Calculus Examples 2025

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Abstract

Here are examples to use as models for problem solving in Calculus. At the same time these examples provide a thorough treatment of results in the subject.

Key words— Algebra, Graphing, Derivatives, Integrals, Change, measures, Limits

Contents

AMS Subject Classifications: Primary 05E05; Secondary 33D52.

1 Introduction

As a student of mathematics I always felt incapable of coming up with a solution to any given problem on the homework. However, if someone showed me how to do that question then I was capable of understanding the solution and reproducing it. I made it through my undergraduate math courses by asking (many) people to show me how to do the various questions that I encountered. Eventually I knew how to do so many of them that I could do well on any exam. It was a good way to succeed.

With that model in mind this is a compendium of examples designed for students that are like I was. Here solutions to many types of problems are presented carefully and the problems are chosen to be representative of most problems that could appear on an exam for a first year Calculus course. The idea is that if you learn how to deliver these solutions yourself then you will have a good command of Calculus.

Later, as a caclulus teacher I found that the greatest part of my task was to show the students how to do problems of the type that might appear on the exam. I found this compendium of examples helped greatly in preparing classes, lectures and problem sessions.

Although I had originally intended to include material from multivariable calculus I ran out of steam and this compendium does not cover the standard multivariable calculus.

An important piece of my philosophy of mathematics and teaching is that the proof (convincing explanation of why something is true) is always part of the endeavor. I think it always important to justify the power rule for the derivative, the chain rule for the derivative, why $\cos\left(\frac{\pi}{3}\right) = \frac{\sqrt{1}}{2}$, etc etc. The examples here are designed to include all those questions and justifications as example problems and to complete them along the way as possible exam questions.

2 Algebra

The **complex numbers** is the number system

$$\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R}\} \qquad \text{with} \quad i^2 = -1,$$

The **exponential**

$$e^x = 1 + x + \tfrac{1}{2!}x^2 + \tfrac{1}{3!}x^3 + \cdots \qquad \text{is the most important function in mathematics.}$$

Let $i^2 = -1$. The **trigonometric and hyperbolic functions** are defined by

$$\cos(x) = \tfrac{1}{2}(-i)(e^{ix} + e^{-ix}), \quad \cosh(x) = \tfrac{1}{2}(e^x + e^{-x}), \quad \sin(x) = \tfrac{1}{2}(e^{ix} - e^{-ix}), \quad \sinh(x) = \tfrac{1}{2}(e^x - e^{-x})$$

and

$$\tan(x) = \frac{\sin(x)}{\cos(x)}, \quad \cot(x) = \frac{1}{\tan(x)}, \quad \sec(x) = \frac{1}{\cos(x)}, \quad \csc(x) = \frac{1}{\sin(x)},$$

and

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)}, \quad \coth(x) = \frac{1}{\tanh(x)}, \quad \mathrm{sech}(x) = \frac{1}{\cosh(x)}, \quad \mathrm{csch}(x) = \frac{1}{\sinh(x)}.$$

The **derivative** $\frac{d}{dx}$ knows what to spit out by always following the rules:

(1) $\frac{dx}{dx} = 1$,

(2) $\frac{d(cf)}{dx} = c\frac{df}{dx}$, if c is a constant,

(3) $\frac{d(f+g)}{dx} = \frac{df}{dx} + \frac{dg}{dx}$,

(4) $\frac{d(fg)}{dx} = f\,\frac{dg}{dx} + \frac{df}{dx}\,g$.

The **integral** is the inverse 'function' to the derivative: the integral undoes the derivative. This means that

$$\int \frac{df}{dx}\,dx = f \quad \text{and} \quad \frac{d}{dx}\Big(\int f\,dx\Big) = f.$$

So

$$\begin{array}{lll}
\frac{d(cf)}{dx} = c\frac{df}{dx} & \text{gives} & \int (cf)\,dx = c\int f\,dx, \\
\frac{d(f+g)}{dx} = \frac{df}{dx} + \frac{dg}{dx} & \text{gives} & \int (f+g)dx = \int f\,dx + \int g\,dx, \\
\frac{d(fg)}{dx} = \frac{df}{dx}g + f\frac{dg}{dx} & \text{gives} & \int f\,dg = fg - \int g\,df, \\
\frac{d(f\circ g)}{dx} = \frac{df}{dg}\frac{dg}{dx} & \text{gives} & \int g\,\frac{du}{dx}\,dx = \int g\,du, \\
\frac{d(f^g)}{dx} = f^g\Big(\frac{g}{f}\frac{df}{dx} + \log f\frac{dg}{dx}\Big) & \text{gives} & \int f^{g-1}\Big(g\frac{df}{dx} + f\log f\frac{dg}{dx}\Big)dx = f^g.
\end{array}$$

Like most inverse 'functions', the integral is not a function.

Other frequently used inverse functions:

$\sqrt{x}$ is the 'function' that undoes x^2. This means that

$$\sqrt{x^2} = x \quad \text{and} \quad (\sqrt{x})^2 = x.$$

$\log(x)$ is the 'function' that undoes e^x. This means that

$$\log(e^x) = x \quad \text{and} \quad e^{\log(x)} = x.$$

$\sin^{-1}(x)$ is the 'function' that undoes $\sin(x)$. This means that

$$\sin^{-1}(\sin(x)) = x \quad \text{and} \quad \sin(\sin^{-1}(x)) = x.$$

$\cos^{-1}(x)$ is the 'function' that undoes $\cos(x)$. This means that

$$\cos^{-1}(\cos(x)) = x \quad \text{and} \quad \cos(\cos^{-1}(x)) = x.$$

$\log_a(x)$ is the 'function' that undoes a^x. This means that

$$\log_a(a^{\sqrt{7}\pi i \sin(32)}) = \sqrt{7}\pi i \sin(32) \qquad \text{and} \qquad a^{\log_a(\sqrt{7}\pi i \sin(32))} = \sqrt{7}\pi i \sin(32).$$

WARNING: $\sin^2(x)$ is VERY DIFFERENT from $\sin(x)^2$. For example,

$$\sin^2\left(\frac{\pi}{4}\right) = \sin(\sin(\frac{\pi}{4})) = \sin(\frac{1}{\sqrt{2}}) \approx 0.6496369, \qquad \text{BUT} \qquad \sin\left(\frac{\pi}{4}\right)^2 = \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}.$$

WARNING: $\sin^{-1}(x)$ is VERY DIFFERENT from $\sin(x)^{-1}$. For example,

$$\sin^{-1}(0) = \sin^{-1}(\sin(0)) = 0, \qquad \text{BUT} \qquad \sin(0)^{-1} = \frac{1}{\sin(0)} = \frac{1}{0} = \text{UNDEFINED IN } \mathbb{C}.$$

2.1 Derivatives and integrals

Example 2.1. Find $\frac{dy}{dx}$ if $y = 5x$.

Proof. $\frac{dy}{dx} = \frac{d(5x)}{dx} = 5\,\frac{dx}{dx} = 5 \cdot 1 = 5.$ □

Example 2.2. Find $\frac{dy}{dx}$ if $y = \pi x$.

Proof. $\frac{dy}{dx} = \frac{d(\pi x)}{dx} = \pi\,\frac{dx}{dx} = \pi \cdot 1 = \pi.$ □

Example 2.3. Prove that $\frac{d1}{dx} = 0$.

Proof.

$$\frac{d1}{dx} = \frac{d(1 \cdot 1)}{dx} = 1 \cdot \frac{d1}{dx} + \frac{d1}{dx} \cdot 1 = \frac{d1}{dx} + \frac{d1}{dx}.$$

Subtract $\frac{d1}{dx}$ from both sides. So $\quad \frac{d1}{dx} = 0.$ □

Example 2.4. Find $\frac{dy}{dx}$ if $y = 5$.

Proof.

$$\frac{dy}{dx} = \frac{d5}{dx} = \frac{d(5 \cdot 1)}{dx} = 5 \cdot \frac{d1}{dx} = 5 \cdot 0 = 0.$$

□

Example 2.5. Find $\frac{dy}{dx}$ if $y = 6342$.

Proof.

$$\frac{dy}{dx} = \frac{d\,6342}{dx} = \frac{d(6342 \cdot 1)}{dx} = 6342 \cdot \frac{d1}{dx} = 6342 \cdot 0 = 0.$$

□

Example 2.6. Prove that if c is a constant then $\frac{dc}{dx} = 0$

Proof.

$$\frac{dc}{dx} = \frac{d(c\cdot 1)}{dx} = c\cdot\frac{d1}{dx} = c\cdot 0 = 0.$$

□

Example 2.7. Find $\frac{dy}{dx}$ if $y = 3x + 12$.

Proof.

$$\frac{dy}{dx} = \frac{d(3x+12)}{dx} = \frac{d(3x)}{dx} + \frac{d12}{dx} = 3\frac{dx}{dx} + 0 = 3\cdot 1 + 0 = 3.$$

□

Example 2.8. Find $\frac{dy}{dx}$ if $y = x^2$.

Proof.

$$\frac{dy}{dx} = \frac{dx^2}{dx} = \frac{d(x\cdot x)}{dx} = x\,\frac{dx}{dx} + \frac{dx}{dx}\,x = x\cdot 1 + 1\cdot x = 2x.$$

□

Example 2.9. Find $\frac{dy}{dx}$ if $y = x^3$.

Proof.

$$\frac{dy}{dx} = \frac{dx^3}{dx} = \frac{d(x^2\cdot x)}{dx} = x^2\,\frac{dx}{dx} + \frac{dx^2}{dx}\,x = x^2\cdot 1 + 2x\cdot x = 3x^2.$$

□

Example 2.10. Find $\frac{dy}{dx}$ if $y = x^4$.

Proof.

$$\frac{dy}{dx} = \frac{dx^4}{dx} = \frac{d(x^3\cdot x)}{dx} = x^3\,\frac{dx}{dx} + \frac{dx^3}{dx}\,x = x^3\cdot 1 + 3x^2\cdot x = 4x^3.$$

□

. . . and we keep on going . . .

Example 2.11. Find $\frac{dy}{dx}$ if $y = x^{6342}$.

Proof.

$$\frac{dy}{dx} = \frac{dx^{6342}}{dx} = \frac{d(x^{6341}\cdot x)}{dx} = x^{6341}\,\frac{dx}{dx} + \frac{dx^{6341}}{dx}\,x = x^{6341}\cdot 1 + 6341x^{6340}\cdot x = 6342x^{6341}.$$

□

. . . and we keep on going . . .

Example 2.12. Find $\dfrac{dx^n}{dx}$ for $n \in \{1, 2, 3, \ldots\}$.

Proof. The base cases are $\frac{dx}{dx} = 1 = 1x^0$ and Example 2.8- Example 2.10. The induction step is:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dx^n}{dx} = \frac{d(x^{n-1} \cdot x)}{dx} = x^{n-1}\frac{dx}{dx} + \frac{dx^{n-1}}{dx}x \\
&= x^{n-1} \cdot 1 + (n-1)x^{n-2} \cdot x, \qquad \text{since we already found } \frac{dx^{n-1}}{dx} = (n-1)x^{n-2}, \\
&= nx^{n-1}.
\end{aligned}$$

□

Example 2.13. Find $\dfrac{dx^n}{dx}$ for $n = 0$.

Proof.

$$\frac{dy}{dx} = \frac{dx^0}{dx} = \frac{d1}{dx} = 0 = 0x^{-1} = 0x^{0-1}.$$

□

Example 2.14. Find $\dfrac{dx^{-6342}}{dx}$.

Proof.

$$\frac{dx^{-6342} \cdot x^{6342}}{dx} = \frac{dx^0}{dx} = \frac{d1}{dx} = 0.$$

On the other hand,

$$\frac{dx^{-6342} \cdot x^{6342}}{dx} = x^{-6342}\frac{dx^{6342}}{dx} + \frac{dx^{-6342}}{dx} \cdot x^{6342} = x^{-6342} \cdot 6342x^{6341} + \frac{dx^{-6342}}{dx} \cdot x^{6342}.$$

So

$$0 = x^{-6342} \cdot 6342x^{6341} + \frac{dx^{-6342}}{dx} \cdot x^{6342}.$$

Solve for $\dfrac{dx^{-6342}}{dx}$.

$$\frac{dx^{-6342}}{dx} = -6342x^{-1}x^{-6342} = (-6342)x^{-6343}.$$

□

Example 2.15. Find $\dfrac{dx^{-n}}{dx}$ for $n \in \{1, 2, 3, \ldots\}$.

Proof. Let $n \in \mathbb{Z}_{>0}$. Then

$$\frac{dx^{-n} \cdot x^n}{dx} = \frac{dx^0}{dx} = \frac{d1}{dx} = 0.$$

On the other hand,

$$\frac{dx^{-n} \cdot x^n}{dx} = x^{-n}\frac{dx^n}{dx} + \frac{dx^{-n}}{dx} \cdot x^n = x^{-n} \cdot nx^{n-1} + \frac{dx^{-n}}{dx} \cdot x^n.$$

So

$$0 = x^{-n} \cdot nx^{n-1} + \frac{dx^{-n}}{dx} \cdot x^n = nx^{-1} + \frac{dx^{-n}}{dx} \cdot x^n.$$

Solve for $\frac{dx^{-n}}{dx}$.

$$\frac{dx^{-n}}{dx} = -nx^{-1}x^{-n} = (-n)x^{-n-1}.$$

and thus we have found $\quad \frac{dx^n}{dx} = nx^{n-1}, \quad$ for *all* integers n. (AMAZING!) □

Example 2.16. Let $y = 3x^3 + 5x^2 + 2x + 7$. Find $\frac{dy}{dx}$.

Proof.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d(3x^3 + 5x^2 + 2x + 7)}{dx} \\
&= \frac{d(3x^3)}{dx} + \frac{d(5x^2 + 2x + 7)}{dx} \\
&= \frac{d(3x^3)}{dx} + \frac{d(5x^2)}{dx} + \frac{d(2x)}{dx} + \frac{d7}{dx} \\
&= 3\frac{dx^3}{dx} + 5\frac{dx^2}{dx} + 2\frac{dx}{dx} + 7\frac{d1}{dx} \\
&= 3 \cdot 3x^2 + 5 \cdot 2x + 2 \cdot 1 + 7 \cdot 0 \\
&= 9x^2 + 10x + 2.
\end{aligned}$$

□

Example 2.17. Let $y = -7x^{-13} + 5x^{-7} + (6+2i)x^{38}$. Find $\frac{dy}{dx}$.

Proof.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d(-7x^{-13} + 5x^{-7} + (6+2i)x^{38})}{dx} \\
&= \frac{d(-7x^{-13})}{dx} + \frac{d(5x^{-7})}{dx} + \frac{d((6+2i)x^{38})}{dx} \\
&= -7\frac{dx^{-13}}{dx} + 5\frac{dx^{-7}}{dx} + (6+2i)\frac{dx^{38}}{dx} \\
&= -7(-13)x^{-13-1} + 5(-7)x^{-7-1} + (6+2i)38x^{38-1} \\
&= 91x^{-14} - 35x^{-8} + (228 + 76i)x^{37}.
\end{aligned}$$

□

Example 2.18. Find $\frac{dy}{dx}$ if $y = g^2$.

Proof.

$$\frac{dy}{dx} = \frac{dg^2}{dx} = \frac{d(g \cdot g)}{dx} = g\,\frac{dg}{dx} + \frac{dg}{dx}\,g = 2g\frac{dg}{dx} = \frac{dy}{dg}\frac{dg}{dx}.$$

□

Example 2.19. Find $\frac{dy}{dx}$ if $y = g^3$.

Proof.

$$\frac{dy}{dx} = \frac{dg^3}{dx} = \frac{d(g^2 \cdot g)}{dx} = g^2 \frac{dg}{dx} + \frac{dg^2}{dx} g = g^2 \cdot \frac{dg}{dx} + 2g \frac{dg}{dx} \cdot g = 3g^2 \frac{dg}{dx} = \frac{dy}{dg}\frac{dg}{dx}.$$

□

Example 2.20. Find $\frac{dy}{dx}$ if $y = g^4$.

Proof.

$$\frac{dy}{dx} = \frac{dg^4}{dx} = \frac{d(g^3 \cdot g)}{dx} = g^3 \frac{dg}{dx} + \frac{dg^3}{dx} g = g^3 \cdot \frac{dg}{dx} + 3g^2 \frac{dg}{dx} \cdot g = 4g^3 \frac{dg}{dx} = \frac{dy}{dg}\frac{dg}{dx}.$$

□

. . . and we keep on going . . .

Example 2.21. Find $\frac{dy}{dx}$ if $y = g^{6342}$.

Proof.

$$\begin{aligned}\frac{dy}{dx} &= \frac{dg^{6342}}{dx} = \frac{d(g^{6341} \cdot g)}{dx} = g^{6341} \frac{dg}{dx} + \frac{dg^{6341}}{dx} g \\ &= g^{6341} \cdot \frac{dg}{dx} + 6341 g^{6340} \frac{dg}{dx} \cdot g = 6342 g^{6341} \frac{dg}{dx} = \frac{dy}{dg}\frac{dg}{dx}.\end{aligned}$$

□

. . . and we keep on going . . .

Example 2.22. Find $\frac{dg^n}{dx}$ for $n \in \{1, 2, 3, \ldots\}$.

Proof. The base cases are $\frac{dg}{dx} = \frac{dg}{dx} = 1g^0 \frac{dg}{dx}$ and Example 2.8- Example 2.20. The induction step: Let $y = g^n$. Then

$$\begin{aligned}\frac{dy}{dx} &= \frac{dg^n}{dx} = \frac{d(g^{n-1} \cdot x)}{dx} = g^{n-1} \frac{dg}{dx} + \frac{dg^{n-1}}{dx} x \\ &= g^{n-1} \cdot \frac{dg}{dx} + (n-1) g^{n-2} \frac{dg}{dx} \cdot g, \qquad \text{since we already found} \quad \frac{dg^{n-1}}{dx} = (n-1) g^{n-2} \frac{dg}{dx}, \\ &= n g^{n-1} \frac{dg}{dx} = \frac{dy}{dg}\frac{dg}{dx}.\end{aligned}$$

□

Example 2.23. Find $\frac{dg^n}{dx}$ for $n = 0$.

Proof. Let $y = g^0 = 1$.

$$\frac{dy}{dx} = \frac{dg^0}{dx} = \frac{d1}{dx} = 0 = 0g^{-1} = 0g^{0-1}\frac{dg}{dx} = \frac{dy}{dg}\frac{dg}{dx}.$$

□

Example 2.24. Find $\dfrac{dg^{-6342}}{dx}$.

Proof.

$$\frac{dg^{-6342}\cdot g^{6342}}{dx} = \frac{dg^0}{dx} = \frac{d1}{dx} = 0.$$

On the other hand,

$$\frac{dg^{-6342}\cdot g^{6342}}{dx} = g^{-6342}\frac{dg^{6342}}{dx} + \frac{dg^{-6342}}{dx}\cdot g^{6342} = g^{-6342}\cdot 6342g^{6341} + \frac{dg^{-6342}}{dx}\cdot g^{6342}.$$

So

$$0 = g^{-6342}\cdot 6342g^{6341} + \frac{dg^{-6342}}{dx}\cdot g^{6342}.$$

Solve for $\dfrac{dg^{-6342}}{dx}$.

$$\frac{dg^{-6342}}{dx} = -6342g^{-1}g^{-6342} = (-6342)g^{-6343}\frac{dg}{dx}.$$

So, if $y = g^{-6342}$ then $\frac{dy}{dx} = \frac{dy}{dg}\frac{dg}{dx}$. □

Example 2.25. Find $\dfrac{dg^{-n}}{dx}$ for $n \in \{1, 2, 3, \ldots\}$.

Proof. . Let $n \in \mathbb{Z}_{>0}$. Then

$$\frac{dg^{-n}\cdot g^{n}}{dx} = \frac{dg^0}{dx} = \frac{d1}{dx} = 0.$$

On the other hand,

$$\frac{dg^{-n}\cdot g^{n}}{dx} = g^{-n}\frac{dg^{n}}{dx} + \frac{dg^{-n}}{dx}\cdot g^{n} = g^{-n}\cdot ng^{n-1}\frac{dg}{dx} + \frac{dg^{-n}}{dx}\cdot g^{n}\frac{dg}{dx}.$$

So

$$0 = g^{-n}\cdot ng^{n-1} + \frac{dg^{-n}}{dx}\cdot g^{n} = ng^{-1}\frac{dg}{dx} + \frac{dg^{-n}}{dx}\cdot g^{n}.$$

Solve for $\dfrac{dg^{-n}}{dx}$.

$$\frac{dg^{-n}}{dx} = -ng^{-1}g^{-n}\frac{dg}{dx} = (-n)x^{-n-1}\frac{dg}{dx}.$$

So, if $y = g^{-6342}$ then $\frac{dy}{dx} = \frac{dy}{dg}\frac{dg}{dx}$.

We have found $\quad \dfrac{dg^n}{dx} = ng^{n-1}\dfrac{dg}{dx}, \quad$ for *all* integers n. (AMAZING!) □

Example 2.26. Find $\dfrac{dy}{dx}$ when $y = (2x-5)^2$.

Proof. If $g = 2x - 5$ then $y = g^2$.

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dg}\frac{dg}{dx} = \frac{dg^2}{dg}\frac{d(2x-5)}{dx} = 2g(2-0) = 2(2x-5)\cdot 2 \\ &= 4(2x-5) = 8x-20.\end{aligned}$$

□

Example 2.27. Find $\dfrac{dy}{dx}$ when $y = (3x-4)^3$.

Proof. If $g = 3x - 4$ then $y = g^3$. Then

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dg}\frac{dg}{dx} = \frac{dg^3}{dg}\frac{d(3x-4)}{dx} = 3g^2(3-0) = 9(3x-4)^2 \\ &= 9(9x^2 - 24x + 16) = 81x^2 - 72x + 144.\end{aligned}$$

□

Example 2.28. Find $\dfrac{dy}{dx}$ when $y = (2x-5)^2(3x-4)^3$.

Proof.

$$\begin{aligned}\frac{dy}{dx} &= \frac{d(2x-5)^2(3x-4)^3}{dx} = (2x-5)^2\frac{d(3x-4)^3}{dx} + \frac{d(2x-5)^2}{dx}(3x-4)^3 \\ &= (2x-5)^2\cdot 3(3x-4)^2\cdot 3 + 2(2x-5)\cdot 2(3x-4)^3 \\ &= (2x-5)(3x-4)^2(9(2x-5)+4(3x-4)) = (2x-5)(3x-4)^2(30x-61).\end{aligned}$$

□

Example 2.29. Find $\dfrac{dy}{dx}$ when $y = \left(\frac{x-3}{x-4}\right)^2$.

Proof.

$$\begin{aligned}\frac{d\,\left(\frac{x-3}{x-4}\right)^2}{dx} &= 2\Big(\frac{x-3}{x-4}\Big)\frac{d\,\Big(\frac{x-3}{x-4}\Big)}{dx} \\ &= 2\Big(\frac{x-3}{x-4}\Big)\frac{d\,\Big((x-3)(x-4)^{-1}\Big)}{dx} \\ &= 2\Big(\frac{x-3}{x-4}\Big)\Big((x-3)\frac{(x-4)^{-1}}{dx} + \frac{d(x-3)}{dx}(x-4)^{-1}\Big) \\ &= 2\Big(\frac{x-3}{x-4}\Big)\Big((x-3)(-1)(x-4)^{-2}\frac{(x-4)}{dx} + 1\cdot(x-4)^{-1}\Big) \\ &= 2\Big(\frac{x-3}{x-4}\Big)\Big(-\frac{(x-3)}{(x-4)^2}\cdot 1 + \frac{1}{x-4}\Big) \\ &= 2\Big(\frac{x-3}{x-4}\Big)\Big(-\frac{(x-3)}{(x-4)^2}\cdot 1 + \frac{x-4}{(x-4)^2}\Big) \\ &= 2\Big(\frac{x-3}{x-4}\Big)\frac{(-1)}{(x-4)^2} = \frac{-2x+6}{(x-4)^3}\end{aligned}$$

□

Example 2.30. Find $\dfrac{d\,x^{m/n}}{dx}$ when m and n are integers and $n \neq 0$.

Proof.

$$\frac{d(x^{m/n})^n}{dx} = \frac{dx^m}{dx} = mx^{m-1}. \qquad \text{On the other hand} \qquad \frac{d(x^{m/n})^n}{dx} = n\big(x^{m/n}\big)^{n-1}\frac{dx^{m/n}}{dx}.$$

So $\quad mx^{m-1} = n\big(x^{m/n}\big)^{n-1}\dfrac{dx^{m/n}}{dx} \quad$ and we can solve for $\dfrac{dx^{m/n}}{dx}$.

$$\begin{aligned}\frac{dx^{m/n}}{dx} &= \frac{mx^{m-1}}{n\big(x^{m/n}\big)^{n-1}} = \frac{mx^{m-1}}{n\big(x^{m/n}\big)^{n}\big(x^{m/n}\big)^{-1}} \\ &= \frac{mx^{m-1}}{nx^m\ \frac{1}{x^{m/n}}} = \left(\frac{m}{n}\right)x^{-1}x^{m/n} = \left(\frac{m}{n}\right)x^{(m/n)-1}.\end{aligned}$$

□

Example 2.31. Evaluate $\int \sqrt{x}\,dx$.

Proof.

$$\int \sqrt{x}\,dx = \int x^{\frac12}\,dx = \frac{x^{\frac32}}{\frac32} + c = \frac23 x^{\frac32} + c, \qquad \text{where } c \text{ is a constant,}$$

since

$$\frac{d\frac23 x^{\frac32}}{dx} = \frac23\cdot\frac32 x^{\frac12} = x^{\frac12}.$$

□

Example 2.32. Evaluate $\int \sqrt[3]{x}\,dx$.

Proof.

$$\int \sqrt[3]{x}\,dx = \int x^{\frac13}\,dx = \frac{x^{\frac43}}{\frac43} + c = \frac34 x^{\frac43} + c, \qquad \text{where } c \text{ is a constant,}$$

since

$$\frac{d\frac34 x^{4/3}}{dx} = \frac34\cdot\frac43 x^{1/3} = x^{1/3}.$$

□

Example 2.33. Evaluate $\int x^{\frac{362}{431}}\,dx$.

Proof.

$$\int x^{\frac{362}{431}}\,dx = \frac{x^{\frac{362}{431}+1}}{\frac{362}{431}+1} + c = \frac{x^{\frac{793}{431}}}{\frac{793}{431}} + c = \frac{431}{793}x^{\frac{793}{431}} + c, \qquad \text{where } c \text{ is a constant,}$$

since

$$\frac{d\frac{431}{793}x^{\frac{793}{431}}}{dx} = \frac{431}{793}\cdot\frac{793}{431}x^{\frac{793}{431}-1} = x^{\frac{362}{431}}.$$

□

Example 2.34. Find $\frac{dy}{dx}$ when $y = \frac{x}{\sqrt{1-2x}}$.

Proof.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d\, \frac{x}{\sqrt{1-2x}}}{dx} = \frac{d\, x\big(\sqrt{1-2x}\big)^{-1}}{dx} = \frac{d\, x\big((1-2x)^{1/2}\big)^{-1}}{dx} \\
&= \frac{d\, x(1-2x)^{-(1/2)}}{dx} = x\,\frac{d\,(1-2x)^{-(1/2)}}{dx} + \frac{dx}{dx}\,(1-2x)^{-(1/2)} \\
&= x\big(-\tfrac{1}{2}\big)(1-2x)^{-3/2}\,\frac{d\,(1-2x)}{dx} + 1\cdot\frac{1}{\sqrt{1-2x}} \\
&= \frac{-x}{2(1-2x)^{3/2}}\cdot(-2) + \frac{1}{(1-2x)^{1/2}} = \frac{x+1-2x}{(1-2x)^{3/2}} = \frac{1-x}{(1-2x)^{3/2}}.
\end{aligned}$$

□

Example 2.35. Find $\frac{dy}{dx}$ when $y = \frac{\sqrt{1+x^2}}{\sqrt{1-x^2}}$.

Proof.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d\,\frac{\sqrt{1+x^2}}{\sqrt{1-x^2}}}{dx} = \frac{d\,\frac{(1+x^2)^{1/2}}{(1-x^2)^{1/2}}}{dx} = \frac{d\left(\frac{1+x^2}{1-x^2}\right)^{1/2}}{dx} \\
&= \frac{1}{2}\cdot\left(\frac{1+x^2}{1-x^2}\right)^{(1/2)-1}\frac{d\left(\frac{1+x^2}{1-x^2}\right)}{dx} = \frac{1}{2}\cdot\left(\frac{1+x^2}{1-x^2}\right)^{-(1/2)}\frac{d\,(1+x^2)(1-x^2)^{-1}}{dx} \\
&= \frac{1}{2}\cdot\left(\frac{1-x^2}{1+x^2}\right)^{1/2}\left((1+x^2)\frac{d\,(1-x^2)^{-1}}{dx} + \frac{d\,(1+x^2)}{dx}(1-x^2)^{-1}\right) \\
&= \frac{1}{2}\left(\frac{1-x^2}{1+x^2}\right)^{1/2}\left((1+x^2)(-1)(1-x^2)^{-2}\frac{d\,(1-x^2)^{-1}}{dx} + 2x(1-x^2)^{-1}\right) \\
&= \frac{1}{2}\cdot\left(\frac{1-x^2}{1+x^2}\right)^{1/2}\left(\frac{(-1)(1+x^2)(-2x)}{(1-x^2)^2} + \frac{2x}{1-x^2}\right) \\
&= \frac{1}{2}\cdot\left(\frac{1-x^2}{1+x^2}\right)^{1/2}\left(\frac{2x(1+x^2)}{(1-x^2)^2} + \frac{2x(1-x^2)}{(1-x^2)^2}\right) \\
&= \frac{1}{2}\cdot\left(\frac{1-x^2}{1+x^2}\right)^{1/2}\left(\frac{2x(1+x^2+1-x^2)}{(1-x^2)^2}\right) \\
&= \frac{1}{2}\cdot\frac{(1-x^2)^{1/2}}{(1+x^2)^{1/2}}\cdot\frac{4x}{(1-x^2)^2} = \frac{2x}{(1+x^2)^{1/2}(1-x^2)^{3/2}}.
\end{aligned}$$

□

Example 2.36. Differentiate $\frac{x^2}{1+x^2}$ with respect to x^2.

