)$ (6) $(2a/3(4-\pi), 2a/3(4-\pi))$

(7) $(\pi/2,\pi/8)$ (8) $(2/5,1)$ (9) $(3/7)h$ (10) $(3/5)h$

(11) On the axis of the cone $3h/4$ from the vertex.

(12) On the axis of the cone $3h/5$ from the vertex.

(13) At $(0,\pi r/4)$ if the semicircle is positioned as in (2).

(14) At $(0,(3/8)r,0)$ if the hemisphere is positioned as in (3).

(15) At $(0,(1/2)r,0)$ if the hemisphere is positioned as in (3).

(16) $(0,2c^2/5)$ (17) $(16/105, 8/15)$ (18) $(0,12/5)$

(19) $(1,-3/5)$ (20) $(3/5,1)$

(21) On the axis of the cone $3h/4$ from the vertex.

(22) $(0,8/3)$ (23) $(4/5,0)$

(24) On the axis of the cone $2h/3$ from the vertex.

(25) $(-r, 3r/(2+\pi))$ (26) $(17\sqrt{17}-1)/12$

(27) $(2r/\pi, 2r/\pi)$

Problem B. Average value of a function.

(2) $50\frac{1}{2}$ (3) 126 (4) 117

(5) 2153693963075557766310 7.46 (10) $2/\pi$ (11) 0

(12) $\frac{1}{2}$ (13) $\frac{1}{2}$ (14) 49/12 (15) $\frac{1}{2}$

(16) $\alpha\left(\frac{a+b}{2}\right)+\beta$ (17a) 200 cases (17b) 1 dollar per day

(18) $\frac{a}{3}(3\sqrt{3}-1)$ (19a) $\frac{2}{3}b^2$ (19b) $\frac{2}{3}b$

(20a) 72 (20b) $82\frac{2}{3}$ (21) $50+28/\pi$

2.27 HW 13 Fall 2006: Selected Answers

Problem A. Motion.

(2) (i) $s = 95$ m, $v = 53$ m/s, $a = 24$ m/s^2 (ii) 42 m

(3) (i) 1.5 sec (ii) $v = 2.5$ cm/s, $x = 4.5$ cm

(4) (i) $s = 18$ m and $a = 0$ (ii) $s = 26$ m, $v = 10$ m/s

(5) (i) $t = 0$, $t = 1$, $t = 3$,
(ii) velocities are 3 m/s, -2 m/s, 6 m/s, and
accelerations are -8 m/s^2, -2 m/s^2, 10 m/s^2

(10) 38 cm

(11) (i) $v = 0$, $a = -6$ m/s^2, (ii) $t = 1$ sec, $t = 2$ sec (iii) 6 m

(12) $v = 9$ cm/s, $a = 40$ cm/s^2

(13) $a = 1$, $b = 1/2$, $c = 2 - \pi/4$

(14) 122.5m (15) $v = 90.2$ m/s, $t = 10.2$ sec, $s = 510.2$ m

(16) $t = 4$ sec and $t = 6$ sec, $v = -29.4$ m/s, and it hits the ground at $t = 10$ sec

(17) 29.4 (18) 49 m (19) 122.5 m

Problem B. Applications of the exponential function.

(1) $y(t) = be^{k(t-a)}$

(6) (a) \$649.80 (b) \$658.40 (c) \$660.49 (d) \$661.53 (e) \$661.56 (f) \$661.56

(7) \$630.08 per month	(8) \$25,167.03	(9) \$119.34
(10) \$1324.13 per month	(11) \$1,984,172	(12) \$1204.01
(13) approx. 137° F	(13b) after approx. 116 min	
(14) approx. 2489 years	(15a) approx. 3.82 days	(15b) approx. 12.68 days
(16a) $200 \cdot 2^{-t/140}$ mg	(16b) approx. 121.9 mg	(16c) approx. 605 days
(17) $N_0 e^k t$	(18) $Q_0 e^{kt}$	(19) $S + (T_0 - S)e^{kt}$
(20) In late 2025	(21) approx. 17.67 years	(22) 95.8%
(23) 3.5 mg	(24) 5×10^9 years	(25) 1890 years
(26) 4800 years	(27) 29.0 years	(28) around 3060 BC

Problem C. Logarithmic differentiation.

(1) $\dfrac{dy}{dx} = \dfrac{-(x^2 - 4x - 42)(x+2)^{3/2}}{3(x+3)^{10/3}(x+6)^{3/2}}$.

(2) $\dfrac{dy}{dx} = y\left(\dfrac{2}{x+1} + \dfrac{3}{x-2} + \dfrac{1}{x+4} + \dfrac{1}{x\ln x}\right)$.

(3) $\dfrac{dy}{dx} = (y/2)\left(\dfrac{1}{x-a} + \dfrac{1}{x-b} - \dfrac{1}{x-p} - \dfrac{1}{x-q}\right)$.

(4) $\dfrac{dy}{dx} = (\sin x)^{\ln x}\left(\dfrac{1}{x}\ln\sin x + \cot x \ln x\right)$.

(5) $\dfrac{dy}{dx} = (\sin x)^{\cos x}(\cot x\cos x - \sin x\ln\sin x)$.

(6) $\dfrac{dy}{dx} = (\sin x)^{\tan x}(1 + \sec^2 x\ln\sin x) + (\tan x)^{\sin x}(\sec x + \cos x\ln\tan x)$.

(7) $\dfrac{dy}{dx} = \dfrac{-yx^{y-1} + y^x\ln y}{x^y\ln x + xy^{x-1}}$.

(8) $\dfrac{dy}{dx} = \dfrac{xy + y^2 - x}{x - x(x+y)\ln x}$.

(9) $\dfrac{dy}{dx} = \dfrac{\ln\sin y + y\tan x}{\ln\cos x - x\cot y}$.

(10) $\dfrac{dy}{dx} = a^x\ln a + e^{\tan x}\sec^2 x + (\cot x)^{\cos x}(\sin x\ln\tan x - \ln\csc x)$.

(11) $\dfrac{dy}{dx} = (\tan x)^{\cot x}\csc^2 x(1 - \ln\tan x)$.

(12) $\dfrac{dy}{dx} = x^x\ln(xe) + \dfrac{1}{x^2}x^{1/x}\ln(e/x)$.

(13) $\dfrac{dy}{dx} = (\sec x)^{\csc x}(\sec x - \csc x\cot x\ln\sec x) + (\csc x)^{\sec x}(\sec x\tan x\ln\csc x - \csc x)$.

(14) $\dfrac{dy}{dx} = \dfrac{1}{x\ln ey}$.

(15) $\dfrac{dy}{dx} = \dfrac{-y^2\tan x}{1 - y\ln\cos x}$.

(16) $\dfrac{dy}{dx} = \dfrac{-y^2}{x(1 - y\ln x)}$.

(17) $\dfrac{dy}{dx} = \ln y + \dfrac{1}{x\ln x}$.

(18) $\dfrac{dy}{dx} = \dfrac{x^{1/x}(1-\ln x)}{x^2}$.

(19) $\dfrac{dy}{dx} = \dfrac{1 + 2\ln x + x^{-2}}{(x^x + x^{-x})^{1/2}(x^x - x^{-x})^{3/2}}$.

Problem D. L'Hôpital's rule.

(6) 5	(7) a/b	(8) 1	(9) 0
(10) 1	(11) 0	(12) $-\infty$	(13) $\ln 3$
(14) $1/6$	(15) $-1/6$	(16) $1/5$	
(17) α	(18) 0	(19) 0	
(20) 0	(21) 0	(22) 1	(23) ∞
(24) 0	(25) 0	(26) 0	(27) 1
(28) e^{-2}	(29) e^3	(30) 1	(31) 1
(32) e^{-1}	(33) 1		

2.28 HW 14 Fall 2006: Selected Answers

Problem A. Derivatives with all functions mixed together.

(1) $\dfrac{dy}{dx} = \dfrac{2(\sec x + \tan x \sin x)}{(\tan x + \cos x)^3}$.

(2) $\dfrac{dy}{dx} = \dfrac{x \cos x \sin x}{2\sqrt{x \sin x}}$.

(3) $\dfrac{dy}{dx} = \dfrac{\cos 3x(1 + 2\cos 2x) + 3\sin 3x(x + \sin 2x)}{\cos^2 3x}$.

(4) $\dfrac{dy}{dx} = 5e^{5x} \ln \sec x + e^{5x} \tan x$.

(5) $\dfrac{dy}{dx} = \dfrac{5x^4 \sin^{-1} 2x - 2x^5(1 - 4x^2)^{-1/2}}{(\sin^{-1} 2x)^2}$.

(6) $\dfrac{dy}{dx} = 2x \cos x^2 - \dfrac{(1 + x^2)\sec^2 x - 2x \tan x}{(1 + x^3)^2}$.

(7) $\dfrac{dy}{dx} = 3(\tan \sqrt{x} + x^2 - \sin x)^2 \left(\dfrac{\sec^2 \sqrt{x}}{2\sqrt{x}} + 2x - \cos x\right)$.

(8) $\dfrac{dy}{dx} = \dfrac{\sin^2 x(2\cos 3x(3\cos^2 x - \sec^2 x) + 3\sin 2x \sin 3x)}{(\cos 3x)^2}$.

(9) $\dfrac{dy}{dx} = e^x(\sec^2 x + \tan x) + \dfrac{\sin x - x \cos x \ln x}{x \sin^2 x}$.

(10) $\dfrac{dy}{dx} = 2a^x(1/x + \ln x \ln a)$.