Proof. This is the same problem as: Find $\frac{dz}{dp}$ when $z = \frac{x^2}{1+x^2}$ and $p = x^2$.

Since $\frac{dz}{dx} = \frac{dz}{dp}\frac{dp}{dx}$ then $\frac{dz}{dp} = \frac{\big(dz/dx\big)}{\big(dp/dx\big)}$.

So

$$\frac{dz}{dp} = \frac{\frac{d}{dx}\left(\frac{x^2}{1+x^2}\right)}{\frac{d}{dx}(x^2)} = \frac{\frac{d\ x^2(1+x^2)^{-1}}{dx}}{\frac{d\ x^2}{dx}} = \frac{x^2\frac{d\ (1+x^2)^{-1}}{dx} + \frac{dx^2}{dx}(1+x^2)^{-1}}{2x}$$
$$= \frac{x^2(-1)(1+x^2)^{-2}\frac{d\ (1+x^2)}{dx} + 2x(1+x^2)^{-1}}{2x} = \frac{\frac{-x^2}{(1+x^2)^2}\cdot 2x + \frac{2x}{1+x^2}}{2x}$$
$$= \frac{-x^2}{(1+x^2)^2} + \frac{1}{1+x^2} = \frac{-x^2+1+x^2}{(1+x^2)^2} = \frac{1}{(1+x^2)^2}.$$

□

Example 2.37. Let $a \in \mathbb{C}$. Find $\frac{dy}{dx}$ when $x^4 + y^4 = 4a^2x^2y^2$.

Proof.

$$\frac{d\,(x^4+y^4)}{dx} = \frac{d\,(4a^2x^2y^2)}{dx}. \qquad \text{So} \quad \frac{dx^4}{dx} + \frac{dy^4}{dx} = 4a^2\frac{dx^2y^2}{dx}.$$

So $\quad 4x^3 + 4y^3\frac{dy}{dx} = 4a^2\left(x^2\frac{dy^2}{dx} + \frac{dx^2}{dx}y^2\right).$

So $\quad 4x^3 + 4y^3\frac{dy}{dx} = 4a^2\left(x^2 2y\frac{dy}{dx} + 2xy^2\right) = 4a^2x^2 2y\frac{dy}{dx} + 4a^2 2xy^2.$

So $\quad 4x^3 - 4a^2 2xy^2 = 4a^2x^2 2y\frac{dy}{dx} - 4y^3\frac{dy}{dx}.$

So $\quad 4x^3 - 4a^2 2xy^2 = \left(4a^2x^2 2y - 4y^3\right)\frac{dy}{dx}.$

So $\quad \frac{4x^3 - 4a^2 2xy^2}{4a^2x^2 2y - 4y^3} = \frac{dy}{dx}.$

So $\quad \frac{dy}{dx} = \frac{x^3 - 2a^2xy^2}{2a^2x^2y - y^3}.$

All we did is take the derivative of both sides and then solve for $\frac{dy}{dx}$. □

Example 2.38. Find $\frac{dy}{dx}\big]_{x=3}$ when $y = (x+1)(x+2)$.

Proof. The notation $\frac{dy}{dx}\big]_{x=3}$ means: find $\frac{dy}{dx}$ and then plug in $x = 3$.

$$\begin{aligned}\frac{dy}{dx}\Big]_{x=3} &= \frac{d\big((x+1)(x+2)\big)}{dx}\Big]_{x=3} \\ &= \left((x+1)\frac{d(x+2)}{dx} + \frac{d(x+1)}{dx}(x+2)\right)\Big]_{x-3} \\ &= ((x+1)+(x+2))\big]_{x=3} = (2x+3)\big]_{x=3} \\ &= 2\cdot 3 + 3 = 9.\end{aligned}$$

□

Example 2.39. Let $a \in \mathbb{C}$. Find $\frac{dy}{dx}$ when $x = \frac{3at}{1+t^3}$ and $y = \frac{3at^2}{1+t^3}$.

Proof. Since $y = \dfrac{3at^2}{1+t^3} = \left(\dfrac{3at}{1+t^3}\right) t = xt$ then $\dfrac{dy}{dx} = x\dfrac{dt}{dx} + \dfrac{dx}{dx}\cdot t = x\dfrac{dt}{dx} + t.$

What is $\dfrac{dt}{dx}$??

Since $\dfrac{dx}{dx} = \dfrac{dx}{dt}\,\dfrac{dt}{dx}$ then $\dfrac{dt}{dx} = \dfrac{(dx/dx)}{(dx/dt)} = \dfrac{1}{dx/dt}.$

So

$$\begin{aligned}
\frac{dt}{dx} &= \frac{1}{dx/dt} = \frac{1}{\frac{d}{dt}\left(\frac{3at}{1+t^3}\right)} = \frac{1}{\frac{d\ (3at)(1+t^3)^{-1}}{dt}} \\
&= \frac{1}{3at(-1)(1+t^3)^{-2}\frac{d(1+t^3)}{dt} + 3a(1+t^3)^{-1}} \\
&= \frac{1}{\frac{-3at}{(1+t^3)^2}3t^2 + \frac{3a}{1+t^3}} = \frac{1}{\frac{-9at^3+3a(1+t^3)}{(1+t^3)^2}} \\
&= \frac{(1+t^3)^2}{-9at^3+3a(1+t^3)} = \frac{(1+t^3)^2}{3a-6at^3}.
\end{aligned}$$

So

$$\begin{aligned}
\frac{dy}{dx} &= x\frac{dt}{dx} + t = \frac{3at}{1+t^3}\,\frac{(1+t^3)^2}{3a(1-2t^3)} + t \\
&= \frac{t(1+t^3)}{1-2t^3} + \frac{t(1-2t^3)}{1-2t^3} = \frac{t+t^4+t-2t^4}{1-2t^3} = \frac{2t-t^4}{1-2t^3}
\end{aligned}$$

□

Example 2.40. Compute $\displaystyle\int \frac{4x-5}{2x^2-5x+1}\,dx.$

Proof. Let $u = 2x^2-5x+1$. Then $\frac{du}{dx} = 4x-5$. So

$$\begin{aligned}
\int \frac{4x-5}{2x^2-5x+1}\,dx &= \int \frac{1}{u}\,\frac{du}{dx}\,dx = \int \frac{1}{u}\,du \\
&= \log(u) + c = \log(2x^2-5x+1) + c,
\end{aligned}$$

where c is a constant. □

Example 2.41. Compute $\displaystyle\int \frac{\tan(\sqrt{x})\sec(\sqrt{x})^2}{\sqrt{x}}\,dx.$

Proof. Let $u = \tan(\sqrt{x})$. Then

$$\frac{du}{dx} = \sec(\sqrt{x})^2\cdot\frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2}\frac{\sec(\sqrt{x})^2}{\sqrt{x}}.$$

So

$$\int \frac{\tan(\sqrt{x})\sec(\sqrt{x})^2}{\sqrt{x}}\,dx = \int 2u\,\frac{du}{dx}\,dx = \int 2u\,du = u^2 + c = \tan(\sqrt{x})^2 + c,$$

where c is a constant. □

Example 2.42. Compute $\int x\sqrt{3x-2}\,dx$.

Proof.

$$\begin{aligned}\int x\sqrt{3x-2}\,dx &= \int \frac{1}{3}3x\sqrt{3x-2}\,dx \\ &= \int \frac{1}{3}(3x-2+2)\sqrt{3x-2}\,dx \\ &= \int \frac{1}{3}\big((3x-2)^{\frac{3}{2}} + 2(3x-2)^{\frac{1}{2}}\big)\,dx \\ &= \frac{1}{3}\Big(\frac{2}{5}\frac{(3x-2)^{\frac{5}{2}}}{3} + 2\frac{(3x-2)^{\frac{3}{2}}}{3}\frac{2}{3}\Big) + c \\ &= \frac{2}{45}(3x-2)^{\frac{5}{2}} + \frac{4}{27}2(3x-2)^{\frac{3}{2}} + c,\end{aligned}$$

where c is a constant. □

Example 2.43. Compute $\int x\sqrt{x^2-1}\,dx$.

Proof. Let $u = x^2-1$. Then $\frac{du}{dx} = 2x$. So

$$\begin{aligned}\int x\sqrt{x^2-1}\,xd = \int \frac{1}{2}2x\sqrt{x^2-1}\,dx = \int \frac{1}{2}\frac{du}{dx}\sqrt{u}\,dx = \int \frac{1}{2}\sqrt{u}\,du = \int \frac{1}{2}u^{\frac{1}{2}}\,du \\ = \frac{1}{2}\frac{2}{3}u^{\frac{3}{2}} + c = \frac{1}{3}u^{\frac{3}{2}} + c = \frac{1}{3}(x^2-1)^{\frac{3}{2}} + c,\end{aligned}$$

where c is a constant. □

Example 2.44. Compute $\int \cos(x)^3\,dx$.

Proof.

$$\begin{aligned}\int \cos(x)^3\,dx &= \int \cos(x)\cos(x)^2\,dx \\ &= \int \cos(x)(1-\sin(x)^2)\,dx \\ &= \int (\cos(x) - \sin(x)^2\cos(x))\,dx \\ &= \sin(x) - \frac{\sin(x)^3}{3} + c,\end{aligned}$$

where c is a constant. □

Example 2.45. Compute $\int \frac{\log(x^2)}{x}\,dx$.

Proof. Let $u = \log(x^2)$. Then

$$\frac{du}{dx} = \frac{1}{x^2}2x = \frac{2}{x}.$$

So

$$\begin{aligned}
\int \frac{1}{2} 2 \frac{\log(x^2)}{x}\,dx &= \int \frac{1}{2}\frac{2}{x}\log(x^2)\,dx \\
&= \int \frac{1}{2}\frac{du}{dx} u\,dx \\
&= \int \frac{1}{2} u\,du \\
&= \frac{1}{2}\frac{u^2}{2} + c \\
&= \frac{u^2}{4} + c \\
&= \frac{\log(x^2)^2}{4} + c,
\end{aligned}$$

where c is a constant. □

Example 2.46. Compute $\displaystyle\int \frac{x}{\sqrt{1+x}}\,dx$.

Proof.

$$\begin{aligned}
\int \frac{x}{\sqrt{1+x}}\,dx = \int \frac{x+1-1}{\sqrt{1+x}}\,dx &= \int \Big(\frac{x+1}{(1+x)^{\frac{1}{2}}} - \frac{1}{(1+x)^{\frac{1}{2}}}\Big)\,dx \\
&= \int \Big((x+1)^{\frac{1}{2}} - (x+1)^{-\frac{1}{2}}\Big)\,dx = \frac{2}{3}(x+1)^{\frac{3}{2}} - 2(x+1)^{\frac{1}{2}} + c,
\end{aligned}$$

where c is a constant. □

Example 2.47. Compute $\displaystyle\int x\sqrt{x-1}\,dx$.

Proof.

$$\begin{aligned}
\int x\sqrt{x-1}\,dx = \int (x-1+1)\sqrt{x-1}\,dx &= \int \big((x-1)\sqrt{x-1} + \sqrt{x-1}\big)\,dx \\
&= \int \big((x-1)^{\frac{3}{2}} + (x-1)^{\frac{1}{2}}\big)\,dx = \frac{2}{5}(x-1)^{\frac{5}{2}} + \frac{2}{3}(x-1)^{\frac{3}{2}} + c,
\end{aligned}$$

where c is a constant. □

Example 2.48. Compute $\displaystyle\int (1-x)\sqrt{1+x}\,dx$.

Proof.

$$\begin{aligned}
\int (1-x)\sqrt{1+x}\,dx &= \int (-(1+x)+2)\sqrt{1+x}\,dx \\
&= \int (-(1+x)\sqrt{1+x} + 2\sqrt{1+x})\,dx \\
&= \int (-(1+x)^{\frac{3}{2}} + 2(1+x)^{\frac{1}{2}})\,dx \\
&= -\frac{2}{5}(1+x)^{\frac{5}{2}} + 2\cdot\frac{2}{3}(1+x)^{\frac{3}{2}} + c \\
&= -\frac{2}{5}(1+x)^{\frac{5}{2}} + \frac{4}{3}(1+x)^{\frac{3}{2}} + c,
\end{aligned}$$

where c is a constant. □

Example 2.49. Compute $\displaystyle\int \frac{\sin(x)}{\sin(x)-\cos(x)}\,dx$.

Proof.

$$\begin{aligned}\int \frac{\sin(x)}{\sin(x)-\cos(x)}\,dx &= \int \frac{\sin(x)-\cos(x)+\sin(x)+\cos(x)}{\sin(x)-\cos(x)}\cdot\frac{1}{2}\,dx \\ &= \frac{1}{2}\int\Big(\frac{\sin(x)-\cos(x)}{\sin(x)-\cos(x)}+\frac{\sin(x)+\cos(x)}{\sin(x)-\cos(x)}\Big)\ ,dx \\ &= \frac{1}{2}\int\Big(1+\frac{\sin(x)+\cos(x)}{\sin(x)-\cos(x)}\Big)\,dx \\ &= \frac{1}{2}\big(x+\log(\sin(x)-\cos(x)\big)+c,\end{aligned}$$

where c is a constant. □

Example 2.50. Compute $\displaystyle\int \frac{x^3}{1+x^8}\,dx$.

Proof.

$$\int \frac{x^3}{1+x^8}\,dx = \int \frac{x^3}{1+(x^4)^2}\,dx \qquad = \int \frac{1}{4}\cdot\frac{4x^3}{1+(x^4)^2}\,dx = \frac{1}{4}\tan^{-1}(x^4)+c,$$

where c is a constant, since

$$\frac{\frac{1}{4}\tan^{-1}(x^4)}{dx} = \frac{1}{4}\frac{1}{1+(x^4)^2}\frac{dx^4}{dx} = \frac{1}{4}\frac{4x^3}{1+(x^4)^2}.$$

□

Example 2.51. Compute $\displaystyle\int \tan^{-1}\Big(\frac{\sin(2x)}{1+\cos(2x)}\Big)\,dx$.

Proof.

$$\begin{aligned}\int \tan^{-1}\Big(\frac{\sin(2x)}{1+\cos(2x)}\Big)\,dx &= \int \tan^{-1}\Big(\frac{2\sin(x)\cos(x)}{1+\cos(x)^2-\sin(x)^2}\Big)\,dx \\ &= \int \tan^{-1}\Big(\frac{2\sin(x)\cos(x)}{\cos(x)^2+\cos(x)^2}\Big)\,dx \\ &= \int \tan^{-1}\Big(\frac{2\sin(x)\cos(x)}{2\cos(x)^2}\Big)\,dx = \int \tan^{-1}\Big(\frac{\sin(x)}{\cos(x)}\Big)\,dx \\ &= \int \tan^{-1}(\tan(x))\,dx = \int x\,dx = \frac{x^2}{2}+c,\end{aligned}$$

where c is a constant. □

Example 2.52. Compute $\displaystyle\int \cos^{-1}(\sin(x))\,dx$.

Proof. Let $x = \sin^{-1}(y)$. Then $\frac{dx}{dy} = \frac{1}{\sqrt{1-y^2}}$. So

$$\begin{aligned}\int \cos^{-1}(\sin(x))\,dx &= \int \cos^{-1}(\sin(x))\,\frac{dx}{dy}\,dy \\ &= \int \cos^{-1}(\sin(\sin^{-1}(y)))\,\frac{1}{\sqrt{1-y^2}}\,dy \\ &= \int \cos^{-1}(y)\,\frac{1}{\sqrt{1-y^2}}\,dy \\ &= -\int \cos^{-1}(y)\,\frac{-1}{\sqrt{1-y^2}}\,dy \\ &= -\frac{(\cos^{-1}(y))^2}{2} + c = -\frac{\cos^{-1}(\sin(x))}{2} + c,\end{aligned}$$

where c is a constant.
Another way: Let $\cos^{-1}(\sin(x)) = y$ then $\sin(x) = \cos(y)$. So

$$y = \frac{\pi}{2} - x.$$

So

$$\cos^{-1}(\sin(x)) = \frac{\pi}{2} - x.$$

So

$$\int \cos^{-1}(\sin(x))\,dx = \int \left(\frac{\pi}{2} - x\right)dx = \frac{\pi}{2} - \frac{x^2}{2} + c,$$

where c is a constant. ☐

2.2 Exponential, trigonometric and hyperbolic functions

Example 2.53. Assume $yx = xy$. Prove that

$$e^{x+y} = e^x e^y.$$

Proof.

$$e^{x+y} = \begin{array}{ll} & 1 \\ + & (x+y) \\ + & \frac{1}{2!}(x+y)^2 \\ + & \frac{1}{3!}(x+y)^3 \\ + & \frac{1}{4!}(x+y)^4 \\ + & \vdots \end{array} = \begin{array}{lc} & 1 \\ + & x+y \\ + & \frac{1}{2!}(x^2+2xy+y^2) \\ + & \frac{1}{3!}(x^3+3x^2y+3xy^2+y^3) \\ + & \frac{1}{4!}(x^4+4x^3y+6x^2y^2+4xy^3+y^4) \\ + & \vdots \end{array}$$

$$= \begin{array}{llllll} & 1 & & & & \\ + & x & +y & & & \\ + & \frac{1}{2!}x^2 & +\frac{1}{2!}2xy & +\frac{1}{2!}y^2 & & \\ + & \frac{1}{3!}x^3 & +\frac{1}{3!}3x^2y & +\frac{1}{3!}3xy^2 & +\frac{1}{3!}y^3 & \\ + & \frac{1}{4!}x^4 & +\frac{1}{4!}4x^3y & +\frac{1}{4!}6x^2y^2 & +\frac{1}{4!}4xy^3 & +\frac{1}{4!}y^4 \\ + & \vdots & & & & \end{array}$$

$$
\begin{array}{rllllll}
 & & 1 & & & & \\
 & + & x & +y & & & \\
 & + & \frac{1}{2!}x^2 & +xy & +\frac{1}{2!}y^2 & & \\
= & + & \frac{1}{3!}x^3 & +\frac{1}{2!}x^2y & +x\frac{1}{2!}y^2 & +\frac{1}{3!}y^3 & \\
 & + & \frac{1}{4!}x^4 & +\frac{1}{3!}x^3y & +\frac{1}{2!}x^2\frac{1}{2!}y^2 & +x\frac{1}{3!}y^3 & +\frac{1}{4!}y^4 \\
 & + & \vdots & & & & \\
= & & e^x & +e^xy & +e^x\frac{1}{2!}y^2 & +e^x\frac{1}{3!}y^3 & +e^x\frac{1}{4!}y^4 \quad \cdots
\end{array}
$$

$$= e^x(1 + y + \tfrac{1}{2!}y^2 + \tfrac{1}{3!}y^3 + \tfrac{1}{4!}y^4 + \cdots)$$

$$= e^x e^y.$$

□

Example 2.54. Let

$$e^x = 1 + x + \tfrac{1}{2!}x^2 + \tfrac{1}{3!}x^3 + \tfrac{1}{4!}x^4 + \cdots .$$

Find $\dfrac{de^x}{dx}$ and $\displaystyle\int e^x\,dx$.

Proof.

$$
\begin{aligned}
\frac{de^x}{dx} &= \frac{d}{dx}\Big(1 + x + \tfrac{1}{2!}x^2 + \tfrac{1}{3!}x^3 + \tfrac{1}{4!}x^4 + \cdots\Big) \\
&= 0 + 1 + \tfrac{1}{2!}2x + \tfrac{1}{3!}3x^2 + \tfrac{1}{4!}4x^3 + \cdots \\
&= 1 + x + \tfrac{1}{2!}x^2 + \tfrac{1}{3!}x^3 + \tfrac{1}{4!}x^4 + \cdots = e^x.
\end{aligned}
$$

Since $\dfrac{de^x}{dx} = e^x$ then

$$\int e^x\,dx = e^x + c, \qquad \text{where } c \text{ is a constant.}$$

□

Example 2.55. Find a polynomial that converts addition into multiplication.

Proof. Let $P(x) = a_0 + a_1x + a_2x^2 + \cdots$.

$$\text{We want} \qquad P(x+y) = P(x)P(y).$$

Well

$$
\begin{aligned}
P(x+y) &= a_0 + a_1(x+y) + a_2(x+y)^2 + \\
&= a_0 + a_1x + a_1y + a_2x^2 + 2a_2xy + a_2y^2 + \cdots, \qquad \text{and} \\
P(x)P(y) &= (a_0 + a_1x + a_2x^2 + \cdots)(a_0 + a_1y + a_2y^2 + \cdots) \\
&= a_0^2 + a_1a_0x + a_0a_1y + a_2a_0x^2 + a_1^2xy + a_0a_2y^2 + \cdots
\end{aligned}
$$

so that, if $a_0 \neq 0$ then

$$a_0 = 1, \quad a_1^2 = 2a_2, \quad a_1a_2 = 3a_3, \quad \ldots$$

and

$$P(x) = 1 + a_1x + \frac{1}{2!}a_1^2x^2 + \frac{1}{3!}a_1^3x^3 + \cdots = e^{a_1x}.$$

□

Example 2.56. Find a polynomial whose derivative is itself.

Proof. Let $Q(x) = c_0 + c_1 x + a_2 x^2 + \cdots \in \mathbb{C}[[x]]$. Then

$$\begin{aligned} \frac{dQ(x)}{dx} &= c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \ldots, \qquad \text{and} \\ Q(x) &= c_0 + c_1 x + c_2 x^2 + \cdots \end{aligned}$$

so that $\frac{dQ(x)}{dx} = Q(x)$ forces

$$c_1 = c_0, \qquad c_2 = \frac{1}{2}c_1 = \frac{1}{2}c_0, \qquad c_3 = \frac{1}{3}c_2 = \frac{1}{3!}c_0, \quad \ldots.$$

So $Q(x) = c_0 + c_1 x + a_2 x^2 + \cdots$ is equal to

$$Q(x) = c_0 + c_0 x + \frac{1}{2!}c_0 x^2 + \frac{1}{3!}c_0 x^3 + \cdots = c_0 e^x, \qquad \text{with } c_0 \in \mathbb{C}.$$

□

Example 2.57. Let $i^2 = -1$. Explain why

$$\begin{aligned} e^{ix} &= \cos(x) + i\sin(x), & e^{x} &= \cosh(x) + \sinh(x), \\ e^{-ix} &= \cos(x) - i\sin(x), & e^{-x} &= \cosh(x) - \sinh(x), \end{aligned}$$

Proof.

$$\begin{aligned} \cosh(x) + \sinh(x) &= \tfrac{1}{2}(e^x + e^{-x}) + \tfrac{1}{2}(e^x - e^{-x}) = \tfrac{1}{2}(2e^x + 0e^{-x}) = e^x, \\ \cosh(x) - \sinh(x) &= \tfrac{1}{2}(e^x + e^{-x}) - \tfrac{1}{2}(e^x - e^{-x}) = \tfrac{1}{2}(0e^x + 2e^{-x}) = e^{-x}, \\ \cos(x) + i\sin(x) &= \tfrac{1}{2}(e^{ix} + e^{-ix}) + i(-\tfrac{1}{2}i(e^{ix} - e^{-ix})) = \tfrac{1}{2}(e^{ix} + e^{-ix} + e^{ix} - e^{-ix}) = e^{ix}, \\ \cos(x) - i\sin(x) &= \tfrac{1}{2}(e^{ix} + e^{-ix}) - i(-\tfrac{1}{2}i(e^{ix} - e^{-ix})) = \tfrac{1}{2}(e^{ix} + e^{ix} - e^{ix} + e^{-ix}) = e^{-ix}. \end{aligned}$$

□

Example 2.58. Explain why

$$\begin{aligned} \cos(-x) &= \cos(x), & \cosh(-x) &= \cosh(x), \\ \sin(-x) &= -\sin(x), & \sinh(-x) &= -\sinh(x). \end{aligned}$$

Proof. Let $i^2 = -1$. Then

$$\begin{aligned} \cos(-x) &= \tfrac{1}{2}(e^{-ix} + e^{ix}) = \tfrac{1}{2}(e^{ix} + e^{-ix}) = \cos(x), \\ \sin(-x) &= -\tfrac{1}{2}i(e^{-ix} - e^{ix}) = -(-\tfrac{1}{2}i(e^{ix} - e^{-ix})) = -\sin(x), \\ \cosh(-x) &= \tfrac{1}{2}(e^{-x} + e^{x}) = \tfrac{1}{2}(e^{x} + e^{-x}) = \cosh(x), \\ \sinh(-x) &= -\tfrac{1}{2}(e^{-x} - e^{x}) = -(-\tfrac{1}{2}(e^{x} - e^{-x})) = -\sin(x). \end{aligned}$$

□

Example 2.59. Explain why

$$\cos^2(x) + \sin^2(x) = 1 \qquad \text{and} \qquad \cosh^2(x) - \sinh^2(x) = 1.$$

Proof. Let $i^2 = -1$. Using that $e^{ix} = \cos(x) + i\sin(x)$ and $e^{-ix} = \cos(x) - i\sin(x)$ from Example 2.57 gives

$$\begin{aligned} 1 = e^0 = e^{ix+(-ix)} \\ &= e^{ix}e^{-ix} \\ &= (\cos(x) + i\sin(x))(\cos(x) - i\sin(x)) \\ &= \cos^2(x) - i\sin(x)\cos(x) + i\sin(x)\cos(x) - i^2\sin^2(x) \\ &= \cos^2(x) - (-1)\sin^2(x) \\ &= \cos^2(x) + \sin^2(x). \end{aligned}$$

Using that $e^x = \cosh(x) + \sinh(x)$ and $e^{-x} = \cosh(x) - \sinh(x)$ from Example 2.57 gives

$$\begin{aligned} 1 = e^0 = e^{x+(-x)} &= e^x e^{-x} \\ &= (\cosh(x) + \sinh(x))(\cosh(x) - \sinh(x)) = \cosh^2(x) - \sinh^2(x). \end{aligned}$$

□

Example 2.60. Explain why

$$\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y), \qquad \sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y),$$
$$\cosh(x + y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y), \qquad \sinh(x + y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y).$$

Proof. Let $i^2 = -1$.

$$\begin{aligned} \cos(x + y) + i\sin(x + y) &= e^{i(x+y)} \\ &= e^{ix+iy} = e^{ix}e^{iy} \\ &= (\cos(x) + i\sin(x))(\cos(y) + i\sin(y)) \\ &= \cos(x)\cos(y) + i\cos(x)\sin(y) + i\sin(x)\cos(y) + i^2\sin(x)\sin(y) \\ &= \big(\cos(x)\cos(y) + (-1)\sin(x)\sin(y)\big) + i\big(\cos(x)\sin(y) + \sin(x)\cos(y)\big). \end{aligned}$$

Comparing terms on each side gives

$$\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y), \qquad \text{and} \qquad \sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y).$$

Next,

$$\begin{aligned} \cosh(x)\cosh(y) + \sinh(x)\sinh(y) &= \left(\frac{e^x + e^{-x}}{2}\right)\left(\frac{e^y + e^{-y}}{2}\right) + \left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^y - e^{-y}}{2}\right) \\ &= \frac{e^x e^y + e^{-x}e^y + e^x e^{-y} + e^{-x}e^{-y}}{4} + \frac{e^x e^y - e^{-x}e^y - e^x e^{-y} + e^{-x}e^{-y}}{4} \\ &= \frac{2e^x e^y + 2e^{-x}e^{-y}}{4} = \tfrac{1}{2}(e^{(x+y)} + e^{-(x+y)}) = \cosh(x + y) \end{aligned}$$

and

$$\begin{aligned} \sinh(x)\cosh(y) + \cosh(x)\sinh(y) &= \left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^y + e^{-y}}{2}\right) + \left(\frac{e^x + e^{-x}}{2}\right)\left(\frac{e^y - e^{-y}}{2}\right) \\ &= \frac{e^x e^y - e^{-x}e^y + e^x e^{-y} - e^{-x}e^{-y}}{4} + \frac{e^x e^y + e^{-x}e^y - e^x e^{-y} - e^{-x}e^{-y}}{4} \\ &= \frac{2e^{x+y} - 2e^{-(x+y)}}{4} = \tfrac{1}{2}(e^{x+y} - e^{-(x+y)}) = \sinh(x + y). \end{aligned}$$