(11) $\dfrac{dy}{dx} = \dfrac{3x^2 - x^{3/2} - 1}{6x\sqrt{x}}$.

(12) $\dfrac{dy}{dx} = \dfrac{x^2 - 4x + 1}{(x^2 - x + 1)^2}$.

(13) $\dfrac{dy}{dx} = \dfrac{\ln \sin x + x \cot x \ln x}{2x\sqrt{1 + \ln x \ln \sin x}}$.

(14) $\dfrac{dy}{dx} = \dfrac{7}{2}x^{-1/2} - \dfrac{35}{2}x^{-9/2} + \dfrac{4x^3}{\sqrt{1 - x^3}} + \dfrac{\csc^2 x}{\cos x}$.

(15) $\dfrac{dy}{dx} = 2\sin x \cos^4 x - 3\sin^3 x \cos^2 x$.

(16) $\dfrac{dy}{dx} = m \cos mx \cos nx - n \sin mx \sin nx$.

(17) $\dfrac{dy}{dx} = \sin^{m-1} x \cos^{n-1} x(m\cos^2 x - n\sin^2 x)$.

(18) $\dfrac{dy}{dx} = \dfrac{2}{\sqrt{1-x^2}}$.

(19) $\dfrac{dy}{dx} = \dfrac{3}{\sqrt{1-x^2}}$.

(20) $\dfrac{dy}{dx} = \dfrac{1}{2x^2}\left((\sqrt{a+x} - \sqrt{a-x})^2 - \dfrac{2x^2}{\sqrt{a^2-x^2}}\right)$.

(21) $\dfrac{dy}{dx} = 96x^3 + 150x^2 + 70x + 10$.

(22) $\dfrac{dy}{dx} = -2\left(\tan\sqrt{1-x^2}\right)\left(\sec^2\sqrt{1-x^2}\right)\dfrac{x}{\sqrt{1-x^2}}$.

(23) $\dfrac{dy}{dx} = \dfrac{x - \sin 2x}{x^3\cos^2 x}$.

(24) $\dfrac{dy}{dx} = \dfrac{e^{2x}(2x\ln x - 1)}{x}(\ln x)^2$.

(25) $\dfrac{dy}{dx} = e^{x^2}\left(1+x^2\right)^{-3/2}\left(x(1+2x^2)\tan^{-1} x + 1\right)$.

(26) $\dfrac{dy}{dx} = \dfrac{e^{\sqrt{x}+2}}{2\sqrt{x}} - \dfrac{e^{\sqrt{x+2}}}{2\sqrt{x+2}}$.

(27) $\dfrac{dy}{dx} = 2(\ln 7)(x+1)7^{x^2+2x}$.

(28) $\dfrac{dy}{dx} = -3\cot^2(e^{3x}x^x)\csc^2(e^{3x}x^x)e^{3x}x^x(4+\ln x)$.

(29) $\dfrac{dy}{dx} = \dfrac{1}{2(1+x^2)}$.

(30) $\dfrac{dy}{dx} = \dfrac{1}{(1+x^2)\tan^{-1} x}$.

(31) $\dfrac{dy}{dx} = \dfrac{2}{1+x^2}$.

(32) $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-x^2}}$.

(33) $\dfrac{dy}{dx} = \dfrac{2}{1+x^2}$.

(34) $\dfrac{dy}{dx} = 0$.

(35) $\dfrac{dy}{dx} = \dfrac{-1}{1+x^2}$.

(36) $\dfrac{dy}{dx} = \sin^{-1} x$.

(37) $\dfrac{dy}{dx} = \cos^{-1} 2x$.

(38) $\dfrac{dy}{dx} = \dfrac{1}{5+3\cos x}$.

(39) $\dfrac{dy}{dx} = 1/2$.

(40) $\dfrac{dy}{dx} = \dfrac{\cos^{-1} x - x\sqrt{1-x^2}}{(1-x^2)^{3/2}}$.

(41) $\dfrac{dy}{dx} = \dfrac{2(a^2-x^2)-x}{4\sqrt{a^2-x^2}} + \dfrac{a^2}{2\sqrt{a^2-x^2}}$.

(42) $\dfrac{dy}{dx} = \dfrac{2}{\sqrt{1-x^2}}$.

(43) $\dfrac{dy}{dx} = \dfrac{2}{1+x^2}$.

(44) $\dfrac{dy}{dx} = 2x^2\cos 2x + 2x\sin 2x - \dfrac{(x+1)\sin x + \cos x}{(x+1)^2}$.

(45) $\dfrac{dy}{dx} = \dfrac{2x-x^4}{(x^2+1)^2} + 2x^4\cos 2x + 4x^3\sin 2x$.

(46) $\dfrac{dy}{dx} = (2\ln x)x^{\ln x - 1}$.