□

Example 2.61. Explain why

$$\frac{de^x}{dx} = e^x, \qquad \begin{matrix} \dfrac{d\cosh(x)}{dx} = \sinh(x), & \dfrac{d\cos(x)}{dx} = -\sin(x), \\ \dfrac{d\sinh(x)}{dx} = \cosh(x), & \dfrac{d\sin(x)}{dx} = \cos(x). \end{matrix}$$

Proof. Let $i^2 = -1$. Then

$$\begin{aligned}\frac{de^x}{dx} &= \frac{d}{dx}(1 + x + \tfrac{1}{2!}x^2 + \tfrac{1}{3!}x^3 + \tfrac{1}{4!}x^4 + \tfrac{1}{5!}x^5 + \tfrac{1}{6!}x^6 + \tfrac{1}{7!}x^7 + \cdots) \\ &= 0 + 1 + \tfrac{1}{2!}2x + \tfrac{1}{3!}3x^2 + \tfrac{1}{4!}4x^3 + \tfrac{1}{5!}5x^4 + \tfrac{1}{6!}6x^5 + \tfrac{1}{7!}7x^6 + \cdots) \\ &= 0 + 1 + x + \tfrac{1}{2!}x^2 + \tfrac{1}{3!}x^3 + \tfrac{1}{4!}x^4 + \tfrac{1}{5!}x^5 + \tfrac{1}{6!}x^6 + \tfrac{1}{7!}x^7 + \cdots \\ &= e^x,\end{aligned}$$

$$\frac{d\cosh(x)}{dx} = \frac{d}{dx}\big(\tfrac{1}{2}(e^x + e^{-x})\big) = \tfrac{1}{2}\big((e^x - e^{-x})\big) = \sinh(x),$$

$$\frac{d\sinh(x)}{dx} = \frac{d}{dx}\big(\tfrac{1}{2}(e^x - e^{-x})\big) = \tfrac{1}{2}\big((e^x + e^{-x})\big) = \cosh(x),$$

$$\begin{aligned}\frac{d\cos(x)}{dx} &= \frac{d}{dx}\big(\tfrac{1}{2}(e^{ix} + e^{-ix})\big) = \tfrac{1}{2}(ie^{ix} - ie^{-ix}) = \tfrac{1}{2}i(e^{ix} - ie^{-ix}) \\ &= -\big(-\tfrac{1}{2}i(e^{ix} - ie^{-ix})\big) = -\sin(x),\end{aligned}$$

$$\begin{aligned}\frac{d\sin(x)}{dx} &= \frac{d}{dx}\big(-\tfrac{1}{2}i(e^{ix} - e^{-ix})\big) = -\tfrac{1}{2}i(ie^{ix} + ie^{-ix})\big) \\ &= -\tfrac{1}{2}i^2(e^{ix} + e^{-ix})\big) = \tfrac{1}{2}(e^{ix} + e^{-ix})\big) = \cos(x).\end{aligned}$$

□

Example 2.62. Explain why

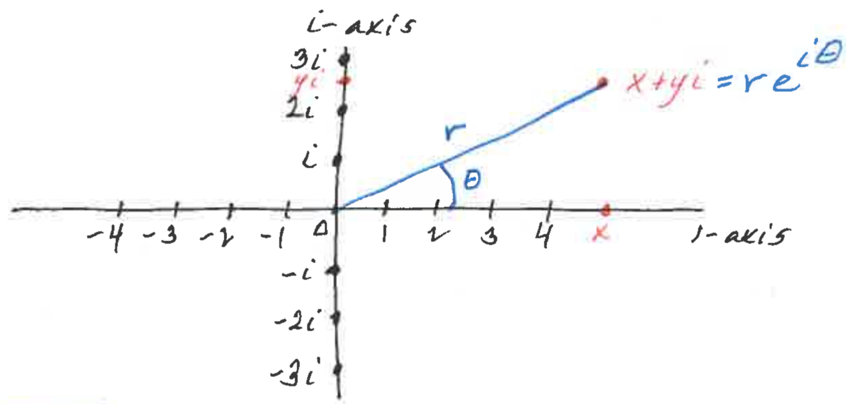
$$\begin{aligned}\cosh(x) &= 1 + \tfrac{1}{2!}x^2 + \tfrac{1}{4!}x^4 + \tfrac{1}{6!}x^6 + \tfrac{1}{8!}x^8 + \cdots, & \sinh(x) &= x + \tfrac{1}{3!}x^3 + \tfrac{1}{5!}x^5 + \tfrac{1}{7!}x^7 + \tfrac{1}{9!}x^9 + \cdots, \\ \cos(x) &= 1 - \tfrac{1}{2!}x^2 + \tfrac{1}{4!}x^4 - \tfrac{1}{6!}x^6 + \tfrac{1}{8!}x^8 - \cdots, & \sin(x) &= x - \tfrac{1}{3!}x^3 + \tfrac{1}{5!}x^5 - \tfrac{1}{7!}x^7 + \tfrac{1}{9!}x^9 - \cdots,\end{aligned}$$

Proof.

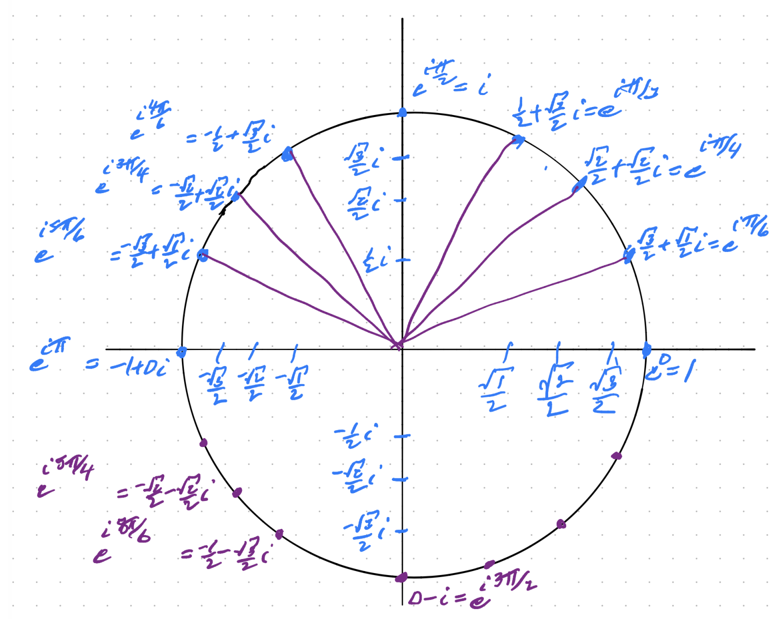
$$\begin{aligned}\cosh(x) &= \tfrac{1}{2}(e^x + e^{-x}) \\ &= \tfrac{1}{2}\begin{pmatrix} 1 + x + \tfrac{1}{2!}x^2 + \tfrac{1}{3!}x^3 + \tfrac{1}{4!}x^4 + \tfrac{1}{5!}x^5 + \tfrac{1}{6!}x^6 + \tfrac{1}{7!}x^7 + \cdots \\ +1 + (-x) + \tfrac{1}{2!}(-x)^2 + \tfrac{1}{3!}(-x)^3 + \tfrac{1}{4!}(-x)^4 + \tfrac{1}{5!}(-x)^5 + \tfrac{1}{6!}(-x)^6 + \tfrac{1}{7!}(-x)^7 + \cdots \end{pmatrix} \\ &= \tfrac{1}{2}\begin{pmatrix} 1 + x + \tfrac{1}{2!}x^2 + \tfrac{1}{3!}x^3 + \tfrac{1}{2!}x^4 + \tfrac{1}{5!}x^5 + \tfrac{1}{6!}x^6 + \tfrac{1}{7!}x^7 + \cdots \\ \\ +1 - x + \tfrac{1}{2!}x^2 - \tfrac{1}{3!}x^3 + \tfrac{1}{4!}x^4 - \tfrac{1}{5!}x^5 + \tfrac{1}{6!}x^6 - \tfrac{1}{7!}x^7 + \cdots \end{pmatrix} \\ &= 1 + \tfrac{1}{2!}x^2 + \tfrac{1}{4!}x^4 + \tfrac{1}{6!}x^6 + \tfrac{1}{8!}x^8 + \cdots.\end{aligned}$$

$$\begin{aligned}
\sinh(x) &= \tfrac{1}{2}(e^x - e^{-x}) \\
&= \tfrac{1}{2}\begin{pmatrix} 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6 + \frac{1}{7!}x^7 + \cdots \\ +\big(1 + (-x) + \frac{1}{2!}(-x)^2 + \frac{1}{3!}(-x)^3 + \frac{1}{4!}(-x)^4 + \frac{1}{5!}(-x)^5 + \frac{1}{6!}(-x)^6 + \frac{1}{7!}(-x)^7 + \cdots\big) \end{pmatrix} \\
&= \tfrac{1}{2}\begin{pmatrix} 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{2!}x^4 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6 + \frac{1}{7!}x^7 + \cdots \\ \\ -1 - x - \frac{1}{2!}x^2 + \frac{1}{3!}x^3 - \frac{1}{4!}x^4 + \frac{1}{5!}x^5 - \frac{1}{6!}x^6 + \frac{1}{7!}x^7 - \cdots \end{pmatrix} \\
&= x + \tfrac{1}{3!}x^3 + \tfrac{1}{5!}x^5 + \tfrac{1}{7!}x^7 + \tfrac{1}{9!}x^9 + \cdots .
\end{aligned}$$

□



Graphing complex numbers



sines and cosines of the favorite angles

Example 2.63. Explain why

$$e^{i0} = \frac{\sqrt{4}}{2} + \frac{\sqrt{0}}{2}i, \qquad \cos(0) = \frac{\sqrt{4}}{2}, \qquad \sin(0) = \frac{\sqrt{0}}{2},$$

$$e^{i\pi/6} = \frac{\sqrt{3}}{2} + \frac{\sqrt{1}}{2}i, \qquad \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}, \qquad \sin\left(\frac{\pi}{6}\right) = \frac{\sqrt{1}}{2},$$

$$e^{i\pi/4} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i, \qquad \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}, \qquad \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2},$$

$$e^{i\pi/3} = \frac{\sqrt{1}}{2} + \frac{\sqrt{3}}{2}i, \qquad \cos\left(\frac{\pi}{3}\right) = \frac{\sqrt{1}}{2}, \qquad \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2},$$

$$e^{i\pi/2} = \frac{\sqrt{0}}{2} + \frac{\sqrt{4}}{2}i, \qquad \cos\left(\frac{\pi}{2}\right) = \frac{\sqrt{0}}{2}, \qquad \sin\left(\frac{\pi}{2}\right) = \frac{\sqrt{4}}{2},$$

Proof. The point at $(1,0)$ is at angle $\theta = 0$ on a circle of radius 1 and so

$$e^{i0} = 1 + 0i, \qquad \text{and} \quad \cos(0) = 1 \quad \text{and} \quad \sin(0) = 0.$$

The point at $(-1,0)$ is at angle $\theta = \pi$ on a circle of radius 1 and so

$$e^{i\pi} = -1 + 0i, \qquad \text{and} \quad \cos(\pi) = -1 \quad \text{and} \quad \sin(\pi) = 0.$$

The point at $(0,1)$ is at angle $\theta = \frac{\pi}{2}$ on a circle of radius 1 and so

$$e^{i\pi/2} = 0 + i, \qquad \text{and} \quad \cos\left(\frac{\pi}{2}\right) = 0 \quad \text{and} \quad \sin\left(\frac{\pi}{2}\right) = 1.$$

The point at $(0,1)$ is at angle $\theta = -\frac{\pi}{2}$ on a circle of radius 1 and so

$$e^{-i\pi/2} = 0 - i, \qquad \text{and} \quad \cos\left(-\frac{\pi}{2}\right) = 0 \quad \text{and} \quad \sin\left(-\frac{\pi}{2}\right) = -1.$$

Place a square with vertices (a,a), $(a,-a)$, $(-a,a)$ and $-a,-a)$ inside a circle of radius 1 so that $a^2 + a^2 = 1^2$. Then $a^2 = \frac{1}{2}$ and $a = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$. The vertex at (a,a) is at angle $\frac{\pi}{4}$ on the circle of radius 1. So

$$e^{i\pi/4} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i, \qquad \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \quad \text{and} \quad \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}.$$

Place a hexagon with vertices $(1,0)$, (a,b), $(-a,b)$, $(-1,0)$, $(-a,-b)$, $(a,-b)$ inside a circle of radius 1 so that $a^2 + b^2 = 1^2$ and a is half way between 0 and 1. Then the point (a,b) is at angle $\frac{\pi}{3}$ on the circle of radius 1 and

$$a = \frac{1}{2} = \frac{\sqrt{1}}{2} \qquad \text{and} \qquad b = \sqrt{1 - a^2} = \sqrt{1 - \left(\frac{1}{2}\right)^2} = \sqrt{1 - \frac{1}{4}} = \frac{\sqrt{3}}{2}.$$

So

$$e^{i\pi/3} = \frac{\sqrt{1}}{2} + \frac{\sqrt{3}}{2}i, \qquad \cos\left(\frac{\pi}{3}\right) = \frac{\sqrt{1}}{2} \quad \text{and} \quad \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}.$$

Flip the previous picture of the inscribed hexagon about the line $y = x$ so that now the hexagon has vertices (b,a), $(0,1)$, $(-b,a)$, $(-b,-a)$, $(0,-1)$ and $(b,-a)$. Then the point (b,a) is at angle $\frac{\pi}{6}$ on the circle of radius 1. So

$$e^{i\pi/6} = \frac{\sqrt{3}}{2} + \frac{\sqrt{1}}{2}i, \qquad \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \quad \text{and} \quad \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{1}}{2}.$$

□

Example 2.64. Simplify $e^{i\pi}+1$.

Proof. Using that $\cos(\pi)=-1$ and $\sin(\pi)=0$ then

$$e^{i\pi}+1=\cos(\pi)+i\sin(\pi)+1=-1+i\cdot 0+1=0+0i=0.$$

□

Example 2.65. Write $z=\sqrt{3}-i$ and $y=\sqrt{3}+3i$ in polar form.

Proof.

$$z=\sqrt{3}-i=2\cdot(\tfrac{\sqrt{3}}{2}-\tfrac{1}{2}i)=2\cdot(\cos\left(\tfrac{-\pi}{6}\right)+\sin\left(\tfrac{-\pi}{6}\right)i)=2e^{-i\pi/6}$$

and

$$y=\sqrt{3}+3i=2\sqrt{3}\cdot(\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}i)=2\sqrt{3}\cdot(\cos\left(\tfrac{\pi}{3}\right)+\sin\left(\tfrac{\pi}{3}\right)i)=2\sqrt{3}e^{i\pi/3}$$

□

Example 2.66. Simplify $(\sqrt{3}-i)(\sqrt{3}+3i)$.

Proof. Using $\sqrt{3}-i=2e^{-i\pi/6}$ and $\sqrt{3}+3i=2\sqrt{3}e^{i\pi/3}$ gives

$$\begin{aligned}(\sqrt{3}-i)(\sqrt{3}+3i)&=2e^{-i\pi/6}\cdot 2\sqrt{3}e^{i\pi/3}=4\sqrt{3}e^{i\pi/6}\\ &=4\sqrt{3}\cdot(\cos(\pi/6)+i\sin(\pi/6))=4\sqrt{3}\cdot(\frac{\sqrt{3}}{2}+\frac{1}{2}i)=6+2\sqrt{3}i.\end{aligned}$$

□

Example 2.67. Evaluate $(\sqrt{3}+3i)^{30}$.

Proof. Using $\sqrt{3}+3i=2\sqrt{3}e^{i\pi/3}$ gives

$$(\sqrt{3}+3i)^{30}=\left(2\sqrt{3}e^{i\pi/3}\right)^{30}=2^303^{30/2}e^{30i\pi/3}=2^{30}3^{15}e^{10\pi i}=2^{30}3^{15}\cdot 1=2^{30}3^{15}.$$

□

Example 2.68. Express $\cos(\theta)^3$ in terms of $\cos(n\theta)$ with $n\in\mathbb{Z}$.

Proof.

$$\begin{aligned}\cos(\theta)^3=\left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)^3&=\tfrac{1}{8}(e^{3i\theta}+3e^{2i\theta}e^{-i\theta}+3e^{i\theta}e^{-2i\theta}+e^{-3i\theta})\\ &=\tfrac{1}{8}(e^{3i\theta}+3e^{i\theta}+3e^{-i\theta}+e^{-3i\theta})=\tfrac{1}{8}(e^{3i\theta}+e^{-3i\theta})+\tfrac{3}{8}(e^{i\theta}+e^{-i\theta})\\ &=\tfrac{1}{4}\cos(3\theta)+\tfrac{3}{4}\cos(\theta).\end{aligned}$$

□

Example 2.69. Evaluate $\dfrac{d^{40}}{dt^{40}}\Big(e^{-t}\cos(t)\Big)\quad\left(\text{and } \dfrac{d^{40}}{dt^{40}}\Big(e^{-t}\sin(t)\Big)\right)$.

Proof.

$$\begin{aligned}
\frac{d^{40}}{dt^{40}}\Big(e^{-t}\cos(t)\Big) + i\frac{d^{40}}{dt^{40}}\Big(e^{-t}\sin(t)\Big) &= \frac{d^{40}}{dt^{40}}\Big(e^{-t}(\cos(t)+i\sin(t))\Big) = \frac{d^{40}}{dt^{40}}\Big(e^{-t}e^{it}\Big) \\
&= \frac{d^{40}}{dt^{40}}\Big(e^{(-1+i)t}\Big) = (-1+i)^{40}e^{(-1+i)t} = \Big(\frac{2}{\sqrt{2}}(-\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i)\Big)^{40}e^{-t}e^{it} \\
&= \Big(\sqrt{2}e^{-i\pi/4}\Big)^{40}e^{-t}e^{it} = 2^{20}e^{-40i\pi/4}e^{-t}e^{it} \\
&= 2^{20}\cdot 1\cdot e^{-t}(\cos(t)+i\sin(t)) = 2^{20}e^{-t}\cos(t) + i2^{20}e^{-t}\sin(t)).
\end{aligned}$$

So

$$\frac{d^{40}}{dt^{40}}\Big(e^{-t}\cos(t)\Big) = 2^{20}e^{-t}\cos(t) \qquad \text{and} \qquad \frac{d^{40}}{dt^{40}}\Big(e^{-t}\sin(t)\Big) = 2^{20}e^{-t}\sin(t).$$

□

Example 2.70. Determine $\dfrac{d\ \tan(x)}{dx}$ and $\dfrac{d\tanh(x)}{dx}$.

Proof.

$$\begin{aligned}
\frac{d\tanh(x)}{dx} &= \frac{d}{dx}\Big(\frac{\sinh(x)}{\cosh(x)}\Big) = \frac{d}{dx}\Big(\sinh(x)(\cosh(x))^{-1}\Big) \\
&= \sinh(x)(-1)\cosh(x))^{-2}\sinh(x) + \cosh(x)(\cosh(x))^{-1} \\
&= \frac{-\sinh^2(x)}{\cosh^2(x)} + 1 = \frac{-\sinh^2(x)+\cosh^2(x)}{\cosh^2(x)} = \frac{1}{\cosh^2(x)} = \mathrm{sech}^2(x)
\end{aligned}$$

and

$$\begin{aligned}
\frac{d\tan(x)}{dx} &= \frac{d}{dx}\Big(\frac{\sin(x)}{\cos(x)}\Big) = \frac{d}{dx}\Big(\sin(x)(\cos(x))^{-1}\Big) \\
&= \sin(x)(-1)\cos(x))^{-2}(-\sin(x)) + \cos(x)(\cos(x))^{-1} \\
&= \frac{\sin^2(x)}{\cos^2(x)} + 1 = \frac{\sin^2(x)+\cos^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} = \sec^2(x).
\end{aligned}$$

□

Example 2.71. Determine $\dfrac{d\ \sec(x)}{dx}$ and $\dfrac{d\ \mathrm{sech}(x)}{dx}$.

Proof.

$$\begin{aligned}
\frac{d\sec(x)}{dx} &= \frac{d}{dx}\Big(\frac{1}{\cos(x)}\Big) = \frac{d}{dx}(\cos(x))^{-1} = (-1)\cos(x))^{-2}\frac{d}{dx}(\cos(x)) \\
&= (-1)\cos(x))^{-2}(-\sin(x)) = \frac{\sin(x)}{\cos(x)}\cdot\frac{1}{\cos(x)} = \tan(x)\sec(x),
\end{aligned}$$

and

$$\begin{aligned}
\frac{d\ \mathrm{sech}(x)}{dx} &= \frac{d}{dx}\Big(\frac{1}{\cosh(x)}\Big) = \frac{d}{dx}(\cosh(x))^{-1} = (-1)(\cosh(x))^{-2}\cdot\sinh(x) \\
&= -\frac{\sinh(x)}{\cosh(x)}\cdot\frac{1}{\cosh(x)} = -\tanh(x)\mathrm{sech}(x).
\end{aligned}$$

□

Example 2.72. Determine $\dfrac{d\ \mathrm{csc}(x)}{dx}$ and $\dfrac{d\ \mathrm{csch}(x)}{dx}$.

Proof.

$$\begin{aligned}\frac{d\csc(x)}{dx} &= \frac{d}{dx}\Big(\frac{1}{\sin(x)}\Big) = \frac{d}{dx}(\sin(x))^{-1} = (-1)\sin(x))^{-2}\frac{d}{dx}(\sin(x)) \\ &= (-1)\sin(x))^{-2}\cdot\cos(x) = -\frac{\cos(x)}{\sin(x)}\cdot\frac{1}{\sin(x)} = -\cot(x)\csc(x),\end{aligned}$$

and

$$\begin{aligned}\frac{d\ \mathrm{csch}(x)}{dx} &= \frac{d}{dx}\Big(\frac{1}{\sinh(x)}\Big) = \frac{d}{dx}(\sinh(x))^{-1} = (-1)(\sinh(x))^{-2}\cdot\cosh(x) \\ &= -\frac{\cosh(x)}{\sinh(x)}\cdot\frac{1}{\sinh(x)} = -\coth(x)\mathrm{csch}(x).\end{aligned}$$

□

Example 2.73. Determine $\int \sin(x)\,dx$ and $\int \tan(x)\,dx$ and $\int \sec(x)\,dx$.

Proof. Since $\frac{d\cos(x)}{dx} = -\sin(x)$ then

$$\int \sin(x)\,dx = \int -(-\sin(x))dx = -\int(-\sin(x))dx = \cos(x)+c, \qquad \text{where } c \text{ is a constant.}$$

Let $z = \cos(x)$. Then

$$\begin{aligned}\int\tan(x)dx &= \int\frac{\sin(x)}{\cos(x)}dx = \int\frac{1}{\cos(x)}\sin(x)dx = -\int\frac{1}{\cos(x)}(-\sin(x))dx = -\int\frac{1}{y}\cdot\frac{dy}{dx}dx \\ &= -\int\frac{1}{y}\,dy = -\log(y)+c,\end{aligned}$$

where c is a constant.
Let $y = \tan(x)+\sec(x)$. Then

$$\begin{aligned}\int\sec(x)dx &= \int\sec(x)\frac{\sec(x)+\tan(x)}{\sec(x)+\tan(x)}dx = \int\frac{1}{\sec(x)+\tan(x)}(\sec^2(x)+\tan(x)\sec(x))dx \\ &= \int\frac{1}{y}\frac{dy}{dx}dx = \int\frac{1}{y}dy = \log(y)+c = \log(\tan(x)+\sec(x))+c,\end{aligned}$$

where c is a constant. □

Example 2.74. Assume $\cosh(x) = \frac{13}{2}$ and $x\in\mathbb{R}_{>0}$. Find $\sinh(x)$ and $\tanh(x)$.

Proof. Using that $\cosh^2(x)-\sinh^2(x) = 1$,

$$\sinh(x) = \sqrt{\sinh^2(x)} = \sqrt{\cosh^2(x)-1} = \sqrt{\left(\tfrac{13}{2}\right)^2-1} = \sqrt{\frac{169}{4}-1} = \sqrt{\frac{169-4}{4}} = \frac{\sqrt{165}}{2},$$

and

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{\frac{\sqrt{165}}{2}}{\frac{13}{2}} = \frac{\sqrt{165}}{13}.$$

□

Example 2.75. Evaluate

$$\int e^x\,dx, \qquad \int e^{10x}\,dx, \qquad \text{and} \qquad \int e^{-31x}\,dx.$$

Proof.

$$\int e^x\,dx = e^x + c \quad \text{where } c \text{ is a constant,} \qquad \text{since} \quad \frac{de^x}{dx} = e^x.$$

$$\int e^{10x}\,dx = \frac{e^{10x}}{10} + c \quad \text{where } c \text{ is a constant,} \qquad \text{since} \quad \frac{d\left(\frac{e^{10x}}{10}\right)}{dx} = \frac{1}{10}e^{10x}\cdot 10 = e^{10x}.$$

$$\int e^{-31x}\,dx = \frac{e^{-31x}}{-31} + c \quad \text{where } c \text{ is a constant,} \qquad \text{since} \quad \frac{d\left(\frac{e^{-31x}}{-31}\right)}{dx} = \frac{1}{-31}e^{-31x}\cdot(-31) = e^{-31x}.$$

□

Example 2.76. Evaluate $\int 2^x\,dx$.

Proof.

$$\int 2^x\,dx = \int \left(e^{\log(2)}\right)^x dx = \int e^{x\log(2)}\,dx = \frac{e^{x\log(2)}}{\log(2)} + c = \frac{2^x}{\log(2)} + c, \quad \text{where } c \text{ is a constant,}$$

since

$$\frac{d\frac{2^x}{\log(2)}}{dx} = \frac{d\frac{e^{x\log(2)}}{\log(2)}}{dx} = \frac{1}{\log(2)}e^{x\log(2)}\cdot\log(2) = e^{x\log(2)} = 2^x.$$

□

Example 2.77. Evaluate $\int 38^x\,dx$.

Proof.

$$\int 38^x\,dx = \int \left(e^{\log(38)}\right)^x dx = \int e^{x\log(38)}\,dx = \frac{e^{x\log(38)}}{\log(38)} + c = \frac{38^x}{\log(38)} + c, \quad \text{where } c \text{ is a constant,}$$

since

$$\frac{d\frac{38^x}{\log(38)}}{dx} = \frac{d\frac{e^{x\log(38)}}{\log(38)}}{dx} = \frac{1}{\log(38)}e^{x\log(38)}\cdot\log(38) = e^{x\log(38)} = 38^x.$$

□

Example 2.78. Evaluate $\int \frac{\cos(x)}{\sin(x)^2}\,dx$.

Proof.

$$\int \frac{\cos(x)}{\sin(x)^2}\,dx = \frac{\cos(x)}{\sin(x)}\cdot\frac{1}{\sin(x)}\,dx = \int \cot(x)\csc(x)\,dx = -\csc(x) + c, \quad \text{where } c \text{ is a constant.}$$

□

Example 2.79. Evaluate $\int \frac{1}{1+\cos(x)}\,dx$.

Proof.

$$\begin{aligned}\int \frac{1}{1+\cos(x)}\,dx &= \int \frac{1}{1+\cos(x)}\frac{(1-\cos(x))}{(1-\cos(x))}\,dx = \int \frac{1-\cos(x)}{\sin(x)^2}\,dx \\ &= \int \Big(\frac{1}{\sin(x)^2} - \frac{\cos(x)}{\sin(x)^2}\Big)\,dx = \int (\csc(x)^2 - \cot(x)\csc(x))\,dx \\ &= -\cot(x) + \csc(x) + c, \quad \text{where } c \text{ is a constant.}\end{aligned}$$

□

Example 2.80. Evaluate $\int e^{3x}\sin(2x)\,dx \quad \Big(\text{and } \int e^{3x}\cos(2x)\,dx\Big)$.

Proof.

$$\begin{aligned}&\int e^{3x}\cos(2x)\,dx + i\int e^{3x}\sin(2x)\,dx = \int e^{3x}(\cos(2x)+i\sin(2x))\,dx = \int e^{3x}e^{i2x}\,dx \\ &= \int e^{(3+2i)x}\,dx = \frac{1}{3+2i}e^{(3+2i)x} + (c_1+c_2 i) = \frac{(3-2i)}{(3+2i)(3-2i)}e^{3x}e^{2ix} + c_1 + ic_2 \\ &= \frac{(3-2i)}{9+4}e^{3x}(\cos(2x)+i\sin(2x)) + c_1 + ic_2 \\ &= \tfrac{1}{13}e^{3x}(3\cos(2x) + i3\sin(2x) - i2\cos(2x) + 2\sin(2x)) + c_1 + ic_2 \\ &= \Big(\tfrac{1}{13}e^{3x}(3\cos(2x)+2\sin(2x)) + c_1\Big) + i\Big(\tfrac{1}{13}e^{3x}(3\sin(2x) - 2\cos(2x)) + c_2\Big),\end{aligned}$$

where c_1 and c_2 constants. So

$$\int e^{3x}\cos(2x)\,dx = e^{3x}(\tfrac{3}{13}\cos(2x) + \tfrac{2}{13}\sin(2x)) + c_1 \qquad \text{and}$$
$$\int e^{3x}\sin(2x)\,dx = e^{3x}(\tfrac{3}{13}\sin(2x) - \tfrac{2}{13}\cos(2x)) + c_2,$$

where c_1 and c_2 constants. □

2.3 Inverse functions

Example 2.81. Explain why

$$\begin{aligned}e^0 &= 1 &&\text{turns into} &&\log(1) = 0, \\ e^x e^y &= e^{x+y} &&\text{turns into} &&\log(ab) = \log(a) + \log(b), \\ e^{-x} &= \frac{1}{e^x} &&\text{turns into} &&\log\Big(\frac{1}{a}\Big) = -\log a, \quad \text{and} \\ (e^x)^y &= e^{yx} &&\text{turns into} &&\log(a^b) = b\log a.\end{aligned}$$

Proof.
(a) $\log(1) = \log(e^0) = 0$.
(b) $\log(ab) = \log(e^{\log(a)}\cdot e^{\log(b)}) = \log(e^{\log(a)+\log(b)}) = \log(a) + \log(b)$.
(c) $\log\Big(\frac{1}{a}\Big) = \log\Big(\frac{1}{e^{\log(a)}}\Big) = \log\Big(e^{-\log(a)}\Big) = -\log(a)$.
(d) $\log(a^b) = \log\Big(\big(e^{\log(a)}\big)^b\Big) = \log\Big(e^{b\log(a)}\Big) = b\log(a)$. □

Example 2.82. Find $\dfrac{d\ \log(x)}{dx}$.

Proof. Let $y = \log(x)$ so that $e^y = x$. Taking the derivative $\frac{d}{dx}$ of both sides gives

$$e^y \frac{dy}{dx} = 1, \qquad \text{and so} \qquad \frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}.$$

So $\dfrac{d\ \log(x)}{dx} = \dfrac{1}{x}$. □

Example 2.83. Find $\lim_{x\to 0} \dfrac{\log(1+x)}{x}$.

Proof. By example 2.82

$$\frac{d\ \log(1+x)}{dx} = \frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots,$$

since

$$\begin{aligned}&(1{+}x)(1 - x + x^2 - x^3 + x^4 - x^5 + \cdots)\\ &\quad = (1 - x + x^2 - x^3 + x^4 - x^5 + \cdots) + (x - x^2 + x^3 - x^4 + x^5 + \cdots) = 1.\end{aligned}$$

So

$$\begin{aligned}\log(1+x) &= \int \frac{d\ \log(1+x)}{dx} dx = \int \frac{1}{1+x} dx = \int (1 - x + x^2 - x^3 + x^4 - x^5 + \cdots) dx\\ &= x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \tfrac{1}{5}x^5 + \cdots.\end{aligned}$$

So

$$\frac{\log(1+x)}{x} = 1 - \tfrac{1}{2}x + \tfrac{1}{3}x^2 - \tfrac{1}{4}x^3 + \tfrac{1}{5}x^4 + \cdots \qquad \text{and} \qquad \lim_{x\to 0} \frac{\log(1+x)}{x} = 1.$$

□

Example 2.84. Prove that

$$\begin{aligned}
\mathrm{arcsinh}(x) &= \log(x + \sqrt{x^2+1}), & \arcsin(x) &= (-i)\log(x + i\sqrt{x^2+1}),\\
\mathrm{arccosh}(x) &= \log(x + \sqrt{x^2-1}), & \arccos(x) &= (-i)\log(x + \sqrt{x^2+1}),\\
\mathrm{arctanh}(x) &= \tfrac{1}{2}\log\Big(\frac{1+x}{1-x}\Big), & \arctan(x) &= (-i)\tfrac{1}{2}\log\Big(\frac{i+x}{i-x}\Big).
\end{aligned}$$

Proof. (a) Let $y = \arcsin(x)$. Then $x = \sinh(y) = (-i)\frac{1}{2}(e^{iy} - e^{-iy})$ and

$$2ix = e^{iy} - e^{-iy} \qquad \text{and} \qquad 2ixe^y = (e^{iy})^2 - 1.$$

So $(e^{iy})^2 - 2ixe^y + 1 = 0$ and

$$e^{iy} = \frac{2ix \pm \sqrt{-4x^2-4}}{2} = x \pm \sqrt{-x^2-1} \qquad \text{and} \qquad iy = \log(x \pm i\sqrt{x^2+1}).$$

So

$$\arcsin(x) = (-i)\log(x \pm i\sqrt{x^2+1}).$$

(b) Let $y = \arccos(x)$. Then $x = \cos(y) = \frac{1}{2}(e^{iy} + e^{-iy})$ and

$$2x = e^{iy} - e^{-iy} \qquad \text{and} \qquad 2xe^{iy} = (e^{iy})^2 + 1.$$

So $(e^{iy})^2 - 2xe^{iy} - 1 = 0$ and

$$e^{iy} = \frac{2x \pm \sqrt{4x^2+4}}{2} = x \pm \sqrt{x^2+1} \qquad \text{and} \qquad iy = \log(x \pm \sqrt{x^2+1}).$$

So

$$\arccos(x) = (-i)\log(x \pm \sqrt{x^2+1}).$$

(c) Let $y = \arctan(x)$ then

$$x = \tan(y) = \frac{\frac{1}{2}(e^{iy} - e^{-iy})}{(-i)\frac{1}{2}(e^{iy} + e^{-iy})} = i\frac{(e^{iy})^2 - 1}{(e^{iy})^2 + 1}.$$

So

$$(e^{iy})^2 x + x = i(e^{iy})^2 - i \qquad \text{and} \qquad (i - x)(e^{iy})^2 = i + x.$$

So

$$e^{2iy} = \frac{i+x}{i-x} \qquad \text{and} \qquad 2iy = \log\Big(\frac{i+x}{i-x}\Big).$$

So

$$\arctan(x) = (-i)\tfrac{1}{2}\log\Big(\frac{i+x}{i-x}\Big).$$

(d) Let $\mathrm{arcsinh}(x) = y$. Then $x = \sinh(y)$ and using $\cosh^2 x - \sinh^2 x = 1$,

$$\begin{aligned}\log\big(x + \sqrt{x^2+1}\big) &= \log\big(\sinh(y) + \sqrt{\sinh^2(y)+1}\big) = \log\big(\sinh(y) + \sqrt{\cosh^2(y)}\big) \\ &= \log\big(\sinh(y) + \cosh(y)\big) = \log\Big(\frac{e^y - e^{-y}}{2} + \frac{e^y + e^{-y}}{2}\Big) \\ &= \log\Big(\frac{2e^y}{2}\Big) = \log(e^y) = y = \mathrm{arcsinh}(x).\end{aligned}$$

□

Example 2.85. Explain why $\dfrac{d\log(x)}{dx} = \dfrac{1}{x}$.

Proof. Since $e^{\log(x)} = x$ then $\dfrac{de^{\log(x)}}{dx} = \dfrac{dx}{dx}$.

So $e^{\log(x)}\dfrac{d\log(x)}{dx} = 1.$ So $x\dfrac{d\log(x)}{dx} = 1.$ So $\dfrac{d\log(x)}{dx} = \dfrac{1}{x}.$ □

Example 2.86. Find $\dfrac{d\arcsin(x)}{dx}$.

Proof. Since $\sin(\arcsin(x)) = x$ then $\dfrac{d\sin(\arcsin(x))}{dx} = \dfrac{dx}{dx}$.

So $\cos(\arcsin(x))\dfrac{d\arcsin(x)}{dx} = 1.$ So $\dfrac{d\arcsin(x)}{dx} = \dfrac{1}{\cos(\arcsin(x))}.$

So we would like to "simplify" $\cos(\arcsin(x))$.

Since $1-\cos^2(\arcsin(x)) = \sin^2(\arcsin(x))$ then $1-\big(\cos(\arcsin(x))\big)^2 = \big(\sin(\arcsin(x))\big)^2$.

So $\quad 1-\big(\cos(\arcsin(x))\big)^2 = x^2.$ $\qquad$ So $\quad 1-x^2 = \big(\cos(\arcsin(x))\big)^2.$

So $\quad \cos(\arcsin(x)) = \sqrt{1-x^2}.$ $\qquad$ So $\quad \dfrac{d\arcsin(x)}{dx} = \dfrac{1}{\cos(\arcsin(x))} = \dfrac{1}{\sqrt{1-x^2}}.$ □

Example 2.87. Find $\dfrac{d\arccos(x)}{dx}$.

Proof. Since $\quad \cos(\arccos(x)) = x$ then $\dfrac{d\cos(\arccos(x))}{dx} = \dfrac{dx}{dx}.$

So $\quad -\sin(\arccos(x))\,\dfrac{d\arccos(x)}{dx} = 1.$ $\qquad$ So $\quad \dfrac{d\arccos(x)}{dx} = \dfrac{-1}{\sin(\arccos(x))}.$

So we would like to "simplify" $\sin(\arccos(x))$.

Since $1-\sin^2(\arccos(x)) = \cos^2(\arccos(x))$ then $1-\big(\sin(\arccos(x))\big)^2 = \big(\cos(\arccos(x))\big)^2$.

So $\quad 1-\big(\sin(\arccos(x))\big)^2 = x^2.$ $\qquad$ So $\quad 1-x^2 = \big(\sin(\arccos(x))\big)^2.$

So $\quad \sin(\arccos(x)) = \sqrt{1-x^2}.$ $\qquad$ So $\quad \dfrac{d\arccos(x)}{dx} = \dfrac{-1}{\sin(\arccos(x))} = \dfrac{-1}{\sqrt{1-x^2}}.$ □

Example 2.88. Find $\dfrac{d\arctan(x)}{dx}$.

Proof. Since $\quad \tan(\arctan(x)) = x$ then $\dfrac{d\tan(\arctan(x))}{dx} = \dfrac{dx}{dx}.$

So $\quad \sec^2(\arctan(x))\,\dfrac{d\arctan(x)}{dx} = 1.$

So $\quad \dfrac{d\arctan(x)}{dx} = \dfrac{1}{\sec^2(\arctan(x))}.$

So we would like to "simplify" $\sec^2(\arctan(x))$.

Since $\quad \sin(x)^2+\cos(x)^2 = 1$ then $\dfrac{\sin(x)^2}{\cos(x)^2} + \dfrac{\cos(x)^2}{\cos(x)^2} = \dfrac{1}{\cos(x)^2}.$

So $\quad \tan(x)^2 + 1 = \sec(x)^2.$

So $\quad \sec^2(\arctan(x)) = \tan^2(\arctan(x)) + 1 = \big(\tan(\arctan(x))\big)^2 + 1 = x^2+1.$

So $\quad \dfrac{d\arctan(x)}{dx} = \dfrac{1}{x^2+1}.$ □

Example 2.89. Find $\dfrac{d\ \mathrm{arccot}(x)}{dx}$.

Proof. Since $\quad \cot(\mathrm{arccot}(x)) = x$ then $\dfrac{d\ \cot(\mathrm{arccot}(x))}{dx} = \dfrac{dx}{dx}.$

So $\quad -\csc^2(\mathrm{arccot}(x))\,\dfrac{d\mathrm{arccot}(x)}{dx} = 1.$

So $\quad \dfrac{d\mathrm{arccot}(x)}{dx} = \dfrac{-1}{\csc^2(\mathrm{arccot}(x))}.$

So we would like to "simplify" $\csc^2(\mathrm{arccot}(x))$.

Since $\quad \sin(x)^2+\cos(x)^2 = 1$ then $\quad \dfrac{\sin(x)^2}{\sin(x)^2} + \dfrac{\cos(x)^2}{\sin(x)^2} = \dfrac{1}{\sin(x)^2}.$

So $\quad 1+\cot(x)^2 = \csc(x)^2.$

So $\quad \csc(\mathrm{arccot}(x))^2 = 1+\cot(\mathrm{arccot}(x))^2 = 1+\big(\cot(\mathrm{arccot}(x))\big)^2 = 1+x^2.$

So $\quad \dfrac{d\mathrm{arccot}(x)}{dx} = \dfrac{-1}{1+x^2}.$ □

Example 2.90. Find $\dfrac{d\ \mathrm{arcsec}(x)}{dx}$.

Proof. Since $\quad \sec(\mathrm{arcsec}(x)) = x$ then $\quad \dfrac{d\sec(\mathrm{arcsec}(x))}{dx} = \dfrac{dx}{dx}.$

So $\quad \tan(\mathrm{arcsec}(x))\sec(\mathrm{arcsec}(x))\,\dfrac{d\ \mathrm{arcsec}(x)}{dx} = 1.$

So $\quad \tan(\mathrm{arcsec}(x))\cdot x\cdot \dfrac{d\ \mathrm{arcsec}(x)}{dx} = 1.$

So $\quad \dfrac{d\ \mathrm{arcsec}(x)}{dx} = \dfrac{1}{x\tan(\mathrm{arcsec}(x))}.$

So we would like to "simplify" $\tan(\mathrm{arcsec}(x))$.

Since $\quad \sin(x)^2+\cos(x)^2 = 1$ then $\quad \dfrac{\sin(x)^2}{\cos(x)^2}+\dfrac{\cos(x)^2}{\cos(x)^2} = \dfrac{1}{\cos(x)^2}.$

So $\quad \tan(x)^2+1 = \sec(x)^2.$

So $\quad \tan(\mathrm{arcsec}(x))^2+1 = \sec(\mathrm{arcsec}(x))^2.$

So $\quad \big(\tan(\mathrm{arcsec}(x))\big)^2+1 = \big(\sec(\mathrm{arcsec}(x))\big)^2.$

So $\quad \big(\tan(\mathrm{arcsec}(x))\big)^2+1 = x^2.$

So $\quad \tan(\mathrm{arcsec}(x)) = \sqrt{x^2-1}.$

So $\quad \dfrac{d\ \mathrm{arcsec}(x)}{dx} = \dfrac{1}{x\sqrt{x^2-1}}.$ □

Example 2.91. Find $\dfrac{d\ \mathrm{arccsc}(x)}{dx}$.

Proof. Since $\quad \csc(\mathrm{arccsc}(x)) = x, \qquad \dfrac{d\csc(\mathrm{arccsc}(x))}{dx} = \dfrac{dx}{dx}.$

So $\quad -\csc(\csc^{-1}x)\cot(\mathrm{arccsc}(x))\,\dfrac{d\ \mathrm{arccsc}(x)}{dx} = 1.$

So $\quad -x\cot(\mathrm{arccsc}(x))\,\dfrac{d\ \mathrm{arccsc}(x)}{dx} = 1.$

So $\quad \dfrac{d\ \mathrm{arccsc}(x)}{dx} = \dfrac{-1}{x\cot(\mathrm{arccsc}(x))}.$

So we would like to "simplify" $\cot(\mathrm{arccsc}(x))$.

Since $\quad \sin^2x+\cos^2x = 1$ then $\quad \dfrac{\sin^2x}{\sin^2x}+\dfrac{\cos^2x}{\sin^2x} = \dfrac{1}{\sin^2x}.$

So $\quad 1+\cot^2x = \csc^2x.$

So $\quad 1+\cot^2(\mathrm{arccsc}(x)) = \csc^2(\mathrm{arccsc}(x)).$

So $\quad 1+\big(\cot(\mathrm{arccsc}(x))\big)^2 = \big(\csc(\mathrm{arccsc}(x))\big)^2.$

So $\quad 1+\big(\cot(\mathrm{arccsc}(x))\big)^2 = x^2.$

So $\quad \cot(\operatorname{arccsc}(x)) = \sqrt{x^2-1}$.

So $\quad \dfrac{d\ \operatorname{arccsc}(x)}{dx} = \dfrac{-1}{x\sqrt{x^2-1}}$. □

Example 2.92. Simplify $\cosh(\operatorname{arcsinh}(x))$.

Proof. Let $\operatorname{arcsinh}(x) = y$. Then $x = \sinh(y)$ and using $\cosh^2 x - \sinh^2 x = 1$,

$$\cosh(\operatorname{arcsinh}(x)) = \cosh y = \sqrt{1+\sinh^2(y)} = \sqrt{1+\sinh^2(\operatorname{arcsinh}(x))} = \sqrt{1+x^2}.$$

□

Example 2.93. Prove that $\dfrac{d}{dx}(\operatorname{arcsinh}(x)) = \dfrac{1}{\sqrt{x^2+1}}$.

Proof. Let $\operatorname{arcsinh}(x) = y$. Then $x = \sinh(y)$ and taking the derivative with respect to x gives

$$1 = \frac{dx}{dx} = \frac{d\sinh(y)}{dx} = \cosh(y)\frac{dy}{dx}.$$

Thus

$$\frac{d}{dx}(\operatorname{arcsinh}(x)) = \frac{dy}{dx} = \frac{1}{\cosh(y)} = \frac{1}{\cosh(\operatorname{arcsinh}(x))} = \frac{1}{\sqrt{x^2+1}},$$

where the last equality uses $\cosh(\operatorname{arcsinh}(x)) = \sqrt{1+x^2}$ from Example 2.92. □

2.4 Integration with square roots

Example 2.94. Let $a \in \mathbb{C}$. Evaluate $\displaystyle\int \sqrt{a^2-x^2}\,dx$.

Proof.

$$\begin{aligned}
\int \sqrt{a^2-x^2}\,dx &= \int a\sqrt{1-\left(\tfrac{x}{a}\right)^2}\,dx = \int a\sqrt{1-\sin^2\theta}\,\frac{dx}{d\theta}d\theta, \qquad \text{where} \quad \tfrac{x}{a} = \sin\theta,\\
&= \int a\sqrt{\cos^2\theta}\,a\cos\theta d\theta = \int a^2\cos^2\theta d\theta = \int \tfrac{a^2}{2}(2\cos^2\theta - 1) + \tfrac{a^2}{2}\big)d\theta\\
&= \int \left(\tfrac{a^2}{2}\cos(2\theta) + \tfrac{a^2}{2}\right)d\theta = \tfrac{a^2}{2}\tfrac{1}{2}\sin(2\theta) + \tfrac{a^2}{2}\theta + c\\
&= \tfrac{a^2}{4}2\sin(\theta)\cos(\theta) + \tfrac{a^2}{2}\theta + c = \tfrac{a^2}{2}\sin(\theta)\sqrt{1-\sin^2\theta} + \tfrac{a^2}{2}\theta + c\\
&= \tfrac{a^2}{2}\frac{x}{a}\sqrt{1-(\tfrac{x}{a})^2} + \tfrac{a^2}{2}\arcsin\left(\tfrac{x}{a}\right) + c = \tfrac{a}{2}x\sqrt{a^2-x^2} + \tfrac{a^2}{2}\arcsin\left(\tfrac{x}{a}\right) + c,
\end{aligned}$$

where c is a constant. □

Example 2.95. Evaluate $\displaystyle\int \frac{1}{\sqrt{a^2-x^2}}\,dx$.

Proof.

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \int \frac{1}{a\sqrt{1-\left(\frac{x}{a}\right)^2}}\,dx = \arcsin\left(\tfrac{x}{a}\right) + c, \qquad \text{where } c \text{ is a constant,}$$

since

$$\frac{d}{dx}\left(\arcsin\left(\tfrac{x}{a}\right)\right) = \frac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}}\cdot\frac{1}{a}.$$

□

Example 2.96. Evaluate $\displaystyle\int \frac{1}{a^2+x^2}\,dx$.

Proof.

$$\int \frac{1}{a^2+x^2}\,dx = \int \frac{1}{a^2(1+\big(\frac{x}{a}\big)^2)}\,dx = \frac{1}{a}\arctan\big(\tfrac{x}{a}\big)+c, \qquad \text{where } c \text{ is a constant,}$$

since

$$\frac{d}{dx}(\arctan\big(\tfrac{x}{a}\big)) = \frac{1}{\sqrt{1+\big(\frac{x}{a}\big)^2}}\cdot\frac{1}{a}.$$

□

Example 2.97. Let $a\in\mathbb{C}$. Evaluate $\displaystyle\int \frac{1}{\sqrt{x^2-a^2}}\,dx$.

Proof.

$$\begin{aligned}\int \frac{1}{\sqrt{x^2-a^2}}\,dx &= \int \frac{\frac{1}{a}1}{\sqrt{\big(\frac{x}{a}\big)^2-1}}\,dx = \int \frac{\frac{1}{a}}{\sqrt{\cosh(\theta)^2-1}}\frac{dx}{d\theta}d\theta, \qquad \text{with} \quad \frac{x}{a}=\cosh(\theta),\\ &= \int \frac{\frac{1}{a}}{\sqrt{\sinh(\theta)^2}}\,a\sinh(\theta)d\theta = \int \frac{1}{\sinh(\theta)}\,\sinh(\theta)d\theta = \int d\theta\\ &= \theta + c = \operatorname{arccosh}\big(\tfrac{x}{a}\big)+c,\end{aligned}$$

where c is a constant. □

Example 2.98. Evaluate $\displaystyle\int \frac{1}{\sqrt{x^2-25}}\,dx$.

Proof.

$$\begin{aligned}\int \frac{1}{\sqrt{x^2-25}}\,dx &= \int \frac{\frac{1}{5}1}{\sqrt{\big(\frac{x}{5}\big)^2-1}}\,dx\\ &= \int \frac{\frac{1}{5}}{\sqrt{\cosh(\theta)^2-1}}\frac{dx}{d\theta}d\theta \qquad \text{with} \quad \frac{x}{5}=\cosh(\theta),\\ &= \int \frac{\frac{1}{5}}{\sqrt{\sinh(\theta)^2}}\,5\sinh(\theta)d\theta = \int \frac{1}{\sinh(\theta)}\,\sinh(\theta)d\theta\\ &= \int d\theta = \theta + c = \operatorname{arccosh}\big(\tfrac{x}{5}\big)+c,\end{aligned}$$

where c is a constant. □

Example 2.99. Evaluate $\displaystyle\int_0^1 \sqrt{9-4x^2}\,dx$.

Proof.

$$\begin{aligned}
\int_0^1 \sqrt{9-4x^2}\,dx &= \int_{x=0}^{x=1} 3\sqrt{1-\tfrac{4}{9}x^2}\,dx = \int_0^1 3\sqrt{1-\left(\tfrac{2x}{3}\right)^2}\,dx \\
&= \int_{x=0}^{x=1} 3\sqrt{1-\left(\tfrac{2x}{3}\right)^2}\,dx = \int_{x=0}^{x=1} 3\sqrt{1-\sin(\theta)^2}\,\frac{dx}{d\theta}d\theta \qquad \text{where} \quad \frac{2x}{3} = \sin(\theta), \\
&= \int_{x=0}^{x=1} 3\sqrt{\cos(\theta)}\,\frac{3}{2}\cos\theta d\theta = \int_{x=0}^{x=1} \frac{9}{2}\cos(\theta)d\theta \\
&= \int_{x=0}^{x=1} \left(\tfrac{9}{4}(2\cos(\theta)^2-1)+\tfrac{9}{4}\right)d\theta = \int_{x=0}^{x=1} \left(\tfrac{9}{4}\cos(2\theta)+\tfrac{9}{4}\right)d\theta \\
&= \left(\tfrac{9}{4}\tfrac{1}{2}\sin(2\theta)+\tfrac{9}{4}\theta\right)\Big]_{x=0}^{x=1} = \left(\tfrac{9}{4}\tfrac{1}{2}2\sin(\theta)\cos(\theta)+\tfrac{9}{4}\theta\right)\Big]_{x=0}^{x=1} \\
&= \left(\tfrac{9}{4}\sin(\theta)\sqrt{1-\sin(\theta)^2}+\tfrac{9}{4}\theta\right)\Big]_{x=0}^{x=1} = \left(\tfrac{9}{4}\tfrac{2}{3}x\sqrt{1-(\tfrac{2}{3}x)^2}+\tfrac{9}{4}\mathrm{arcsin}\left(\tfrac{2}{3}x\right)\right)\Big]_{x=0}^{x=1} \\
&= \tfrac{3}{2}\sqrt{1-\tfrac{9}{4}}+\tfrac{9}{4}\mathrm{arcsin}(\tfrac{2}{3})-0-0 = \frac{\sqrt{5}}{2}+\tfrac{9}{4}\mathrm{arcsin}(\tfrac{2}{3}).
\end{aligned}$$

□

2.5 Partial fractions and integration

Example 2.100. Find a, b such that $\dfrac{2x^4+3x^2}{(x^2+1)^2(x^2+2)} = \dfrac{a}{(x^2+1)^2} + \dfrac{b}{x^2+2}$.

Proof. Since $(x^2+1)^2 = x^2(x^2+2)+1$ then

$$1 = (-x^2)(x^2+2)+(x^2+1)^2.$$

So

$$\begin{aligned}
\frac{2x^4+3x^2}{(x^2+1)^2(x^2+2)} &= \frac{(2x^2-1)(x^2+2)+2}{(x^2+1)^2(x^2+2)} = \frac{2x^2-1}{(x^2+1)^2} + \frac{2}{(x^2+1)^2(x^2+2)} \\
&= \frac{2x^2-1}{(x^2+1)^2} + \frac{2((-x^2)(x^2+2)+(x^2+1)^2)}{(x^2+1)^2(x^2+2)} \\
&= \frac{2x^2-1}{(x^2+1)^2} + \frac{-2x^2}{(x^2+1)^2} + \frac{2}{x^2+2} \\
&= \frac{-1}{(x^2+1)^2} + \frac{2}{x^2+2}
\end{aligned}$$

□

Example 2.101. Find a, b, c such that $\dfrac{3x^2-2x+1}{(x+1)(x^2+2x+2)} = \dfrac{a}{x+1} - \dfrac{b(2x+2)}{x^2+2x+2} - \dfrac{c}{(x+1)^2+1}$.

Proof. Since $x^2+2x+2 = (x+1)(x+1)+1$ then

$$1 = (x^2+2x+2)-(x+1)(x+1).$$

So

$$\begin{aligned}\frac{3x^2-2x+1}{(x+1)(x^2+2x+2)} &= \frac{3(x^2+2x+2)-8x-5}{(x+1)(x^2+2x+2)} = \frac{3}{x+1} + \frac{-8(x+1)+3}{(x+1)(x^2+2x+2)} \\ &= \frac{3}{x+1} - \frac{8}{x^2+2x+2} + \frac{3((x^2+2x+2)-(x+1)(x+1))}{(x+1)(x^2+2x+2)} \\ &= \frac{3}{x+1} - \frac{8}{x^2+2x+2} + \frac{3}{x+1} - \frac{3(x+1)}{x^2+2x+2} \\ &= \frac{6}{x+1} - \frac{\frac{3}{2}(2x+2)}{x^2+2x+2} - \frac{8}{(x+1)^2+1}.\end{aligned}$$

□

Example 2.102. Find a, b such that $\dfrac{9x+1}{(x-3)(x+1)} = \dfrac{a}{x-3} + \dfrac{b}{x+1}$.

Proof. Since $x+1 = (x-3)+4$ then $4 = (x+1)-(x-3)$ and

$$1 = \tfrac{1}{4}(x+1) - \tfrac{1}{4}(x-3).$$

So

$$\begin{aligned}\frac{9x+1}{(x-3)(x+1)} &= \frac{9(x+1)-8}{(x-3)(x+1)} = \frac{9}{x-3} - \frac{8}{(x-3)(x+1)} \\ &= \frac{9}{x-3} - \frac{8(\frac{1}{4}(x+1) - \frac{1}{4}(x-3)}{(x-3)(x+1)} \\ &= \frac{9}{x-3} - \frac{2}{x-3} + \frac{2}{x+1} \\ &= \frac{7}{x-3} + \frac{2}{x+1}.\end{aligned}$$

□

Example 2.103. Find a, b, c such that $\dfrac{4}{x^2(x+2)} = \dfrac{a}{x+2} + \dfrac{b}{x} + \dfrac{c}{x^2}$.

Proof. Since $x^2 - (x-1)(x-2) = 4$ then

$$\begin{aligned}\frac{4}{x^2(x+2)} &= \frac{x^2-(x-2)(x+2)}{x^2(x+2)} = \frac{x^2}{x^2(x+2)} - \frac{(x-2)(x+2)}{x^2(x+2)} \\ &= \frac{1}{x+2} - \frac{x-2}{x^2} = \frac{1}{x+2} - \frac{1}{x} + \frac{2}{x^2}\end{aligned}$$

□

Example 2.104. Evaluate $\displaystyle\int \frac{4}{x^2(x+2)}\,dx$.