(47) $\dfrac{dy}{dx} = (\tan x)^{\cot x}(\csc^2 x)(1-\ln\tan x)$.

(48) $$\frac{dy}{dx} = x^{\cos^{-1} x}\left(\frac{\cos^{-1} x}{x} - \frac{\ln x}{\sqrt{1-x^2}}\right) + (\tan x)^{\cot x}(\csc^2 x)(1-\ln\tan x) + (\cot x)^{\tan x}(\sec^2 x)(\ln\cot x - 1).$$

(49) $$\begin{aligned}\frac{dy}{dx} &= (\cos x)(e^x)(\ln x)(x^x)(x^{\cos^{-1} x}) + (\sin x)(e^x)(\ln x)(x^x)(x^{\cos^{-1} x}) \\ &\quad + (\sin x)(e^x)(1/x)(x^x)(x^{\cos^{-1} x}) + (\sin x)(e^x)(\ln x)(1+\ln x)(x^x)(x^{\cos^{-1} x}) \\ &\quad + (\sin x)(e^x)(\ln x)(x^x) + x^{\cos^{-1} x}\left(\frac{\cos^{-1} x}{x} - \frac{\ln x}{\sqrt{1-x^2}}\right).\end{aligned}$$

(50) $\dfrac{dy}{dx} = \dfrac{\ln y - (y/x)}{\ln x - (x/y)}$.

(51) $\dfrac{dy}{dx} = -\left(\dfrac{y}{x}\right)^3$.

(52) $\dfrac{dy}{dx} = -\dfrac{y}{x}$.

(53) $\dfrac{dy}{dx} = \dfrac{\cos(x+y) - y}{x - \cos(x+y)}$.

(54) $\dfrac{dy}{dx} = -\left(\dfrac{b}{a}\right)^m \left(\dfrac{x}{y}\right)^{m-1}$.

(55) $\dfrac{dy}{dx} = y/x$.

(56) $\dfrac{dy}{dx} = \dfrac{x-y}{x(1+\ln x)}$.

Problem B. Integrals with mixed functions.

(1)	$(1/2)(\sin^{-1} x)^2 + c$	(2)	$(-1/2)\cos(2\tan^{-1} x) + c$
(3)	$\sin(\ln x) + c$	(4)	$-\cot(\ln x) + c$
(5)	$e^{\tan x} + c$	(6)	$-e^{\cos^2 x} + c$
(7)	$(1/2)(\ln(\sin x))^2 + c$	(8)	$\ln\lvert\ln\sin x\rvert + c$
(9)	$(1/2)(\ln(\sec x + \tan x))^2 + c$	(10)	$(1/4)(\tan^{-1} x^2)^2 + c$
(11)	$(1/4)(\sin^{-1} x^2)^2 + c$	(12)	$\ln\lvert\sin^{-1} x\rvert + c$
(13)	$\ln\lvert x + \ln(\sec x)\rvert + c$	(14)	$\ln\lvert\ln(\tan x)\rvert + c$
(15)	$\tan(1 + \ln x) + c$	(16)	$(1/2)\cot(2e^{-x} + 5) + c$
(17)	$(1/3)\sin(e^{x^3}) + c$	(18)	$\dfrac{e^{m\tan^{-1} x}}{m} + c$
(19)	$\tan(xe^x) + c$	(20)	$2\sin(e^{\sqrt{x}}) + c$

Problem C. Areas of regions.

(1)	πab sq. units	(2)	$15/2$ sq. units
(3)	6π sq. units	(4)	$2\sqrt{2}/3 + 9\pi/2 - 9\sin^{-1}(1/3)$ sq. units
(5)	$\pi/2 + 1/3$ sq. units	(6)	$\pi a^2/4 - 2a^2/3$ sq. units

(7) $\sqrt{2}/6 + 9\pi/8 - (9/4)\sin^{-1}(1/3)$ sq. units

(8)	$\pi/4 - 1/2$ sq. units	(9)	$23/6$ sq. units
(10)	$1/3$ sq. units		

Problem D. Different types of volume problems.

(1) $f(x) = \sqrt{\dfrac{2x-a}{\pi}}$	(2) $f(x) = \pm\sqrt{\dfrac{2x+1}{\pi}}$
(3) $32\pi/3$ cubic units	(4a) $2\pi a^3$ cubic units
(4b) $16\pi a^3/15$ cubic units	(4c) $8\pi a^3/5$ cubic units
(5) $(8\pi/3)(2-\sqrt{2})$ cubic units	(6) $(\pi/15)(88\sqrt{2}+107)$ cubic units
(7) $112\pi a^3/15$ cubic units	(8) hs^2 cubic units
(9) hs^2 cubic units	(10) $8r^3/3$ cubic units
(11) $\pi^2/4$ cubic units	(12) $28\pi/3$ cubic units

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