Proof. Using $x^2 - (x-2)(x+2) = 4$ gives

$$\begin{aligned}
\int \frac{4}{x^2(x+2)}\,dx &= \int \frac{x^2-(x-2)(x+2)}{x^2(x+2)}\,dx \\
&= \int \frac{x^2}{x^2(x+2)}\,dx - \int \frac{(x-2)(x+2)}{x^2(x+2)}\,dx \\
&= \int \frac{1}{x+2}\,dx - \int \frac{x-2}{x^2}\,dx \\
&= \int \frac{1}{x+2}\,dx - \int \frac{1}{x}\,dx + \int \frac{2}{x^2}\,dx \\
&= \log(x+2)\ dx - \log(x) + 2\cdot\frac{1}{-1}x^{-1} + c \\
&= \log(x+2)\ dx - \log(x) - \frac{2}{x} + c, \qquad \text{where } c \text{ is a constant.}
\end{aligned}$$

□

Example 2.105. Evaluate $\displaystyle\int \frac{5x^4+13x^3+6x^2+4}{x^3+2x^2}\,dx.$

Proof.

$$\begin{aligned}
\int \frac{5x^4+13x^3+6x^2+4}{x^3+2x^2}\,dx &= \int \frac{5x(x^3+2x^2)+3x^3+6x^2+4}{x^3+2x^2}\,dx \\
&= \int \frac{5x(x^3+2x^2)+3(x^3+2x^2)+4}{x^3+2x^2}\,dx \\
&= \int (5x+3)\,dx + \int \frac{4}{x^2(x+2)}\,dx \\
&= \frac{5}{2}x^2 + 3x + \log(x+2) - \log(x) - \frac{2}{x} + c, \qquad \text{where } c \text{ is a constant,}
\end{aligned}$$

and the last line uses Example 2.104. □

Example 2.106. Evaluate $\displaystyle\int \frac{4x}{(x^2+4)(x-2)}\,dx.$

Proof. Using $(x^2+4) - (x-2)(x-2) = 4x$ gives

$$\begin{aligned}
\int \frac{4x}{(x^2+4)(x-2)}\,dx &= \int \frac{(x^2+4)-(x-2)(x-2)}{(x^2+4)(x-2)}\,dx \\
&= \int \frac{(x^2+4)}{(x^2+4)(x-2)}\,dx - \int \frac{(x-2)(x-2)}{(x^2+4)(x-2)}\,dx \\
&= \int \frac{1}{x-2}\,dx - \int \frac{x-2}{(x^2+4)}\,dx \\
&= \int \frac{1}{x-2}\,dx - \int \frac{1}{2}\frac{2x}{(x^2+4)}\,dx - \int \frac{-2}{x^2+4}dx \\
&= \int \frac{1}{x-2}\,dx - \tfrac{1}{2}\int \frac{2x}{(x^2+4)}\,dx + 2\int \frac{\frac{1}{4}}{\left(\frac{x}{2}\right)^2+1}dx \\
&= \int \frac{1}{x-2}\,dx - \tfrac{1}{2}\int \frac{2x}{(x^2+4)}\,dx + \int \frac{\frac{1}{2}}{\left(\frac{x}{2}\right)^2+1}dx \\
&= \log(x-2) - \tfrac{1}{2}\log(x^2+4) + \arctan\left(\tfrac{x}{2}\right) + c,
\end{aligned}$$

where c is a constant. $\square$

2.6 Other integration examples

Example 2.107. Evaluate $\displaystyle\int (6x^2+10)\sinh(x^3+5x-2)\,dx$.

Proof.

$$\begin{aligned}
\int (6x^2+10)\sinh(x^3+5x-2)\,dx &= \int 2\sinh(x^3+5x-2)(3x^2+5)\,dx \\
&= \int 2\sinh(y)\frac{dy}{dx}\,dx \qquad (\text{with } y = x^3+5x+2) \\
&= \int 2\sinh(y)\,dy \\
&= 2\cosh(y)+c = 2\cosh(x^3+5x+2)+c,
\end{aligned}$$

where c is a constant. $\square$

Example 2.108. Evaluate $\displaystyle\int \frac{\operatorname{sech}^2(3x)}{10+2\tanh(3x)}\,dx$.

Proof.

$$\begin{aligned}
\int \frac{\operatorname{sech}^2(3x)}{10+2\tanh(3x)}\,dx = \int \frac{\frac{1}{3}3\cdot 2\operatorname{sech}^2(3x)}{10+2\tanh(3x)}\,dx &= \int \tfrac{1}{3}\cdot\frac{1}{10+2\tanh(3x)}\cdot 3\cdot 2\operatorname{sech}^2(3x)\,dx \\
&= \tfrac{1}{3}\int \frac{1}{y}\cdot\frac{dy}{dx}dx \qquad (\text{with} \quad y = 10+2\tanh(3x)) \\
&= \tfrac{1}{3}(\log(y)+c') = \tfrac{1}{3}\log(y)+c = \tfrac{1}{3}\log(10+2\tanh(3x))+c,
\end{aligned}$$

where c and c' are constants. $\square$

Example 2.109. Evaluate $\displaystyle\int xe^{5x}\,dx$.

Proof.

$$\begin{aligned}
\int xe^{5x}\,dx = \tfrac{1}{5}\int x5e^{5x}\,dx = \tfrac{1}{5}\int (x\cdot 5e^{5x} + e^{5x} - e^{5x})dx &= \tfrac{1}{5}\int (x\cdot 5e^{5x} + e^{5x})\,dx - \tfrac{1}{5}\int e^{5x}dx \\
&= \tfrac{1}{5}\int \frac{d(xe^{5x})}{dx}\,dx - \tfrac{1}{5}\int e^{5x}dx = \tfrac{1}{5}(xe^{5x}) - \tfrac{1}{5}\cdot\tfrac{1}{5}e^{5x}+c \\
&= \tfrac{1}{5}xe^{5x} - \tfrac{1}{25}\,e^{5x}+c,
\end{aligned}$$

where c is a constant. $\square$

Example 2.110. Evaluate $\displaystyle\int x^2\log(x)\,dx$.

Proof. Using backwards of the product rule,

$$\begin{aligned}\int x^2\log(x)\,dx &= \tfrac{1}{3}\int 3x^2\log(x)\,dx = \tfrac{1}{3}\int (3x^2\log(x) + x^3\tfrac{1}{x} - x^3\tfrac{1}{x})\,dx \\ &= \tfrac{1}{3}\int (3x^2\log(x) + x^3\tfrac{1}{x})\,dx - \tfrac{1}{3}\int x^3\tfrac{1}{x}\,dx \\ &= \tfrac{1}{3}\int (3x^2\log(x) + x^3\tfrac{1}{x})\,dx - \tfrac{1}{3}\int x^2\,dx \\ &= \tfrac{1}{3}x^3\log(x) - \tfrac{1}{3}\cdot\tfrac{1}{3}x^3 + c = \tfrac{1}{3}x^3\log(x) - \tfrac{1}{9}x^3 + c,\end{aligned}$$

where c is a constant.

An alternate method would be to put $x = e^z$ so that $\frac{dx}{dz} = e^z$. Then the integral becomes

$$\int x^2\log(x)dx = \int e^{2z}\log(e^z)\frac{dx}{dz}dz = \int e^{2z}ze^z dz = \int ze^{3z}dz = \tfrac{1}{3}ze^{3z} - \tfrac{1}{9}e^{3z} + c.$$

□

Example 2.111. Evaluate $\int \log(x)\,dx$.

Proof. Using backwards of the product rule,

$$\begin{aligned}\int \log(x)\,dx &= \int (\log(x) + x\cdot\frac{1}{x} - x\frac{1}{x})\,dx = \int (\log(x) + x\cdot\frac{1}{x})\,dx - \int x\frac{1}{x}\,dx \\ &= \int (\log(x) + x\cdot\frac{1}{x})\,dx - \int 1\,dx = x\log(x) - x + c,\end{aligned}$$

where c is a constant. □

Example 2.112. Evaluate $\int e^{3x}\sin(2x)\,dx$.

Proof. This is a repeat of Example 2.80. This time do it by backwards of the product rule.

$$\begin{aligned}\int e^{3x}\cos(2x)\,dx &= \tfrac{1}{3}\int 3e^{3x}\cos(2x)\,dx = \tfrac{1}{3}\int \big(3e^{3x}\cos(2x) - 2e^{3x}\sin(2x)) + 2e^{3x}\sin(2x)\big)\,dx \\ &= \tfrac{1}{3}\int \big(3e^{3x}\cos(2x) - 2e^{3x}\sin(2x)\big)\,dx + \tfrac{2}{9}\int 3e^{3x}\sin(2x)\,dx \\ &= \tfrac{1}{3}e^{3x}\cos(2x) + \tfrac{2}{9}\int \big((3e^{3x}\sin(2x) + 2e^{3x}\cos(2x)) - 2e^{3x}\cos(2x)\big)\,dx \\ &= \tfrac{1}{3}e^{3x}\cos(2x) + \tfrac{2}{9}e^{3x}\sin(2x) - \tfrac{4}{9}\int e^{3x}\cos(2x)\,dx\end{aligned}$$

then

$$\tfrac{13}{9}\int e^{3x}\cos(2x)\,dx = \tfrac{1}{3}e^{3x}\cos(2x) + \tfrac{2}{9}e^{3x}\sin(2x) + c', \qquad \text{where } c' \text{ is a constant.}$$

So

$$\int e^{3x}\cos(2x)\,dx = \tfrac{3}{13}e^{3x}\cos(2x) + \tfrac{2}{13}e^{3x}\sin(2x) + c, \qquad \text{where } c \text{ is a constant.}$$

□

Example 2.113. Evaluate $\int \cosh^4(\theta)\,d\theta$.

Proof.

$$
\begin{aligned}
\int \cosh^4(\theta)\, d\theta &= \int \big(\frac{1}{2}(e^\theta + e^{-\theta})\big)^4\, d\theta \\
&= \int \frac{1}{2^4}\big(e^{4\theta} + 4e^{(3-1)\theta} + 6e^{(2-2)\theta} + 4e^{(1-3)\theta} + e^{-4\theta}\big)\, d\theta \\
&= \int \frac{1}{2^4}\big(2\cosh(4\theta) + 4\cdot 2\cosh(2\theta) + 6\big)\, d\theta \\
&= \frac{1}{2^4}\big(2\frac{1}{4}\sinh(4\theta) + 4\cdot 2\cdot \frac{1}{2}\sinh(2\theta) + 6\theta\big) + c \\
&= \frac{1}{2^5}\sinh(4\theta) + \frac{1}{2^2}\sinh(2\theta) + \frac{3}{2^3}\theta + c, \qquad \text{where } c \text{ is a constant.}
\end{aligned}
$$

□

Example 2.114. Evaluate $\int \sinh^5(x)\cosh^6(x)\, dx$.

Proof.

$$
\begin{aligned}
\int \sinh^5(x)\cosh^6(x)\, dx &= \int \sinh x \sinh^4(x)\cosh^6(x)\, dx \\
&= \int \sinh x(\sinh^2(x))^2\cosh^6(x)\, dx \\
&= \int \sinh x(1-\cosh^2(x))^2\cosh^6(x)\, dx \\
&= \int \sinh x(1 - 2\cosh^2(x) + \cosh^4(x))\cosh^6(x)\, dx \\
&= \int \sinh x(\cosh^6(x) - 2\cosh^8(x) + \cosh^{10}(x))\, dx \\
&= \frac{1}{7}\cosh^7(x) - \frac{2}{9}\cosh^9(x) + \frac{1}{11}\cosh^{11}(x) + c, \qquad \text{where } c \text{ is a constant.}
\end{aligned}
$$

□

Example 2.115. Evaluate $\int \sinh^5(x)\cosh^7(x)\, dx$.

Proof.

$$
\begin{aligned}
\int \sinh^5(x)\cosh^7(x)\, dx &= \int \sinh(x)\sinh^4(x)\cosh^7(x)\, dx \\
&= \int \sinh(x)(1-\cosh^2(x))^2\cosh^7(x)\, dx \\
&= \int \sinh(x)(1 - 2\cosh^2(x) + \cosh^4(x))\cosh^7(x)\, dx \\
&= \int \sinh(x)(\cosh^7(x) - 2\cosh^9(x) + \cosh^{11}(x))\, dx \\
&= \frac{1}{8}\cosh^8(x) - \frac{2}{10}\cosh^{10}(x) + \frac{1}{12}\cosh^{12}(x) + c, \qquad \text{where } c \text{ is a constant.}
\end{aligned}
$$

□

Example 2.116. Evaluate $\int e^x \cos(3x)dx$.

Proof.

$$\begin{aligned}\int e^x \cos(3x)dx &= \int e^x \frac{1}{2}(e^{3ix}+e^{-3ix})dx = \frac{1}{2}\int (e^{(1+3i)x}+e^{(1-3i)x})dx \\ &= \frac{1}{2(1+3i)}e^{(1+3i)x} + \frac{1}{2(1-3i)}e^{(1-3i)x} + c \\ &= \frac{(1-3i)}{2(1+9)}e^{(1+3i)x} + \frac{(1+3i)}{2(1+9)}e^{(1-3i)x} + c \\ &= \frac{1}{20}e^x\Big((1-3i)e^{3ix} + (1+3i)e^{-3ix}\Big) + c \\ &= \frac{1}{20}e^x\Big(e^{3ix}+e^{-3ix} - 3i(e^{3ix}-e^{-3ix})\Big) + c \\ &= \frac{1}{10}e^x\Big(\cos(3x) + 3\sin(3x)\Big) + c \\ &= \frac{1}{10}e^x\cos(3x) + \frac{3}{10}e^x\sin(3x) + c\end{aligned}$$

Check:

$$\frac{d}{dx}(\frac{1}{10}e^x\cos(3x) + \frac{3}{10}e^x\sin(3x)) = \frac{1}{10}e^x(-3\sin(3x)+9\cos(3x)) + \frac{1}{10}e^x(\cos(3x)+3\sin(3x)$$

□

Example 2.117. Evaluate $\int e^{-2x}\sin(11x)dx$

Proof.

$$\begin{aligned}\int e^{-2x}\sin(11x)dx &= \int e^{-2x}\frac{1}{2}(-i)(e^{11ix}-e^{-11ix})dx = (-i)\frac{1}{2}\int (e^{(-2+11i)x}-e^{(-2-11i)x})dx \\ &= (-i)\frac{1}{2(-1+11i)}e^{(-2+11i)x} - \frac{1}{2(-2-11i)}e^{(-2-11i)x} + c \\ &= (-i)\frac{(-2-11i)}{2(4+121)}e^{(-2+11i)x} - \frac{(-2+11i)}{2(4+121)}e^{(-2-11i)x} + c \\ &= (-i)\frac{1}{125}\cdot\frac{1}{2}e^{-2x}\Big((-2-11i)e^{11ix} - (-2+11i)e^{-11ix}\Big) + c \\ &= (-i)\frac{1}{125}\cdot\frac{1}{2}e^{-2x}\Big((-2)(e^{11ix}-e^{-11ix}) - 11i(e^{11ix}+e^{-11ix})\Big) + c \\ &= \frac{1}{125}e^{-2x}\Big((-2)\sin(11x) - 11\cos(11x)\Big) + c \\ &= \frac{-2}{125}e^{-2x}\sin(11x) - \frac{11}{125}e^{-2x}\cos(11x) + c\end{aligned}$$

Check:

$$\begin{aligned}&\frac{d}{dx}(\frac{-2}{125}e^{-2x}\sin(11x) + \frac{-11}{125}e^{-2x}\cos(11x)) \\ &\quad = \frac{1}{125}e^{-2x}(-22\cos(11x)+121\sin(11x)) + \frac{1}{125}e^{-2x}(-2)(-2\sin(11x)-11\cos(11x)\end{aligned}$$

□

Example 2.118. Evaluate $\int e^{5x}\cos(7t)dt$.

2.7 Additional derivative examples

Example 2.119. Find $\frac{dy}{dx}$ when $y = \log_x 10$.

Proof.

$$x^y = x^{\log_x 10} = 10.$$

Take the derivative:

$$\frac{d\ x^y}{dx} = \frac{d\ (e^{\log(x)})^y}{dx} = \frac{d\ e^y \log(x)}{dx} = e^{y \ln x}\left(y \cdot \frac{1}{x} + \frac{dy}{dx}\log(x)\right)$$
$$= \frac{d10}{dx} = 0.$$

So $\quad e^{y\log(x)}\left(y \cdot \frac{1}{x} + \frac{dy}{dx}\log(x)\right) = 0.$

Solve for $\frac{dy}{dx}$.

$$e^{y\log(x)}\frac{dy}{dx}\log(x) = \frac{-e^{y\log(x)}y}{x}. \qquad \text{So} \qquad \frac{dy}{dx} = \frac{-e^{y\log(x)}y}{xe^{y\log(x)}\log(x)} = \frac{-y}{x\log(x)} = \frac{\log_x 10}{x\log(x)}.$$

□

Example 2.120. Find the third derivative of 2^x with respect to x.

Proof. $y = 2^x$.

$$\frac{dy}{dx} = \frac{d2^x}{dx} = \frac{2(e^{\log(2)})^x}{dx} = \frac{de^{x\log(2)}}{dx} = e^{x\log(2)}(\log(2)) = (e^{\log(2)})^x\log(2) = 2^x\log(2).$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d2^x\log(2)}{dx} = \log(2)\cdot 2^x\log(2) = (\log(2))^2 2^x.$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d}{dx}\big(\log(2))^2 2^x\big) = (\log(2))^2 2^x\log(2) = (\log(2))^3 2^x.$$

□

Example 2.121. Let $a, b \in \mathbb{C}$. Show that if $y = a\cos(\log(x)) + b\sin(\log(x))$ then

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0.$$

Proof.

$$\begin{aligned}\frac{dy}{dx} &= a(-\sin(\log(x)))\frac{1}{x} + b\cos(\log(x))\frac{1}{x}\\ &= -a\sin(\log(x))x^{-1} + b\cos(\log(x))x^{-1},\end{aligned}$$

$$\begin{aligned}\frac{d^2y}{dx^2} &= -a\cos(\log(x))\frac{1}{x}x^{-1} + -a\sin(\log(x))(-1)x^{-2} + \ -b\sin(\log(x))\frac{1}{x}x^{-1} + b\cos(\log(x))(-1)x^{-2}\\ &= \frac{-a\cos(\log(x)) + a\sin(\log(x)) - b\sin(\log(x)) - b\cos(\log(x))}{x^2}\\ &= \frac{1}{x^2}\big((a-b)\sin(\log(x)) - (a+b)\cos(\log(x))\big).\end{aligned}$$

So

$$\begin{aligned}
LHS &= x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y \\
&= x^2\,\frac{1}{x^2}\big((a-b)\sin(\log(x)) - (a+b)\cos(\log(x))\big) \\
&\qquad + x\big(-a\sin(\log(x))x^{-1} + b\cos(\log(x))x^{-1}\big) \\
&\qquad + a\cos(\log(x)) + b\sin(\log(x)) \\
&= (a-b)\sin(\log(x)) - (a+b)\cos(\log(x)) \\
&\quad - a\sin(\log(x)) + b\cos(\log(x)) \\
&\quad + b\sin(\log(x)) + a\cos(\log(x)) \\
&= 0.
\end{aligned}$$

□

Example 2.122. Let $a, b \in \mathbb{C}$. Find $\frac{dy}{dx}$ when $a\sin(xy) + b\cos\left(\frac{x}{y}\right) = 0$.

Proof. Take the derivative:

$$\begin{aligned}
0 &= a\cos(xy)\left(x\frac{dy}{dx} + 1\cdot y\right) + \; -b\sin\left(\frac{x}{y}\right)\left(x(-1)y^{-2}\frac{dy}{dx} + 1\cdot y^{-1}\right) \\
&= a\cos(xy)x\frac{dy}{dx} + a\cos(xy)y + b\sin\left(\frac{x}{y}\right)\frac{x}{y^2}\frac{dy}{dx} - b\sin\left(\frac{x}{y}\right)y^{-1}.
\end{aligned}$$

Solve for $\frac{dy}{dx}$.

$$a\cos(xy)x\frac{dy}{dx} + b\sin\left(\frac{x}{y}\right)\frac{x}{y^2}\frac{dy}{dx} = a\cos(xy)y - b\sin\left(\frac{x}{y}\right)y^{-1}.$$

So

$$\frac{dy}{dx} = \frac{a\cos(xy)y - b\sin\left(\frac{x}{y}\right)y^{-1}}{a\cos(xy)x + b\sin\left(\frac{x}{y}\right)\frac{x}{y^2}} = \frac{a\cos(xy)y^3 - b\sin\left(\frac{x}{y}\right)y}{a\cos(xy)xy^2 + b\sin\left(\frac{x}{y}\right)x}$$

□

Example 2.123. Let $a \in \mathbb{C}$. Find $\frac{dy}{dx}$ when $y = \tan^{-1}\left(\frac{a}{x}\right)\cdot\cot^{-1}\left(\frac{x}{a}\right)$.

Proof.

$$\frac{dy}{dx} = \tan^{-1}\left(\frac{a}{x}\right)\left(\frac{-1}{1+\left(\frac{x}{a}\right)^2}\right)\frac{1}{a} + \frac{1}{1+\left(\frac{x}{a}\right)^2}(-1)ax^{-2}\cot^{-1}\left(\frac{x}{a}\right)$$

$$= \frac{-\tan^{-1}\left(\frac{a}{x}\right)}{a+\frac{x^2}{a}} + \frac{-\cot^{-1}\left(\frac{x}{a}\right)a}{x^2+a^2}$$

$$= \frac{-\tan^{-1}\left(\frac{a}{x}\right)a}{a^2+x^2} + \frac{-\cot^{-1}\left(\frac{x}{a}\right)a}{x^2+a^2}$$

$$= \left(\frac{-a}{a^2+x^2}\right)\left(\tan^{-1}\left(\frac{a}{x}\right)+\cot^{-1}\left(\frac{x}{a}\right)\right).$$

If $\frac{a}{x} = \tan z$ then $\frac{x}{a} = \cot z$ and $z = \tan^{-1}\left(\frac{a}{x}\right) = \cot^{-1}\left(\frac{x}{a}\right)$.

So

$$\frac{dy}{dx} = \left(\frac{-a}{a^2+x^2}\right)\left(\tan^{-1}\left(\frac{a}{x}\right)+\tan^{-1}\left(\frac{a}{x}\right)\right) = \frac{-2a\tan^{-1}\left(\frac{a}{x}\right)}{a^2+x^2}.$$

□

Example 2.124. Find $\frac{dy}{dx}$ when $y = \frac{(x+2)^{\frac{5}{2}}}{(x+6)^{\frac{1}{2}}(x+3)^{\frac{7}{2}}}$.

Proof. Sometimes it can simplfy calculations to take the log of both sides before taking the derivative.

$$\begin{aligned}\log(y) &= \log\left(\frac{(x+2)^{\frac{5}{2}}}{(x+6)^{\frac{1}{2}}(x+3)^{\frac{7}{2}}}\right)\\ &= \log\left((x+2)^{\frac{5}{2}}\right) - \log\left((x+6)^{\frac{1}{2}}\right) - \log\left((x+3)^{\frac{7}{2}}\right)\\ &= \frac{5}{2}\log(x+2) - \frac{1}{2}\log(x+6) - \frac{7}{2}\log(x+3).\end{aligned}$$

So, by taking the derivative with respect to x,

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{5}{2}\frac{1}{(x+2)} - \frac{1}{2}\cdot\frac{1}{(x+6)} - \frac{7}{2}\frac{1}{(x+3)}.$$

So

$$\begin{aligned}\frac{dy}{dx} &= y\Big(\frac{5}{2}\frac{1}{(x+2)} - \frac{1}{2}\cdot\frac{1}{(x+6)} - \frac{7}{2}\frac{1}{(x+3)}\Big)\\ &= \frac{(x+2)^{\frac{5}{2}}}{(x+6)^{\frac{1}{2}}(x+3)^{\frac{7}{2}}}\Big(\frac{5}{2}\frac{1}{(x+2)} - \frac{1}{2}\cdot\frac{1}{(x+6)} - \frac{7}{2}\frac{1}{(x+3)}\Big)\end{aligned}$$

□

Example 2.125. If $x^m y^n = (x+y)^{m+n}$ show that $\frac{dy}{dx} = yx$.

Proof. Sometimes it can simplfy calculations to take the log of both sides before taking the derivative. Since

$$\log(x^m y^n) = \log((x+y)^{m+n}) \qquad \text{then} \qquad \log(x^m) + \log(y^n) = (m+n)\log(x+y)$$

and

$$m\log(x) + n\log(y) = (m+n)\log(x+y).$$

Take the derivative with respect to x.

$$\frac{m}{x} + \frac{n}{y}\frac{dy}{dx} = (m+n)\frac{1}{x+y}\Big(1 + \frac{dy}{dx}\Big).$$

So

$$\frac{m}{x} + \frac{n}{y}\frac{dy}{dx} = \frac{(m+n)}{x+y} + \frac{(m+n)}{x+y}\frac{dy}{dx}.$$

So

$$\Big(\frac{n}{y} - \frac{m+n}{x+y}\Big)\frac{dy}{dx} = \frac{m+n}{x+y} - \frac{m}{x}.$$

So

$$\Big(\frac{nx+ny-my-ny}{y(x+y)}\Big)\frac{dy}{dx} = \frac{mx+nx-mx-my}{x(x+y)}.$$

So

$$\Big(\frac{nx-my}{y}\Big)\frac{dy}{dx} = \frac{nx-my}{x}.$$

So

$$\frac{dy}{dx} = \frac{y}{x}.$$

□

Example 2.126. Let $a \in \mathbb{C}$. Find $\dfrac{dy}{dx}$ when $y = a^x + e^{\tan(x)} + (\cot(x))^{\cos(x)}$.

Proof. Since

$$y = (e^{\log(a)})^x + e^{\tan(x)} + \big(e^{\log(\cot(x))}\big)^{\cos(x)} = e^{x\log(a)} + e^{\tan(x)} + e^{\cos(x)\log(\cot(x))}$$

then

$$\begin{aligned}\frac{dy}{dx} &= e^{x\log(a)}\log(a) + e^{\tan(x)}\sec(x)^2 + e^{\cos(x)\log(\cot(x))}\Big(\frac{\cos(x)(-\csc(x)^2)}{\cot(x)} + (-\sin(x))\log(\cot(x))\Big) \\ &= e^{x\log(a)}\log(a) + e^{\tan(x)}\sec(x)^2 + e^{\cos(x)\log(\cot(x))}\Big(\frac{\cos(x)\frac{-1}{\sin(x)^2}}{\frac{\cos(x)}{\sin(x)}} + (-\sin(x))\log(\cot(x))\Big) \\ &= a^x\log(a) + e^{\tan(x)}\sec(x)^2 + (\cot(x))^{\cos(x)}\big(-\csc(x) - \sin(x)\log(\cot(x))\big).\end{aligned}$$

□

Example 2.127. Find $\frac{dy}{dx}$ when $y = x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}$.

Proof. Since $y = x^y$ then $y = (e^{\log(x)})^y = e^{y\log(x)}$. So

$$\frac{dy}{dx} = e^{y\log(x)}\frac{d(y\log(x)}{dx} = e^{y\log(x)}\Big(\frac{y}{x} + \frac{dy}{dx}\log(x)\Big).$$

So

$$\frac{dy}{dx} = \frac{y}{x}e^{y\log(x)} + \log(x)e^{y\log(x)}\frac{dy}{dx}.$$

So

$$(1 - \log(x)e^{y\log(x)})\frac{dy}{dx} = \frac{y}{x}x^y.$$

So

$$\frac{dy}{dx} = \frac{\frac{yx^y}{x}}{1 - \log(x)\cdot x^y} = \frac{yx^y}{x(1 - x^y\log(x))}.$$

□

2.8 Equations

Example 2.128. Solve the equation $\frac{d^2y}{dx^2} = 0$.

Proof. If $y = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \cdots$ then

$$0 = \frac{d^2y}{dx^2} = 2c_2 + 3\cdot 2c_3x + 4\cdot 3c_4x^2 + \cdots \qquad \text{so that} \qquad y = c_0 + c_1x,$$

since $c2c_2 = 0$, $6c_3 = 0$, $12c_4 = 0$, ... So

$$y = c_0 + c_1x, \qquad \text{where } c_0 \text{ and } c_1 \text{ are constants.}$$

□

Example 2.129. Solve the equation $\frac{dy}{dx} = \cos(x)$.

Proof. Since

$$\int \frac{dy}{dx}\,dx = \int \cos(x)dx \qquad \text{then} \qquad y = \sin(x) + C, \qquad \text{where } C \text{ is a constant.}$$

Let's check this answer: Let $y = \sin(x) + C$. Then

$$\frac{dy}{dx} = \frac{d}{dx}(\sin(x) + C) = \cos(x) + 0 = \cos(x).$$

□

Example 2.130. Solve the equation $\frac{dy}{dx} = y$.

Proof. Since

$$\frac{1}{y}\cdot\frac{dy}{dx} = 1 \qquad \text{then} \qquad \int \frac{1}{y}\frac{dy}{dx}\,dx = \int 1\,dx.$$

So

$$\log(y) = x + C, \qquad \text{where } C \text{ is a constant.}$$

So

$$y = e^{x+C} = e^C e^x = ce^x, \qquad \text{where } c \text{ is a constant.}$$

where c is a constant.

Let's check this answer: Let c be a constant and let $y = ce^x$. Then

$$\frac{dy}{dx} = \frac{d}{dx}(ce^x) = c\frac{de^x}{dx} = ce^x = y.$$

□

Example 2.131. Solve the equation $\frac{dy}{dx} = y^{\frac{1}{3}}$.

Proof. Since

$$y^{-\frac{1}{3}}\frac{dy}{dx} = 1 \qquad \text{then} \qquad \int y^{-\frac{1}{3}}\frac{dy}{dx}\,dx = \int 1\,dx.$$

So

$$\tfrac{3}{2}y^{2/3} = x + C, \qquad \text{where } C \text{ is a constant.}$$

So

$$y^{2/3} = \tfrac{2}{3}x + c \qquad \text{and} \qquad y = (\tfrac{2}{3}x + c)^{3/2},$$

where c is a constant.

Let's check this answer: Let c be a constant and let $y = (\frac{2}{3}x + c)^{3/2}$. Then

$$\frac{dy}{dx} = \frac{d}{dx}\Big((\tfrac{2}{3}x + c)^{3/2}\Big) = \tfrac{3}{2}(\tfrac{2}{3}x + c)^{\frac{1}{2}} \cdot \tfrac{2}{3} = (\tfrac{2}{3}x + c)^{\frac{1}{2}} \qquad \text{and}$$
$$y^{1/3} = ((\tfrac{2}{3}x + c)^{3/2})^{\frac{1}{3}} = (\tfrac{2}{3}x + c)^{1/2}.$$

So $\frac{dy}{dx} = y^{\frac{1}{3}}$. □

Example 2.132. Verify by substitution that $y = x^2 + \frac{2}{x}$ is a solution of the equation $\frac{dy}{dx} + \frac{y}{x} = 3x$.

Proof. Letting $y = x^2 + \frac{y}{x}$ then

$$\frac{dy}{dx} + \frac{y}{x} = (2x - 2x^{-2}) + \frac{1}{x}(x^2 + \frac{2}{x}) = 2x - 2x^{-2} + x + 2x^{-2} = 3x,$$

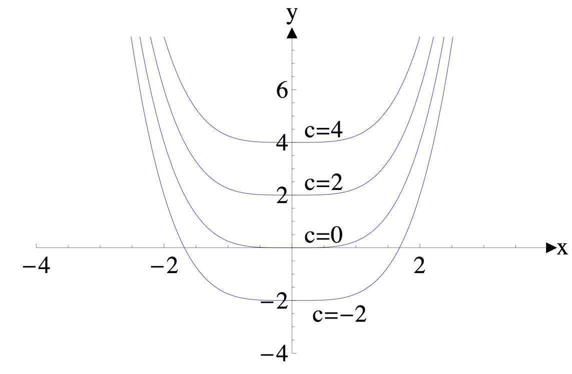
which verifies that y satisfies the equation $\dfrac{dy}{dx} + \dfrac{y}{x} = 3x$. □

Example 2.133. Solve the equation $\frac{dy}{dx} = x^3$.

Proof.

$$y = \int \frac{dy}{dx}\,dx = \int x^3\,dx = \tfrac{1}{4}x^4 + c, \qquad \text{where } c \text{ is a constant.}$$

Sample graphs of the real solutions of these equations for $c \in \{-2, 0, 2, 4\}$ are



Graphs of $y = \frac{1}{4}x^4 + c$

□

Example 2.134. Solve the equation $\frac{dy}{dx} = x^3$ given that $y(0) = 2$.

Proof. Since $\frac{dy}{dx} = x^3$ then

$$y = \int \frac{dy}{dx}\,dx = \int x^3\,dx = \tfrac{1}{4}x^4 + c, \qquad \text{where } c \text{ is a constant.}$$

Since $y(0) = 2$ then $\frac{1}{4}0^4 + c = 2$ and $c = 2$.
So $y = \frac{1}{4}x^4 + 2$. □

Example 2.135. Solve the equation $\frac{dy}{dx} = \frac{y}{1+x}$.

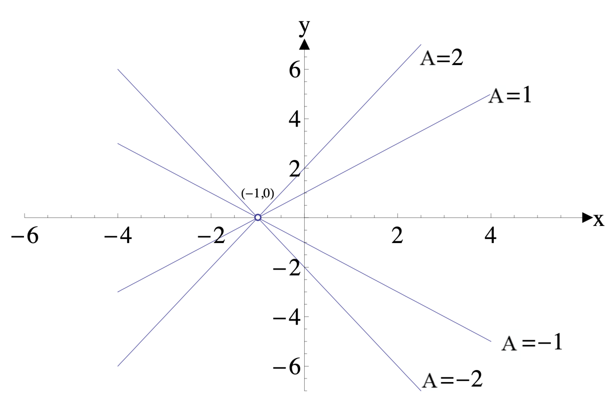
Proof. Integrating both sides of $\frac{1}{y}\frac{dy}{dx} = \frac{1}{1+x}$ gives

$$\int \frac{1}{y}\frac{dy}{dx}\,dx = \int \frac{1}{1+x}\,dx \qquad \text{and} \qquad \log(y) = \log(1+x) + c, \quad \text{where } c \text{ is a constant.}$$

So

$$y = e^{\log(y)} = e^{\log(1+x)+c} = e^c e^{\log(1+x)} = A(1+x), \qquad \text{where } A \text{ is a constant.}$$

Sample graphs of the real solutions of these equations for $A \in \{-2, -1, 1, 2\}$ are



Real solutions of $y = A(1+x)$

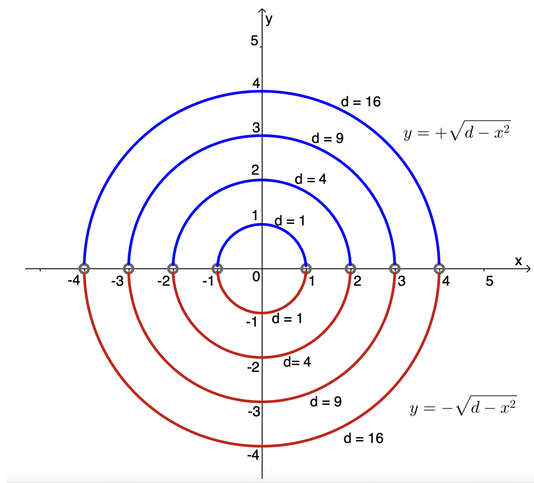
□

Example 2.136. Solve the equation $\frac{dy}{dx} = \frac{-x}{y}$ with $y(0) = 3$.

Proof. Integrating both sides of $y\frac{dy}{dx} = -x$ gives

$$\int y\frac{dy}{dx}\,dx = \int -x\,dx \qquad \text{and} \qquad \frac{1}{2}y^2 = -\frac{1}{2}x^2 + c, \quad \text{where } c \text{ is a constant.}$$

So $y^2 = -x^2 + d$, where d is a constant. Sample graphs of real solutions these equations for $d \in \{1, 2, 3, 4\}$ are



Real solutions of $y^2 = -x^2 + d$

Since $y(0) = 3$ then $-0^2 + 2c = 3^2$ and $2c = 9$. So $y = \sqrt{-x^2 + 2} = \sqrt{9 - x^2}$. The graph of this solution is the blue semicircle going through $(0, 3)$. □

Example 2.137. Solve $x\frac{dy}{dx} + y = e^x$.

Proof. Since $\frac{d}{dx}(xy) = x\frac{dy}{dx} + y = e^x$ then integrating both sides with respect to x gives

$$\int \frac{d}{dx}(xy)\,dx = \int e^x\,dx \qquad \text{and} \qquad xy = e^x + c, \qquad \text{where } c \text{ is a constant.}$$

So $y = \frac{1}{x}e^x + c\frac{1}{x}$, where c is a constant. □

Example 2.138. Solve the equation $\frac{dy}{dx} + \frac{y}{x} = \sin x$.

Proof. Since $x\frac{dy}{dx} + y = x\sin x$ then integrating both sides gives

$$xy = \int x\sin x\,dx \int \big((x\sin x - \cos x) + \cos x\big)\,dx = -x\cos x + \int \cos x\,dx = -x\cos x + \sin x + c,$$

where c is a constant. So

$$y = -\cos x + \frac{1}{x}\sin x + c\frac{1}{x}, \qquad \text{where } c \text{ is a constant.}$$

□

Example 2.139. Solve $\frac{1}{2}\frac{dy}{dx} - xy = x$ with $y(0) = -3$.

Proof. Since $\frac{1}{2}\frac{dy}{dx} = x(y+1)$ then $\frac{1}{2}\frac{1}{(y+1)}\frac{dy}{dx} = x$ and integrating both sides with respect to x gives

$$\int \frac{1}{2}\frac{1}{(y+1)}\frac{dy}{dx}\,dx = \int x\,dx \quad \text{and} \quad \tfrac{1}{2}\log(y+1) = \tfrac{1}{2}x^2 + c, \quad \text{where } c \text{ is a constant.}$$

Since $y(0) = -3$ then $\frac{1}{2}\log(-3+1) = \frac{1}{2}0^2 + c$ and $2c = \log(-3+1)$. Thus

$$\log(y+1) = x^2 + \log(-2) \quad \text{and} \quad y + 1 = e^{x^2+\log(-2)} = e^{x^2}e^{\log(-2)} = e^{x^2}(-2) = -2e^{x^2}.$$

Thus

$$y = -1 - 2e^{x^2}.$$

□

Example 2.140. Solve the differential equation $\frac{dy}{dx} = \frac{y}{x} + \cos\left(\frac{y}{x}\right)^2$ by substituting $u = \frac{y}{x}$.

Proof. If $u = \frac{y}{x}$ then $xu = y$ and $\frac{dy}{dx} = x\frac{du}{dx} + u$. So

$$\frac{dy}{dx} = \frac{y}{x} + \cos\left(\frac{y}{x}\right)^2 \qquad \text{is the same as} \qquad x\frac{du}{dx} + u = u + \cos(u)^2.$$

So

$$\frac{1}{\cos(u)^2}\frac{du}{dx} = \frac{1}{x}, \qquad \text{or equivalently} \qquad \sec(u)^2\frac{du}{dx} = \frac{1}{x},$$

and integrating both sides with respect to x gives

$$\tan(u) = \log(x) + c, \quad \text{where } c \text{ is a constant.}$$

So $u = \arctan(\log(x) + c)$ and $\frac{y}{x} = \arctan(\log(x) + c)$ and

$$y = x\arctan(\log(x) + c), \quad \text{where } c \text{ is a constant.}$$

□

Example 2.141. Let $y' = \frac{dy}{dx}$. Solve the equation $y' + 3y = 0$.

Proof. Let $D = \frac{d}{dx}$. Then the equation is

$$(D+3)y = 0 \qquad \text{or, equivalently,} \quad Dy = -3y.$$

Solutions are

$$y = c_2e^{-3x}, \qquad \text{where } c_2 \text{ is a constant,}$$

Let's check this answer: Let c_2 be a constant and let $y = c_2e^{-3x}$. Then

$$\frac{dy}{dx} = \frac{d}{dx}(c_2e^{-3x}) = (-3)e^{-3x} = -3y.$$

□

Example 2.142. Let $y' = \frac{dy}{dx}$. Solve the equation $y' + 4y = 0$.

Proof. Let $D = \frac{d}{dx}$. Then the equation is

$$(D+4)y = 0 \qquad \text{or, equivalently,} \quad Dy = -4y.$$

Solutions are

$$y = c_1 e^{-4x}, \qquad \text{where } c_1 \text{ is a constant,}$$

Let's check this answer: Let c_1 be a constant and let $y = c_1 e^{-4x}$. Then

$$\frac{dy}{dx} = \frac{d}{dx}(c_1 e^{-4x}) = c_1(-4)e^{-4x} = -4y.$$

□

Example 2.143. Let $y' = \frac{dy}{dx}$ and $y'' = \frac{d^2y}{dx^2}$. Solve the equation $y'' + 7y' + 12y = 0$.

Proof. Let $D = \frac{d}{dx}$. Then $D^2 = \frac{d^2}{dx^2}$ and the equation is $(D^2 + 7D + 12)y = 0$. So the equation is

$$(D+4)(D+3)y = 0, \qquad \text{or, equivalently} \qquad (D+3)(D+4)y = 0,$$

From Example 2.141 and Example 2.142, the equation $(D+3)(D+4)y = 0$ has solutions

$$y = c_1 e^{-4x} + c_2 e^{-3x}, \qquad \text{where } c_1 \text{ and } c_2 \text{ are constants.}$$

Let's check this answer: Let c_1 and c_2 be constants and let $y = c_1 e^{-4x} + c_2 e^{-3x}$. Then

$$\begin{aligned}(D+4)(D+3)y &= (D+4)(D+3)(c_1 e^{-4x} + c_2 e^{-3x}) \\ &= (D+4)(D+3)c_1 e^{-4x} + (D+3)(D+4)c_2 e^{-3x} \\ &= (D+4)0 + (D+3)0 = 0 + 0 = 0.\end{aligned}$$

□

Example 2.144. Solve the equation $y'' + 2y' + y = 0$.

Proof. Let $D = \frac{d}{dx}$. Then $D^2 = \frac{d^2}{dx^2}$ and the equation is

$$(D^2 + 2D + 1)y = 0, \qquad \text{or, equivalently,} \quad (D+1)^2 y = 0.$$

Let $z = e^x y$ so that $y = e^{-x} z$. By the product rule

$$(D+1)e^{-x}z = e^{-x}(Dz) + (D(e^{-x}))z + e^{-x}z = e^{-x}Dz - e^{-x}z + e^{-x}z = e^{-x}Dz.$$

So

$$0 = (D+1)^2 y = (D+1)(D+1)e^x z = (D+1)e^x Dz = e^x DDz = e^x D^2 z.$$

So the equation

$$(D+1)^2 y = 0 \qquad \text{is equivalent to} \qquad D^2 z = 0.$$

Since the equation $D^2 z = 0$ has solutions $z = c_1 + c_2 x$ where c_1 and c_2 are constants then the equation $(D+1)^2 y = 0$ has solutions

$$y = e^{-x} z = c_1 e^{-x} + c_2 x e^{-x}, \qquad \text{where } c_1 \text{ and } c_2 \text{ are constants.}$$

□

Example 2.145. Let $y' = \frac{dy}{dx}$ and $y'' = \frac{d^2y}{dx^2}$. Solve the equation

$$y'' - 4y' + 13y = 0 \qquad \text{subject to } y(0) = 1 \text{ and } y'(0) = 6.$$

Proof. Let $D = \frac{d}{dx}$. Then $D^2 = \frac{d^2}{dx^2}$ and the equation $(D^2 - 4D + 13)y = 0$ is

$$(D - (2+3i))(D - (2-3i))^2 y = 0, \qquad \text{which has solutions} \quad y = c_1 e^{(2+3i)x} + c_2 e^{(2-3i)x}$$

with $c_1, c_2 \in \mathbb{C}$. Then

$$y' = (2+3i)c_1 e^{(2+3i)x} + (2-3i)c_2 e^{(2-3i)x}$$

and

$$1 = y(0) = c_1 e^0 + c_2 e^0 = c_1 + c_2 \qquad \text{and} \qquad 6 = y'(0) = (2+3i)c_1 + (2-3i)c_2.$$

Solving for c_1 and c_2 gives $6 = 2 \cdot 1 + 3i(c_1 - c_2)$ so that $c_1 + c_2 = 1$ and $c_1 - c_2 = -\frac{4}{3}i$.
So $c_1 = \frac{1}{2} - \frac{2}{3}i$ and $c_2 = \frac{1}{2} + \frac{2}{3}i$ and

$$\begin{aligned} y &= (\tfrac{1}{2} - \tfrac{2}{3}i)e^{(2+3i)x} + (\tfrac{1}{2} + \tfrac{2}{3}i)e^{(2-3i)x} = \tfrac{1}{2}e^2(e^{3ix} + e^{-3ix}) - \tfrac{2}{3}ie^2(e^{3ix} - e^{-3ix}) \\ &= e^2\cos(3x) - \tfrac{4}{3}e^2\sin(3x). \end{aligned}$$

□

Example 2.146. Solve the equation $y'' + 2y' - 8y = 1 - 8x^2$.

Proof. Let $D = \frac{d}{dx}$. Then the equation $(D^2 + 2D - 8)y = 0$ is

$$(D-2)(D+4)y = 0 \qquad \text{which has solutions} \quad y = c_1 e^{2x} + c_2 e^{-4x},$$

where $c_1, c_2 \in \mathbb{C}$.

Let $y = ax^2 + bx + c$ so that $y' = 2ax + b$ and $y'' = 2a$. Then

$$1 - 8x^2 = y'' + 2y' - 8y = 2a + 2(2ax + b) - 8(ax^2 + bx + c) = -8ax^2 + (4a - 8b)x + (2a + 2b - 8c)$$

so that $a = 1$ and $4a - 8b = 0$ and $2a + 2b - 8c = 1$. So $a = 1$ and $b = \frac{1}{2}$ and $c = \frac{1}{4}$ and

$$y = x^2 + \tfrac{1}{2}x + \tfrac{1}{4} \qquad \text{is a particular solution.}$$

So the general solution to $y'' + 2y' - 8y = 1 - 8x^2$ is

$$y = c_1 e^{2x} + c_2 e^{-4x} + x^2 + \tfrac{1}{2}x + \tfrac{1}{4}, \qquad \text{with } c_1, c_2 \in \mathbb{C}.$$

□

Example 2.147. Solve the equation $y'' + 2y' - 8y = e^{3x}$.

Proof. Let $D = \frac{d}{dx}$. Then the equation $(D^2 + 2D - 8)y = 0$ is

$$(D-2)(D+4)y = 0 \qquad \text{which has solutions} \quad y = c_1 e^{2x} + c_2 e^{-4x},$$

where $c_1, c_2 \in \mathbb{C}$.

Let $y = ae^{3x}$ so that $y' = 3ae^{3x}$ and $y'' = 9ae^{3x}$. Then

$$e^3x = y'' + 2y' - 8y = 9ae^{3x} + 6ae^{3x} - 8ae^{3x} = 7ae^{3x} \qquad \text{gives} \quad a = \tfrac{1}{7}.$$

So

$$y = \tfrac{1}{7}e^{3x} \qquad \text{is a particular solution.}$$

So the general solution to $y'' + 2y' - 8y = e^{3x}$ is

$$y = c_1 e^{2x} + c_2 e^{-4x} + \tfrac{1}{7}e^{3x}, \qquad \text{with } c_1, c_2 \in \mathbb{C}.$$

□

Example 2.148. Solve the equation $y'' + 2y' - 8y = 85\cos(x)$.

Proof. Let $D = \frac{d}{dx}$. Then the equation $(D^2 + 2D - 8)y = 0$ is

$$(D-2)(D+4)y = 0 \qquad \text{which has solutions} \quad y = c_1e^{2x} + c_2e^{-4x},$$

where $c_1, c_2 \in \mathbb{C}$.

(c) Let $y = a\cos(x) + b\sin(x)$ so that $y' = -a\sin(x) + b\cos(x)$ and $y'' = -a\cos(x) - b\cos(x)$ and

$$\begin{aligned}85\cos(x) = y'' + 2y' - 8y &= -a\cos(x) - b\sin(x) - 2a\sin(x) + 2b\cos(x) - 8a\cos(x) - 8b\sin(x)\\ &= (-a + 2b - 8a)\cos(x) + (-b - 2a - 8b)\sin(x) = (-9a + 2b)\cos(x) + (-2a - 9b)\sin(x),\end{aligned}$$

giving $9b = -2a$ and $85 = -9a + 2b = (-9)\frac{-9}{2}b + 2b = \frac{85}{2}b$. So $b = 2$ and $a = \frac{-9}{2} \cdot 2 = -9$. So

$$y = -9\cos(x) + 2\sin(x) \qquad \text{is a particular solution.}$$

The general solution to $y'' + 2y' - 8y = 85\cos(x)$ is

$$y = c_1e^{2x} + c_2e^{-4x} - 9\cos(x) + 2\sin(x), \qquad \text{with } c_1, c_2 \in \mathbb{C}.$$

□

Example 2.149. Solve the equation $y'' + 2y' - 8y = 3 - 24x^2 + 7e^{3x}$.

Proof. Let $D = \frac{d}{dx}$. Then the equation $(D^2 + 2D - 8)y = 0$ is

$$(D-2)(D+4)y = 0 \qquad \text{which has solutions} \quad y = c_1e^{2x} + c_2e^{-4x},$$

where $c_1, c_2 \in \mathbb{C}$.

Since $3 - 24x^2 + 7e^{3x} = 3(1 - 8x^2) + 7e^3x$, the particular solutions for Example 2.146 and Example 2.147 give that

$$y = 3(\tfrac{1}{2}x + \tfrac{1}{4}) + 7 \cdot \tfrac{1}{7}e^{3x} \qquad \text{is a particular solution}$$

of $y'' + 2y' - 8y = 3(1 - 8x^2) + 7e^{3x}$. The general solution to $y'' + 2y' - 8y = 3(1 - 8x^2) + 7e^{3x}$ is

$$y = c_1e^{2x} + c_2e^{-4x} + \tfrac{3}{2}x + \tfrac{3}{4} + e^{3x}, \qquad \text{with } c_1, c_2 \in \mathbb{C}.$$

□

Example 2.150. Solve the equation $y'' - y = e^x$.

Proof. Let $D = \frac{d}{dx}$. Then the equation $(D^2 - 1)y = 0$ is

$$(D-1)(D+1)y = 0 \qquad \text{which has solutions} \quad y = c_1e^{x} + c_2e^{-x},$$

where $c_1, c_2 \in \mathbb{C}$. If $y = axe^x$ then $y' = axe^x + ae^x$ and $y'' = axe^x + ae^x + ax^e$ and

$$e^x = y'' - y = axe^x + 2ae^x - axe^x \qquad \text{gives} \quad a = \tfrac{1}{2},$$

so that

$$y = \tfrac{1}{2}xe^x \qquad \text{is a particular solution.}$$

The general solution to $y'' - y = e^x$ is

$$y = c_1e^{x} + c_2e^{-x} + \tfrac{1}{2}xe^x, \qquad \text{with } c_1, c_2 \in \mathbb{C}.$$

□

Example 2.151. Solve the equation $y'' + 2y' + y = e^{-x}$.

Proof. Let $D = \frac{d}{dx}$. Then the equation $(D^2 + 2D + 1))y = 0$ is

$$(D+1)^2 y = 0 \qquad \text{which has solutions} \quad y = c_1 e^{-x} + c_2 x e^{-x},$$

where $c_1, c_2 \in \mathbb{C}$. If $y = ax^2 e^{-x}$ then $y' = 2axe^{-x} - ax^2 e^{-x} = (2ax - ax^2)e^{-x}$ and $y'' = (2a - 2ax)e^{-x} - (2ax - ax^2)e^{-e} = (2a - 4ax + ax^2)e^{-x}$ and

$$e^{-x} = y'' + 2y' + y = (2a - 4ax + ax^2 + 4ax - 2ax^2 + ax^2)e^{-x} = 2ae^{-x} \qquad \text{gives} \quad a = \tfrac{1}{2},$$

so that

$$y = \tfrac{1}{2} x^2 e^{-x} \qquad \text{is a particular solution.}$$

The general solution to $y'' + 2y' + y = e^{-x}$ is

$$y = c_1 e^{-x} + c_2 x e^{-x} + \tfrac{1}{2} x^2 e^{-x}, \qquad \text{with } c_1, c_2 \in \mathbb{C}.$$

□

Example 2.152. Solve the equation $y'' + 49y = 28\sin(7t)$.

Proof. Let $D = \frac{d}{dx}$. Then the equation $(D^2 + 2D + 1))y = 0$ is

$$(D + 7i)(D - 7i)y = 0 \qquad \text{which has solutions} \quad y = c_1 e^{7it} + c_2 e^{-7it},$$

where $c_1, c_2 \in \mathbb{C}$. Another way to write $y = c_1 e^{7it} + c_2 e^{-7it}$ is

$$y = A\cos(7t) + B\sin(7t), \qquad \text{where } A \text{ and } B \text{ are constants.}$$

If $y = at\cos(7t) + bt\sin(7t)$ then

$$\begin{aligned}
y &= at\cos(7t) + bt\sin(7t), \\
y' &= -7at\sin(7t) + a\cos(7t) + 7bt\cos(7t) + b\sin(7t) \\
&= (7bt + a)\cos(7t) + (-7at + b)\sin(7t), \\
y'' &= -7(7bt + a)\sin(7t) + 7b\cos(7t) + 7(-7at + b)\cos(7t) + (-7a)\sin(7t) \\
&= (-49at + 14b)\cos(7t) + (-49bt - 14a)\sin(7t),
\end{aligned}$$

so that

$$y'' + 49y = 14b\cos(7t) - 14a\sin(7t) \qquad \text{giving } b = 0 \text{ and } a = -2.$$

Thus

$$y = -2t\cos(7t) \qquad \text{is a particular solution}$$

and the general solution to $y'' + 49y = 28\sin(7t)$ is

$$y = A\cos(7t) + B\sin(7t) - 2t\cos(7t), \qquad \text{where } A \text{ and } B \text{ are constants.}$$

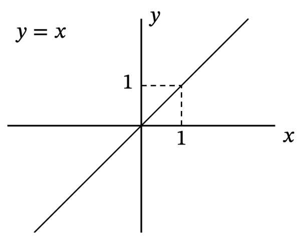
□

3 Graphing

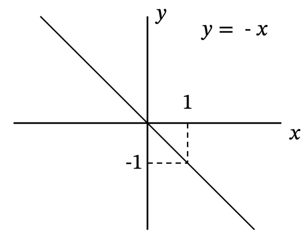
3.1 Basic graphs

Example 3.1. Graph the lines $\{(x,y)\in\mathbb{R}^2 \mid y=x\}$ and $\{(x,y)\in\mathbb{R}^2 \mid y=-x\}$.

Proof.



The line $y=x$

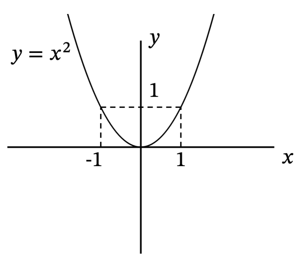


the line $y=-x$

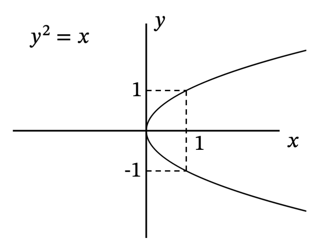
□

Example 3.2. Graph the parabolas $\{(x,y)\in\mathbb{R}^2 \mid y=x^2\}$ and $\{(x,y)\in\mathbb{R}^2 \mid y^2=x\}$.

Proof. These two graphs are flips of each other around the line $y=x$. The first graph is obtained by plotting the points $(0,0)$, $(1,1)$, $(-1,1)$, $(2,4)$, $(-2,4)$ and connecting these points with a smooth continuous curve.



The parabola $y=x^2$

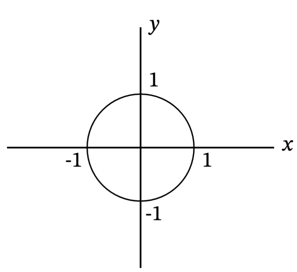


the parabola $y^2=x$

□

Example 3.3. Graph the circle $\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2=1\}$.

Proof. The set $\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2=1\}$ is the set of points in $\mathbb{R}^2$ that are distance 1 from the origin.



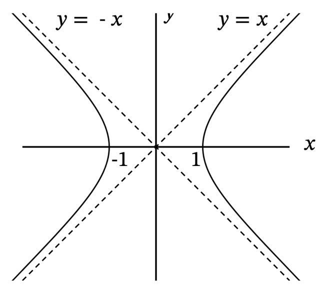
All points in $\mathbb{R}^2$ that are distance 1 from the origin.

The circle $x^2+y^2=1$

□

Example 3.4. Graph the hyperbola $\{(x,y)\in\mathbb{R}^2 \mid x^2-y^2=1\}$.

Proof.



The hyperbola $x^2 - y^2 = 1$

Graphing notes:

(a) If $y = 0$ then $x^2 = 1$. So $x = \pm 1$.

(b) If $x = 0$ then $-y^2 = 1$ which is impossible for $y \in \mathbb{R}$.

(c) The equation is $1 - \left(\frac{y}{x}\right)^2 = \left(\frac{1}{x}\right)^2$.

If x gets very big then $\frac{1}{x}$ gets closer and closer to 0 and the equation gets closer and closer to $1 - \left(\frac{y}{x}\right)^2 = 0$. This is the same as $\left(\frac{y}{x}\right)^2 = 1$, which is the same as $\frac{y}{x} = \pm 1$, i.e. $y = \pm x$. So, as x gets very large the equation gets closer and closer to $y = x$ and $y = -x$. As x gets very negative the basic hyperbola gets closer and closer to $y = x$ and $y = -x$.

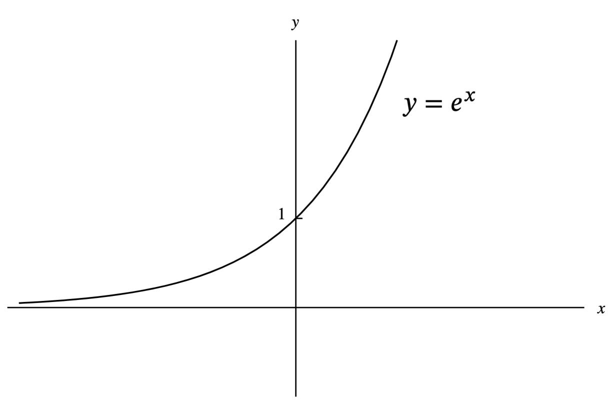
Asymptotes:

$y = x$	is an asymptote of the basic hyperbola as $x \to +\infty$
$y = -x$	is an asymptote of the basic hyperbola as $x \to +\infty$
$y = x$	is an asymptote of the basic hyperbola as $x \to -\infty$
$y = -x$	is an asymptote of the basic hyperbola as $x \to -\infty$.

□

Example 3.5. Graph $\{(x, y) \in \mathbb{R}^2 \mid y = e^x\}$ and $\{(x, y) \in \mathbb{R}^2 \mid y = \log(x)\}$.

Proof. The graph of solutions of $y = e^x$ is obtained by plotting the points $(-2, e^{-2})$, $(-1, e^{-1})$, $(0, 1)$, $(1, e)$, $(2, e^2)$ and connecting these points with a smooth continuous curve. As $x \to -\infty$ the value of e^x is positive and gets closer and closer to 0.

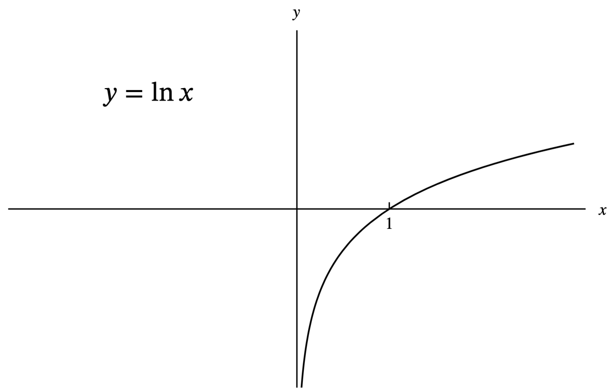


Real solutions of $y = e^x$

The functions e^x and $\log(x)$ are inverse functions since

$$e^{\log(x)} = x \qquad \text{and} \qquad \log(e^x) = x.$$

Since e^x and $\log(x)$ are inverse functions then the graph of solutions of $y = \log(x)$ is the graph of solutions of $y = e^x$ except flipped about the line $y = x$.

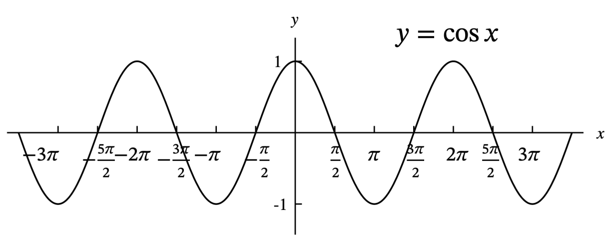


Real solutions of $y = log(x)$

□

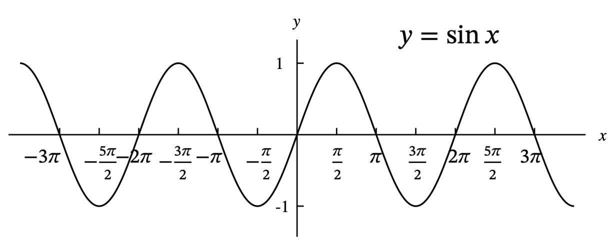
Example 3.6. Graph $\{(x, y) \in \mathbb{R}^2 \mid y = \cos(x)\}$ and $\{(x, y) \in \mathbb{R}^2 \mid y = \sin(x)\}$.

Proof. The value of $\cos(x)$ is the x-coordinate of the point at angle x on a circle of radius one. If x starts at 0 and increases then $\cos(x)$ starts at 1 and oscillates between 1 and -1, returning to 1 each time x reaches a multiple of 2π completing a revolution around the circle.



Real solutions of $y = \cos(x)$

Since $\sin(x) = \cos(x + \frac{\pi}{2})$ the graph of solutions of $y = \sin(x)$ is the same as the graph of solutions of $y = \cos(x)$ except which the x-axis shifted by $\frac{\pi}{2}$.



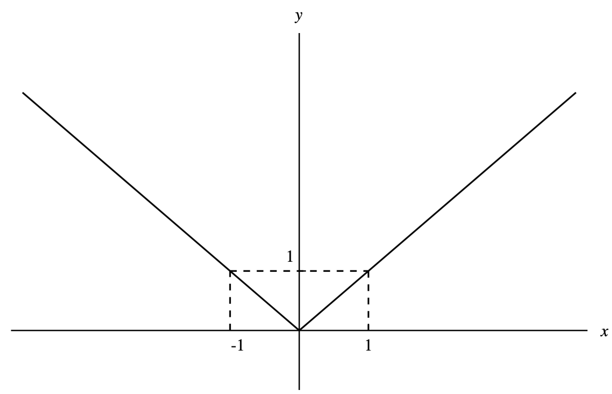
Real solutions of $y = \sin(x)$

□

Example 3.7. Graph $\{(x, f(x)) \in \mathbb{R}^2\}$ where $f\colon \mathbb{R} \to \mathbb{R}$ is given by

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0, \\ -x, & \text{if } x \leq 0. \end{cases}$$

Proof. Use the line $y = x$ for $x \in \mathbb{R}_{\geq 0}$ and the line $y = -x$ for $x \in \mathbb{R}_{\leq 0}$ to obtain the graph of solutions of $y = |x|$.

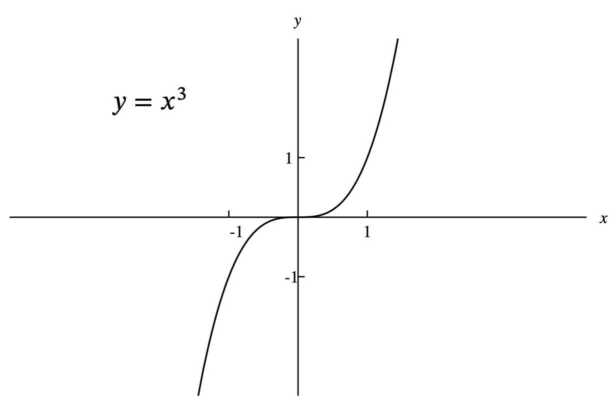


Real solutions of $y = |x|$

□

Example 3.8. Graph $\{(x, y) \in \mathbb{R}^2 \mid y = x^3\}$ and $\{(x, y) \in \mathbb{R}^2 \mid y = x^{1/3}\}$.

Proof. Determine the graph of solutions of $y = x^3$ by connecting the points $(-2, -8)$, $(-1, -1)$, $(0, 0)$, $(1, 1)$, $(2, 8)$ with a smooth continuous curve. As $x \to \infty$ the value of x^3 gets very large positive and as $x \to -\infty$ the value of x^3 is very large and negative.

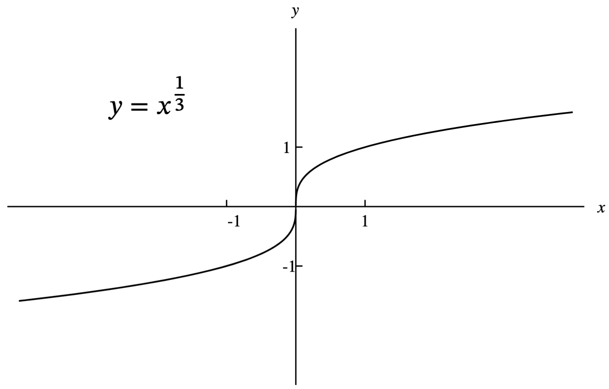


Real solutions of $y = x^3$

The functions x^3 and $x^{\frac{1}{3}}$ are inverse 'function's since

$$(x^{\frac{1}{3}})^3 = x \qquad \text{and} \qquad (x^3)^{\frac{1}{3}} = x.$$

Since x^3 and $x^{\frac{1}{3}}$ are inverse 'function's then the graph of solutions of $y = x^{\frac{1}{3}}$ is the graph of solutions of $y = x^3$ except flipped about the line $y = x$.

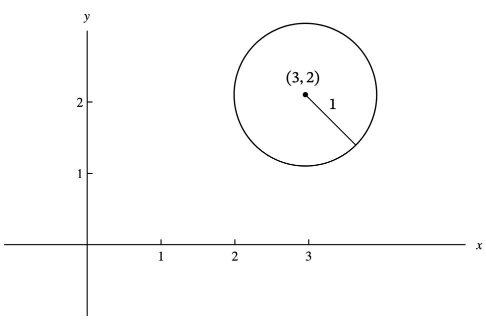


Real solutions of $y = x^{\frac{1}{3}}$

□

Example 3.9. Graph $\{(x, y) \in \mathbb{R}^2 \mid (x-3)^2 + (y-2)^2 = 1\}$.

Proof.



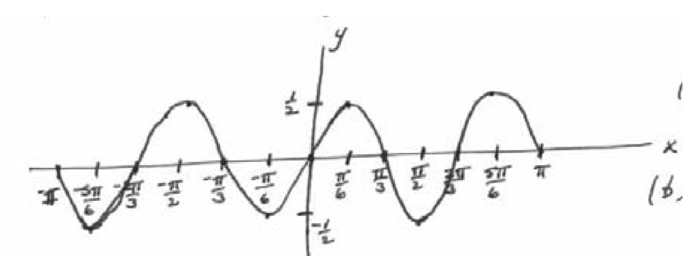
A circle of radius 1 and center $(3, 2)$

To graph solutions of $(x-3)^2 + (y-2)^2 = 1$:
(a) $x^2 + y^2 = 1$ is a basic circle of radius 1.
(b) The center is shifted by
3 to the right in the x-direction,
2 upwards in the y-diection.

□

Example 3.10. Graph $\{(x, y) \in \mathbb{R}^2 \mid 2y = \sin 3x\}$.

Proof.



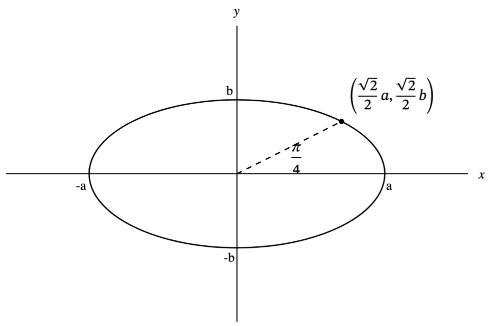
Solutions of $2y = \sin(3x)$ in $\mathbb{R}^2$

To graph solutions of $2y = \sin(3x)$:
(a) $y = \sin x$ is the basic graph.
(b) The x-axis is scaled (squished) by 3.
(c) The y-axis is scaled by 2.

□

Example 3.11. Let $a, b \in \mathbb{R}_{>0}$. Graph $\{(x, y) \in \mathbb{R}^2 \mid \frac{1}{a^2}x^2 + \frac{1}{b^2}y^2 = 1$.

Proof.



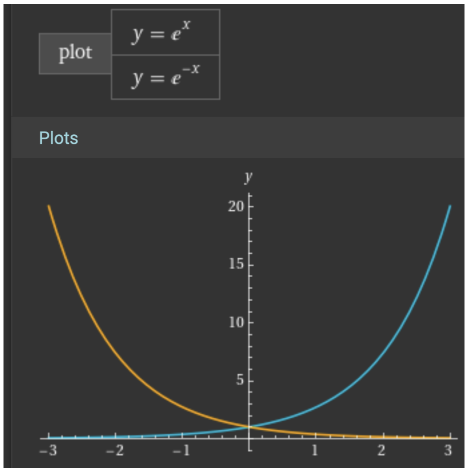
An ellipse with width $2a$ and hieght $2b$

To graph solutions of $\frac{1}{a^2}x^2 + \frac{1}{b^2}y^2 = 1$:
(a) $x^2 + y^2 = 1$ is a basic circle of radius 1.
(b) The x-axis is scaleld by a.
(c) The y-axis is scaleld by b.

□

Example 3.12. Graph $\{(x, y) \in \mathbb{R}^2 \mid y = e^{-x}\}$.

Proof. The solutions of $y = e^{-x}$ in $\mathbb{R}^2$ are the solutions of $y = e^x$ flipped about the line $y = 0$.

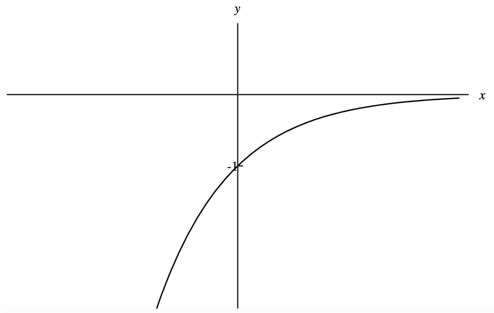


the solutions of $y = e^x$ and $y = e^{-x}$ from Wolfram alpha

□

Example 3.13. Graph $\{(x, y) \in \mathbb{R}^2 \mid y = -e^{-x}\}$.

Proof.



Solutions of $y = -e^{-x}$

To graph solutions of $y = -e^{-x}$:
(a) $y = e^x$ is the basic graph.
(b) $y = -e^{-x}$ is the same as $-y = e^{-x}$.
(c) The x-axis is flipped (around $x = 0$).
(d) The y-axis is flipped (around $y = 0$).

□

Example 3.14. Graph $\{(x, y) \in \mathbb{R}^2 \mid y = \sinh(x)\}$ and $\{(x, y \in \mathbb{R}^2 \mid y = \cosh(x)\}$.

Proof. Since

$$\cosh(x) = \tfrac{1}{2}(e^x + e^{-x}) \qquad \text{and} \qquad \sinh(x) = \tfrac{1}{2}(e^x e^{-x})$$

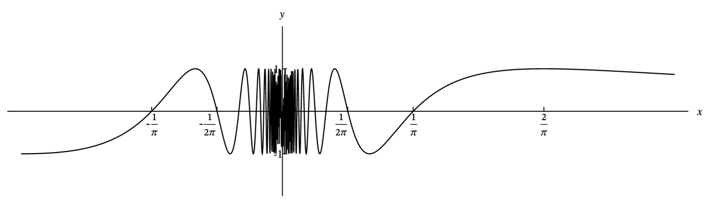
then the graph of $\cosh x$ is halfway between the graph of e^x and the graph of e^{-x} and The graph of $\sinh x$ is halfway between the graph of e^x and the graph of $-e^{-x}$.

PICTURES

□

Example 3.15. Graph $\{(x, y) \in \mathbb{R}^2 \mid y = \sin\left(\frac{1}{x}\right)$.

Proof.



The graph of $y = \sin\left(\frac{1}{x}\right)$

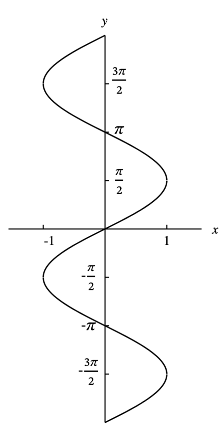
To graph $y = \sin\left(\frac{1}{x}\right)$:

(a) $y = \sin x$ is the basic graph.
(b) The positive x axis is flipped (around $x = 1$).
(c) The negative x axis is flipped (around $x = -1$).
(d) As $x \to \infty$ then $\sin\left(\frac{1}{x}\right)$ is positive and gets closer and closer to 0.
(e) As $x \to -\infty$ then $\sin\left(\frac{1}{x}\right)$ is negative and gets closer and closer to 0.
(f) As $x \to 0$ and is positive then $\sin\left(\frac{1}{x}\right)$ oscillates between $+1$ and -1.

□

Example 3.16. Graph $\{(x, y) \in \mathbb{R}^2 \mid y = \arcsin(x)\}$.

Proof.



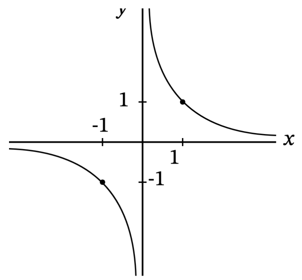
Solutions of $y = \arcsin(x)$

To graph solutions of $y = \arcsin(x)$:
(a) The graph of solutions of $y = \sin(x)$ is the basic graph.
(b) Solutions of $y = \arcsin(x)$ is the same as solutions of $\sin(y) = x$.
So the x and y axis are switched from $y = \sin(x)$.
So flip the graph of solutions of $y = \sin(x)$ across the line $x = y$.

□

Example 3.17. Graph $\{(x,y)\in\mathbb{R}^2 \mid y=\frac{1}{x}\}$.

Proof.



Solutions of $y=\frac{1}{x}$

(a) As x gets large $\frac{1}{x}$ gets closer and closeer to 0.
(b) As x gets closer to 0 (from the positive side) then $\frac{1}{x}$ gets larger and larger.
(c) As x gets closer to 0 (from the negative side) then $\frac{1}{x}$ gets more and more negative.
(d) As x gets more and more negative $\frac{1}{x}$ gets closer and closer to 0.
(e) If $x=1$ then $y=1$.
(f) If $x=-1$ then $y=-1$.

Asymptotes:

$y=0$ (the x axis) is an asymptote to real solutions of $y=\frac{1}{x}$ as $x\to+\infty$
$y=0$ (the x axis) is an asymptote to $y=\frac{1}{x}$ as $x\to-\infty$
$x=0$ (the y axis) is an asymptote to solutions of $y=\frac{1}{x}$ as $x\to 0^+$
$x=0$ (the y axis) is an asymptote to real solutions of $y=\frac{1}{x}$ as $x\to 0^-$.

□

Example 3.18. Graph $\{(x,g(x))\in\mathbb{R}^2\}$ and $\{(x,h(x))\in\mathbb{R}^2\}$ where

$$g\colon \mathbb{R}\to\mathbb{R} \text{ is given by } g(x)=2x \qquad \text{and} \qquad h\colon \mathbb{R}_{\neq 1}\to\mathbb{R} \text{ is given by } h(x)=2x.$$

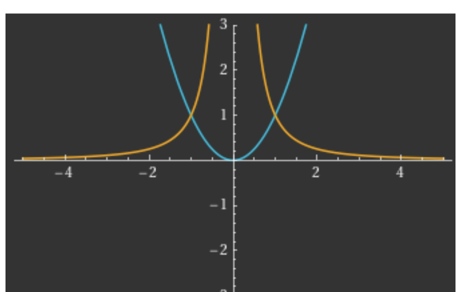
Proof. Removing the point $x=1$ from the source of the function $g\colon\mathbb{R}\to\mathbb{R}$ given by $g(x)=2x$ gives the function $h\colon\mathbb{R}_{\neq 1}\to\mathbb{R}$ given by $h(x)=2x$.

PICTURE	*PICTURE*
the graph of $g\colon\mathbb{R}\to\mathbb{R}$ given by $g(x)=2x$	the graph of $h\colon\mathbb{R}_{\neq 1}\to\mathbb{R}$ given by $h(x)=2x$

□

Example 3.19. Graph $\{(x,y)\in\mathbb{R}^2 \mid y=x^2\}$ and $\{(x,y)\in\mathbb{R}^2 \mid y=\frac{1}{x^2}\}$.

Proof. If $a\in\mathbb{R}_{\geq 1}$ then $\frac{1}{a}\in\mathbb{R}_{(0,1]}$. The graph of $\frac{1}{x^2}$ is the same as the graph of x^2 with the region $\mathbb{R}_{\geq 1}$ flipped with the region $\mathbb{R}_{(0,1]}$ on the y-axis.



the graph of $\frac{1}{x^2}$ and the graph of x^2 with
the region $\mathbb{R}_{\geq 1}$ flipped with the region $\mathbb{R}_{(0,1]}$ on the y-axis
screenshot from Wolfram alpha
`plot x^2 and 1/x^2 with x from -5 to 5 and y from -3 to 3`

□

Example 3.20. Let $c \in \mathbb{R}$ and graph $\{(x, y) \in \mathbb{R}^2 \mid y = (x-3)^2 + c\}$.

Proof. The graph of $(x-3)^2$ is the graph of x^2 shifted 3 units to the right and the graph of $(x-3)^2+c$ is the graph of $(x-3)^2$ shifted c units up.

PICTURE
The graph of x^2, the graph of $(x-3)^2$,
and the graph of $(x-3)^2 + c$

□

3.2 Additional graphing examples

Example 3.21. Let $f\colon \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} \dfrac{1-\cos(x)}{x^2}, & \text{if } x \neq 0, \\ 1 & \text{if } x = 0. \end{cases}$$

Graph $\{(x, f(x)) \in \mathbb{R}^2\}$ and determine if $f(x)$ is continuous at $x = 0$.

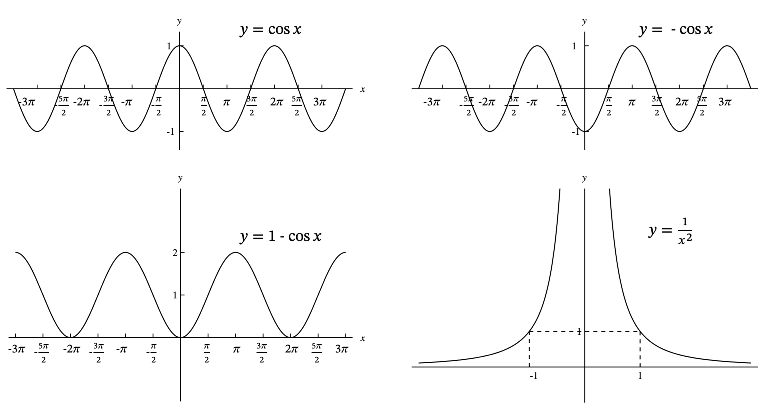
Proof. Since

$$\begin{aligned} \lim_{x\to 0} \frac{1-\cos(x)}{x^2} &= \lim_{x\to 0} \frac{1 - \left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots\right)}{x^2} \\ &= \lim_{x\to 0} \frac{\frac{1}{2!}x^2 - \frac{1}{4!}x^4 + \frac{1}{6!}x^6 - \cdots}{x^2} \\ &= \lim_{x\to 0} \left(\tfrac{1}{2!} - \tfrac{1}{4!}x^2 + \tfrac{1}{6!}x^4 - \cdots\right) \\ &= \tfrac{1}{2} - 0 + 0 - 0 + \cdots = \tfrac{1}{2} \end{aligned}$$

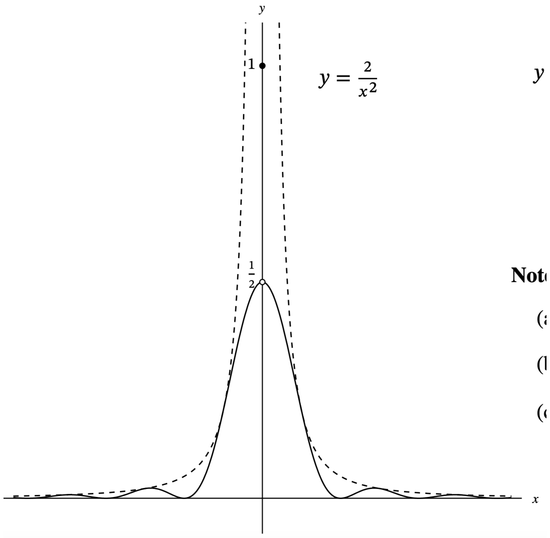
then

$$\lim_{x\to 0} f(x) = \tfrac{1}{2}. \qquad \text{Since } f(0) = 1 \text{ then} \quad \lim_{x\to 0} f(x) \neq f(0).$$

So $f(x)$ is **not** continuous at $x = 0$. Use the graphs



to build the graph of $y = f(x)$.



The graph of $y = f(x)$

To graph $y = f(x)$:
(a) As $x \to 0$ then $\frac{1-\cos(x)}{x^2} \to \frac{1}{2}$.
(b) $f(0) = 1$.
(c) At the peaks of $1 - \cos(x)$ there is an equality $\frac{1-\cos(x)}{x^2} = \frac{2}{x^2}$.
(d) The dotted curve is the graph of $y = \frac{2}{x^2}$.

□

Example 3.22. Graph $\left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; y = \dfrac{x^2-1}{x^3-4x} \right\}$.

Proof. Notes:

(a) $y = \dfrac{x^2-1}{x^3-4x} = \dfrac{(x+1)(x-1)}{x(x^2-4)} = \dfrac{(x+1x)(x-1)}{x(x+2)(x-2)}$.

(b) If $x = 1$ then $y = 0$.

(c) If $x = -1$ then $y = 0$.

(d) If $x > 2$ and x is close to 2 then y is very large and positive . $\left(\dfrac{\text{pos}\cdot\text{pos}}{\text{pos}\cdot\text{pos}\cdot\text{pos}}\right)$

(e) If $x < 2$ and x is close to 2 then y is very large and negative. $\left(\dfrac{\text{pos}\cdot\text{pos}}{\text{pos}\cdot\text{pos}\cdot\text{neg}}\right)$

(f) If $x > 0$ and x is close to 0 then y is very large and positive. $\left(\dfrac{\text{pos}\cdot\text{neg}}{\text{pos}\cdot\text{pos}\cdot\text{neg}}\right)$

(g) If $x < 0$ and x is close to 0 then y is very large and negative. $\left(\dfrac{\text{pos}\cdot\text{neg}}{\text{neg}\cdot\text{pos}\cdot\text{neg}}\right)$

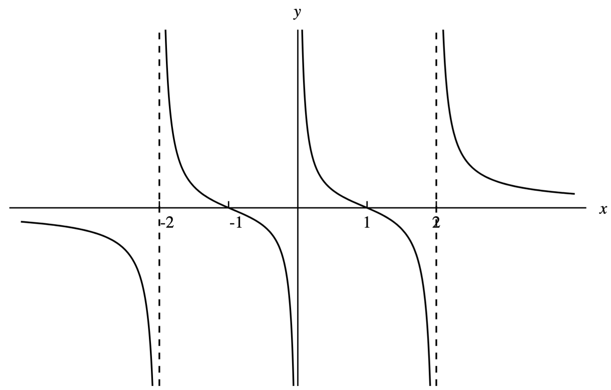
(h) If $x > -2$ and x is close to -2 then y is very large and positive. $\left(\dfrac{\text{neg}\cdot\text{neg}}{\text{neg}\cdot\text{pos}\cdot\text{neg}}\right)$

(i) If $x < -2$ and x is close to -2 then y is very large and negative. $\left(\dfrac{\text{neg}\cdot\text{neg}}{\text{neg}\cdot\text{neg}\cdot\text{neg}}\right)$

(j) $y = \dfrac{x^2-1}{x^3-4x}$ is the same as $(-y) = \dfrac{(-x)^2-1}{(-x)^3-4(-x)}$ so if y is flipped to $-y$ and x is flipped to $-x$ then the graph stays the same.

(k) As $x \to \infty$ then y is positive and gets close to 0

(k) As $x \to -\infty$ then y is negative and gets close to 0



Real solutions of $y = \dfrac{x^2-1}{x^3-4x}$

□

Example 3.23. Graph $\{(x,y) \in \mathbb{R}^2 \mid \sqrt{x} + \sqrt{y} = 1\}$.

Proof. Notes:

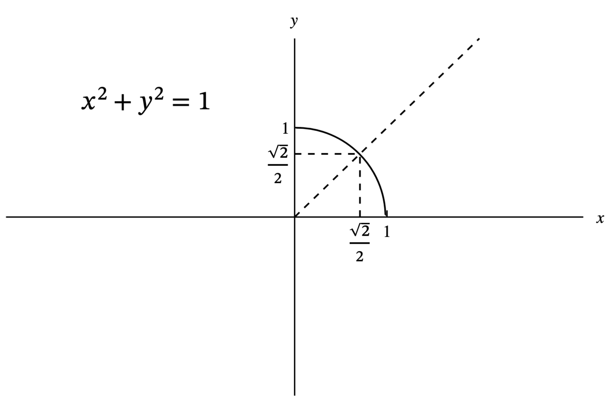
(a) If x and y are switched this graph stays the same.

(b) If $x = 0$ then $\sqrt{y} = 1$ and $y = 1^1 = 1$.

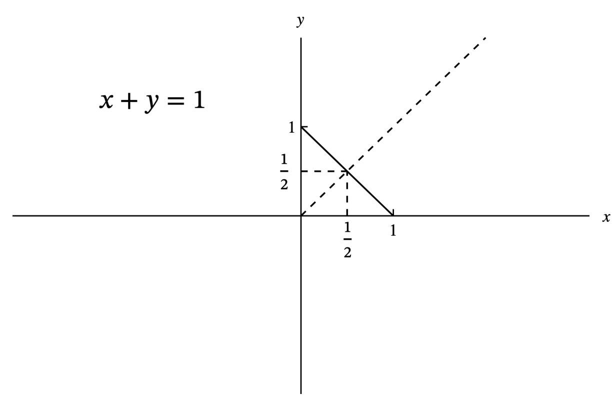
(c) If $y = 0$ then $x = 1$.

(d) If $x = y$ then $\sqrt{x} + \sqrt{x} = 1$ and $\sqrt{x} = \frac{1}{2}$ so that $x = \frac{1}{4}$.

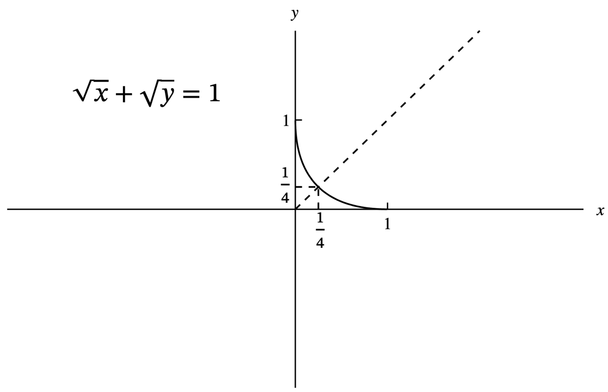
(e) This graph should be similar to $x^2 + y^2 = 1$ and $x + y = 1$.



Real solutions of $y = x^2 + y^2 = 1$



Real solutions of $y = x + y = 1$



Real solutions of $y = \sqrt{x} + \sqrt{y} = 1$

□

Example 3.24. Graph $\left\{(x,y) \in \mathbb{R}^2 \;\middle|\; y = \dfrac{x^2-1}{x^2+1}\right\}$.

Proof. Notes:

(a) $y = \dfrac{x^2-1}{x^2+1} = \dfrac{x^2+1-2}{x^2+1} = 1 - \dfrac{2}{x^2+1}$.

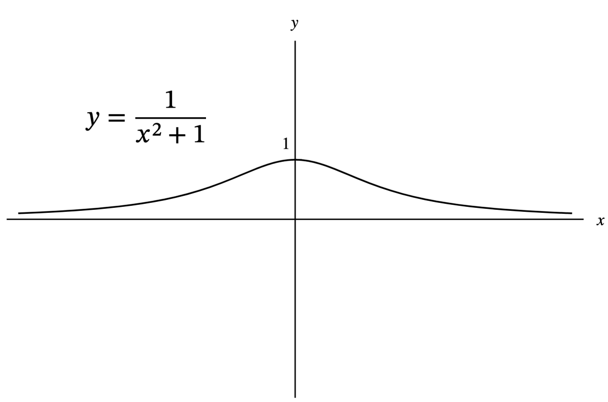
Notes for the graph of $\left\{(x,y) \in \mathbb{R}^2 \;\middle|\; y = \dfrac{1}{x^2+1}\right\}$:

(a) If $x = 0$ then $y = \frac{1}{0^2+1} = \frac{1}{1} = 1$.

(b) If $x \to \infty$ then y is positive and close to 0.

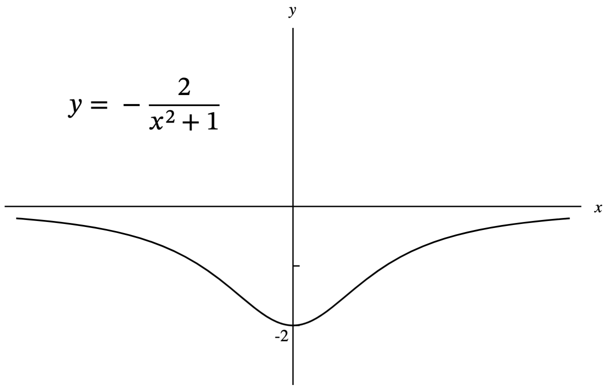
(c) If $x \to -\infty$ then y is positive and close to 0.

(d) Since $y = \frac{1}{x^2+1} = \frac{1}{(-x)^2+1}$ then the graph stays the same if x is flipped to $-x$.

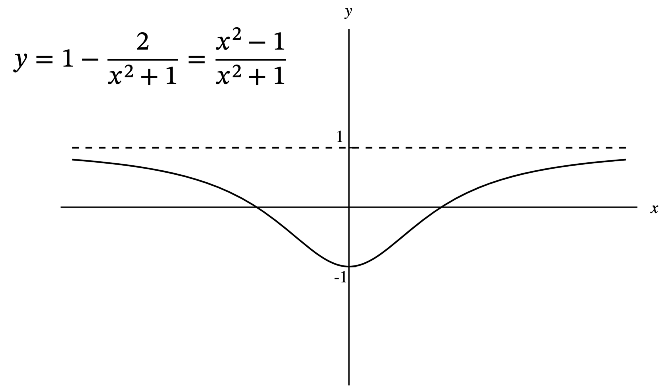


Real solutions of $y = \dfrac{1}{x^2+1}$

Then



Real solutions of $y = \dfrac{-2}{x^2+1}$



Real solutions of $y = \dfrac{x^2-1}{x^2+1}$

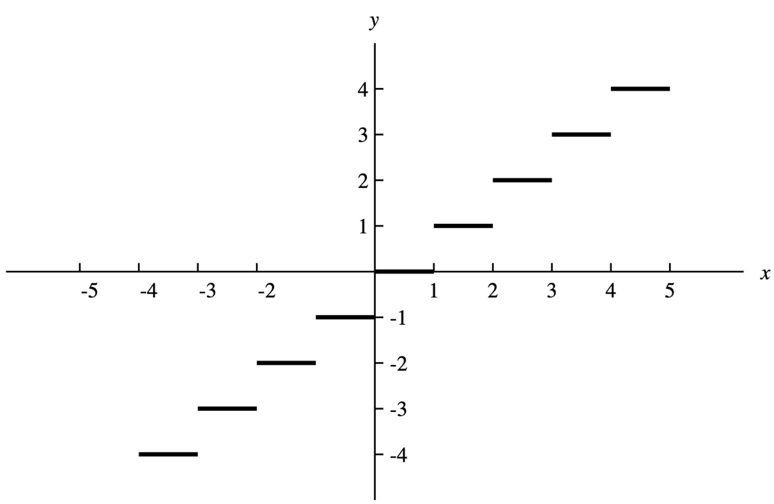
□

Example 3.25. Graph $\{(x, f(x)) \in \mathbb{R}\times\mathbb{Z}\}$ where $f\colon \mathbb{R}\to\mathbb{R}$ is $f(x) = \lfloor x \rfloor$, the *round down function*, given by

$$f(x) = (\text{maximal integer such that } n \le x).$$

For example $f(3.2) = \lfloor 3.2 \rfloor = 3$.

Proof.



The function $f(x)$ is continuous if $x \notin \mathbb{Z}$. Then

$$\lim_{x\to 1^-} \lfloor x \rfloor = 0 \qquad \text{and} \qquad \lim_{x\to 1^+} \lfloor x \rfloor = 1.$$

□

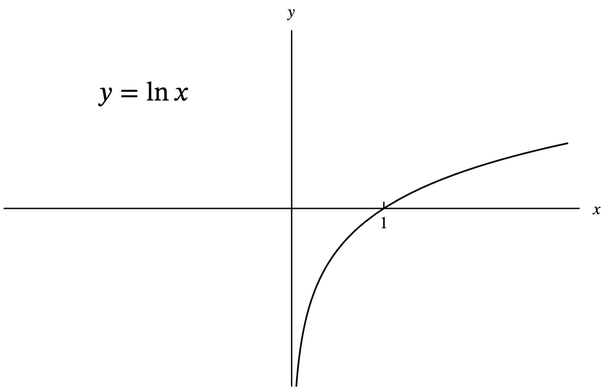
The function $g\colon \mathbb{R}\to\mathbb{R}$ denoted $g(x) = \lceil x \rceil$ is the *round up function* having $\lceil 3.2 \rceil = 4$.

Example 3.26. Graph $\{(x,y)\in\mathbb{R}^2 \mid y = \log(4-x^2)\}$.

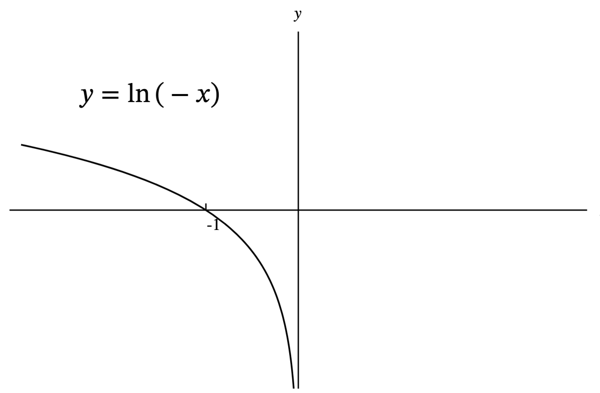
Proof. Notes:

(a) $y = \log(4 - x^2) = \log((2 + x)(2 - x)) = \log(2 + x) + log(2 - x)$.

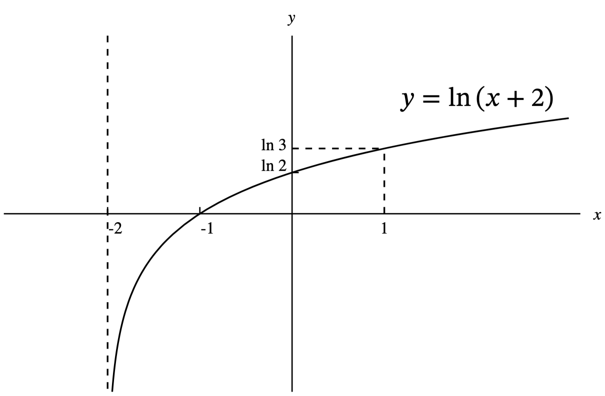
(b) Since $y = \log(4 - x^2) = \log(4 - (-x)^2)$ then the graph stays the same when x is flipped to $-x$.



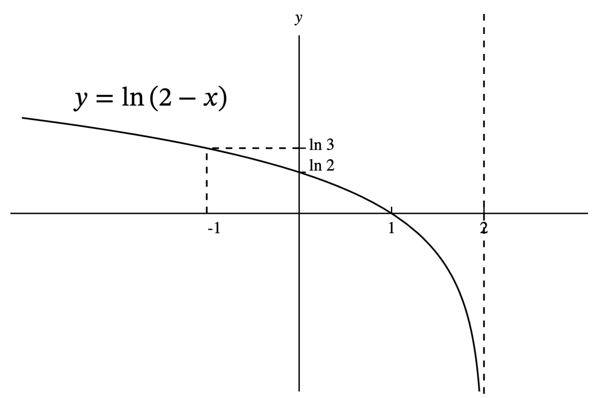
Real solutions of $y = \log(x)$



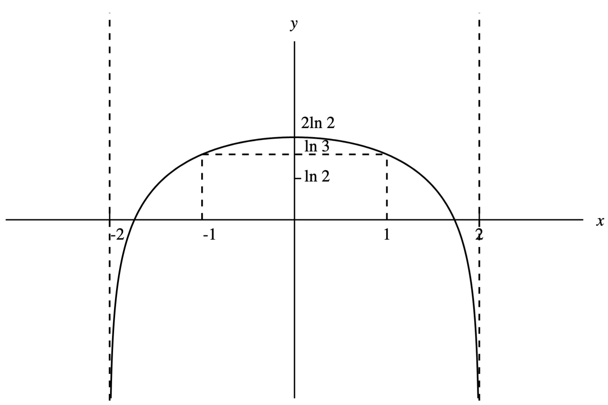
Real solutions of $y = \log(-x)$



Real solutions of $y = \log(x + 2)$



Real solutions of $y = \log(2 - x)$



Real solutions of $y = \log(4 - x^2)$

□

Example 3.27. Graph $\{(x, y) \in \mathbb{R}^2 \mid y = x^{\frac{2}{3}}(6 - x)^{\frac{1}{3}}\}$.

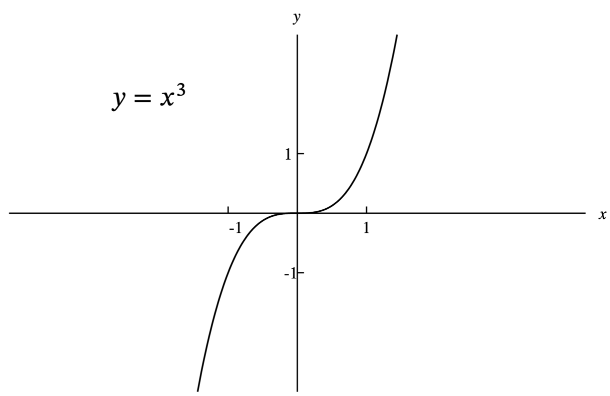
Proof. Notes:

(a) If $x = 0$ then $y = 0$.

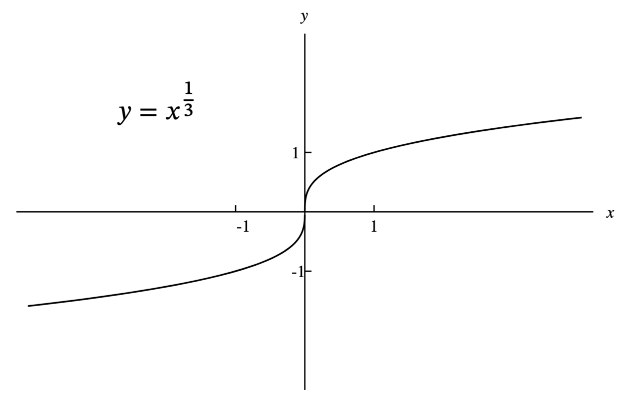
(b) If $x = 6$ then $y = 0$.

(c) If $x \to \infty$ then y gets close to $x^{\frac{2}{3}}(-x)^{\frac{1}{3}} = . - x$.

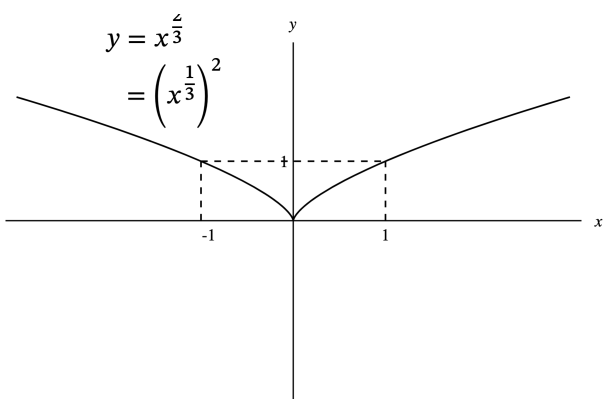
(d) If $x \to -\infty$ then $y \to \infty$ (and y gets close to $-x$ again).



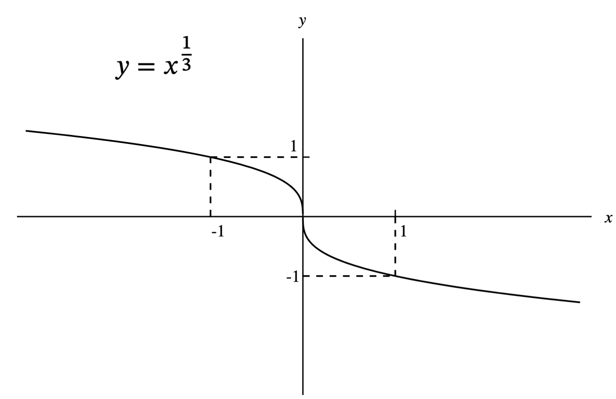
Real solutions of $y = x^3$



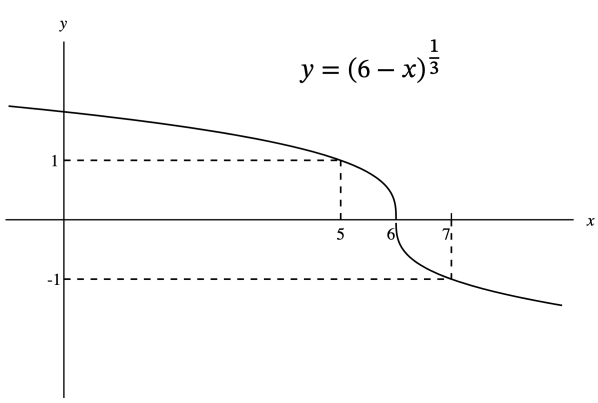
Real solutions of $y = x^{\frac{1}{3}}$



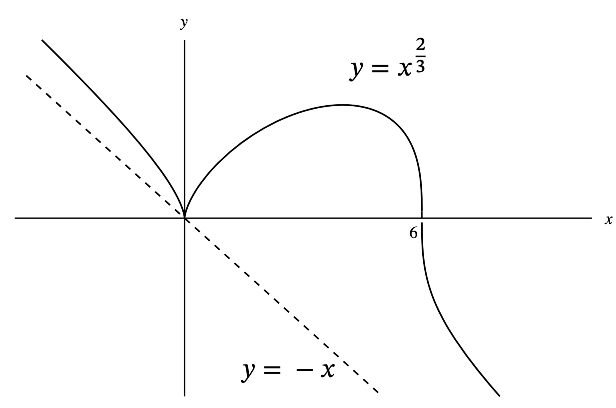
Real solutions of $y = x^{\frac{2}{3}}$



Real solutions of $y = (-x)^{\frac{1}{3}}$



Real solutions of $y = (6 - x)^{\frac{1}{3}}$



Real solutions of $y = x^{\frac{2}{3}}(6 - x)^{\frac{1}{3}}$

□

Example 3.28. Graph $\{(x, f(x)) \in \mathbb{R}^2\}$ where $f\colon \mathbb{R} \to \mathbb{R}$ is given by $f(x) = 3x^2 - 2x - 1$.

Proof. Notes:

(a) The x^2 indicates this is a parabola.

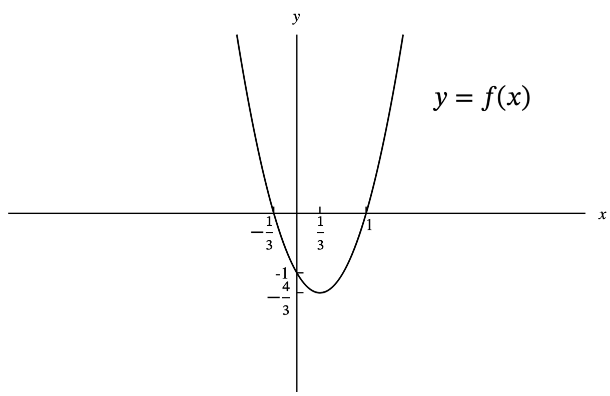
(b) Since the coefficient of x^2 is positive this is a concave up parabola.

(c) There is a factorization: $3x^2 - 2x - 1 = (x-1)(3x+1)$. We know $x - 1$ should be. factor since when you plug in 1 you get $3 \cdot 1^2 - 2 \cdot 1 - 1 = 0$.

(d) The value $f(x)$ is 0 if $x - 1$ or if $x = -\frac{1}{3}$.

(e) The minimum will be where $\frac{df}{dx}\big]_{x=a}$ is 0. Since $\frac{df}{dx}\big]_{x=a} = (6x-2)\big]_{x=a} - 6a - 2$. So $\frac{df}{dx}\big]_{x=a}$ is 0 when $a = \frac{1}{3}$. Then

$$f(\tfrac{1}{3}) = 3(\tfrac{1}{3})^2 - 2(\tfrac{1}{3}) - 1 = \tfrac{1}{3} - \tfrac{2}{3} - 1 = -\tfrac{4}{3}.$$



Real solutions of $y = 3x^2 - 2x - 1$

□

Example 3.29. Graph $\{(x, f(x)) \in \mathbb{R}^2\}$ where $f \colon \mathbb{R} \to \mathbb{R}$ is given by $f(x) = 2x^3 - 21x^2 + 36x - 20$.

Proof. Notes:

(a) If $x \to \infty$ then $f(x) \to \infty$.

(b) If $x \to -\infty$ then $f(x) \to -\infty$.

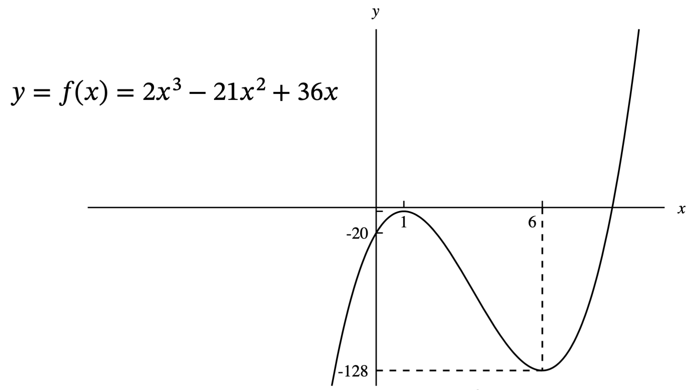
(c) Since $\frac{df}{dx} = 6x^2 - 42x + 36 = 6(x^2 - 7x + 6) = 6(x-6)(x-1)$ then $\frac{df}{dx}$ is 0 when $x = 6$ and when $x = 1$.

$$f(6) = 2 \cdot 6^3 - 21 \cdot 6^2 + 36 \cdot 6 - 20 = 6^2(12 - 21 + 6) - 20 = 6^2(-3) - 20 - 128,$$
$$f(1) = 2 - 21 + 36 - 20 = 38 - 41 = -3.$$

(d) Since

$$\frac{d^2 f}{dx^2}\Big]_{x=6} = (12x - 42)\big]_{x=6} = 72 - 42 = 30 > 0,$$
$$\frac{d^2 f}{dx^2}\Big]_{x=1} = (12x - 42)\big]_{x=1} = 12 - 42 = -30 < 0,$$

so that the graph is concave up when $x = 6$ and the graph is concave down when $x = 1$.



Real solutions of $y = 2x3 - 21x^2 + 36x - 20$

□

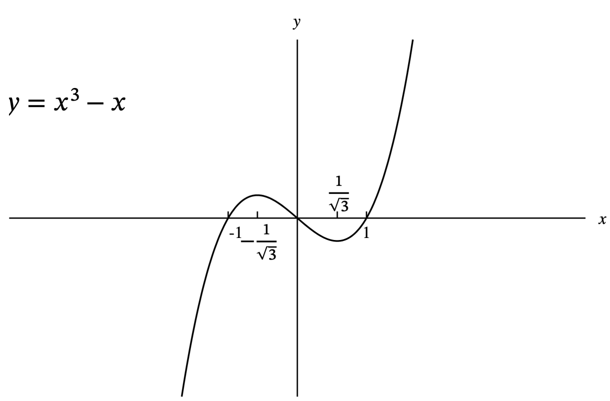
Example 3.30. Graph $\{(x, f(x)) \in \mathbb{R}^2\}$ where $f\colon \mathbb{R} \to \mathbb{R}$ is given by $f(x) = x^3 - x + 1$.

Proof. Notes:

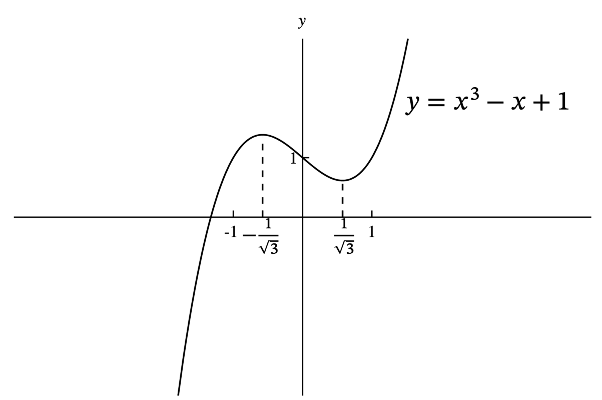
(a) This graph is the graph of solutions of $y = x^3 - x$ shifted up by 1.

(b) $x^3 - x = x(x^2 - 1) = x(x + 1)(x - 1)$.

(c) Since $\frac{d(x^3-x)}{dx} = 3x^2 - 1$ then $\frac{d(x^3-x)}{dx}\Big]_{x=a}$ is 0 when $a = \pm\frac{1}{\sqrt{3}}$.



Real solutions of $y = x^3 - x$



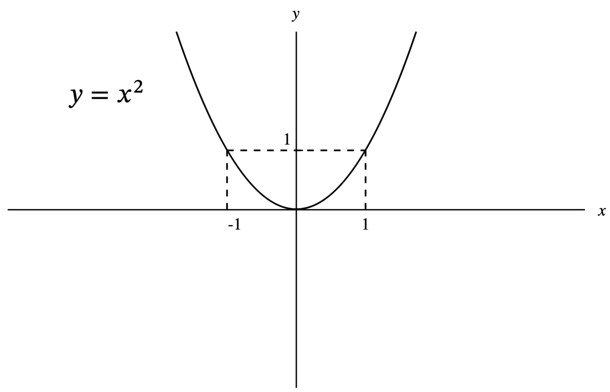
Real solutions of $y = x^3 - x + 1$

□

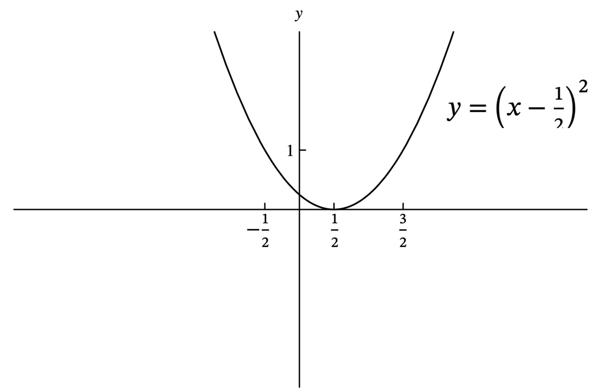
Example 3.31. Graph $\{(x, y) \in \mathbb{R}^2 \mid y = x - x^2 - 27\}$.

Proof. Notes: The $-x^2$ indicates to us that this graph is a concave down parabola.

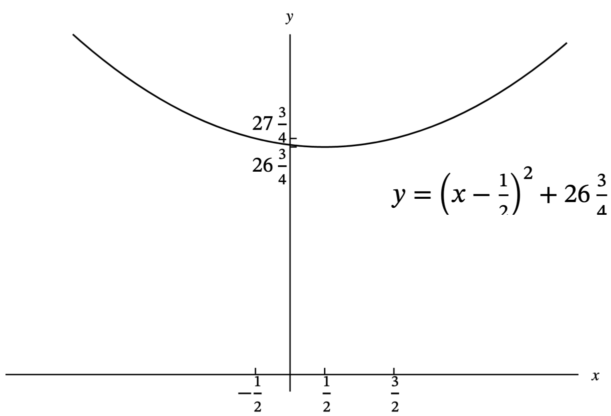
$$x - x^2 - 27 = -(x^2 - x + 27) = -(x^2 - x + \tfrac{1}{4} - \tfrac{1}{4} + 27) = -((x - \tfrac{1}{2})^2 + 26\tfrac{3}{4}).$$



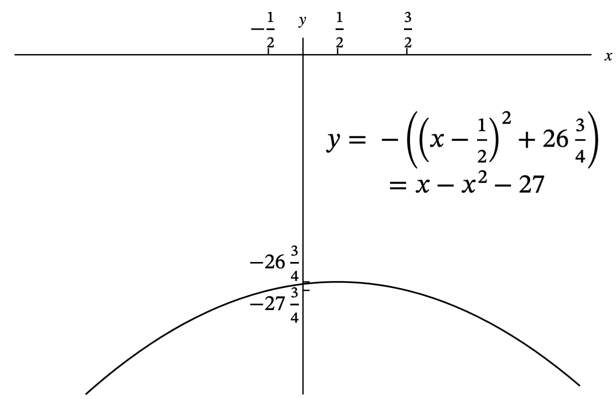
Real solutions of $y = x^2$



Real solutions of $y = (x - \frac{1}{2})^2$



Real solutions of $y = (x - \frac{1}{2})^2 + 26\frac{3}{4}$



Real solutions of $y = x - x^2 - 27$

□

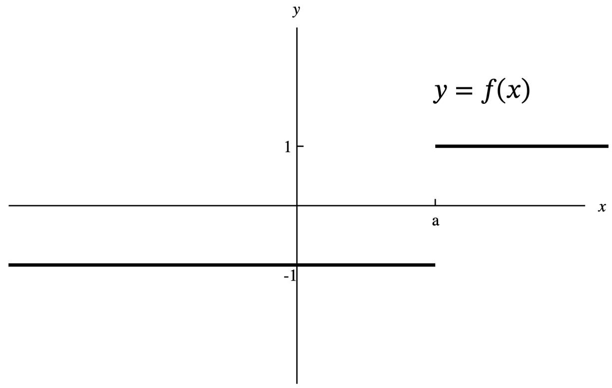
Example 3.32. Let $a \in \mathbb{R}_{>0}$. Graph $\{(x, f(x)) \in \mathbb{R}^2\}$ where $f: \mathbb{R} \to \mathbb{R}$ is given by

$$f(x) = \begin{cases} \dfrac{|x-a|}{x-a}, & \text{if } x \neq a, \\ 1, & \text{if } x = a. \end{cases}$$

For which values of x is $f(x)$ continuous?

Proof.

$$\begin{aligned} f(x) &= \begin{cases} \dfrac{|x-a|}{x-a}, & \text{if } x \neq a, \\ 1, & \text{if } x = a, \end{cases} \\ &= \begin{cases} \dfrac{x-a}{x-a}, & \text{if } x \neq a \text{ and } x-a \in \mathbb{R}_{>0}, \\ -\dfrac{(x-a)}{x-a}, & \text{if } x \neq a \text{ and } x-a \in \mathbb{R}_{<0}, \\ 1, & \text{if } x = a, \end{cases} \\ &= \begin{cases} 1, & \text{if } x \neq a \text{ and } x-a \in \mathbb{R}_{>0}, \\ -1, & \text{if } x \neq a \text{ and } x-a \in \mathbb{R}_{<0}, \\ 1, & \text{if } x = a, \end{cases} \end{aligned}$$



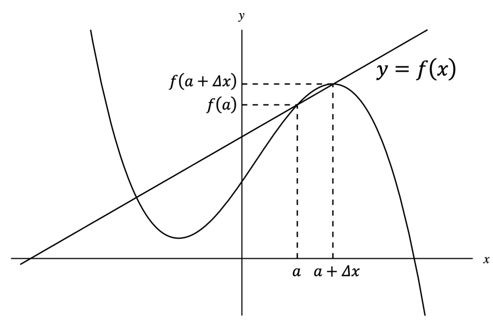
Graph of $\{(x, f(x) \in \mathbb{R}^2\}$

The graph has a jump at $x = a$. So $f(x)$ is not continuous at $x = a$. $\square$

3.3 Graphing: Slope and areas

Example 3.33. (The fundamental theorem of change) For a smooth continuous function $f\colon \mathbb{R} \to \mathbb{R}$ let

$$D_f(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$



So that $D_f(a)$ is the slope of f at $x = a$ (the rate of change of f with respect to x at $x = a$).

Let c be a constant and let f and g be functions and assume that D_f and D_g exist. Show that

(a) $D_x = 1$,

(b) $D_{cf} = cD_f$,

(c) $D_{f+g} = D_f + D_g$,

(d) $D_{fg} = D_f \cdot g + f \cdot D_g$.

Proof.

$$D_x(x) = \lim_{h\to 0} \frac{(x+h) - x}{h} = \lim_{h\to 0} \frac{h}{h} = \lim_{h\to 0} 1 = 1.$$

$$D_{cf}(x) = \lim_{h\to 0} \frac{(cf)(x+h) - (cf)(x)}{h} = \lim_{h\to 0} \frac{c \cdot f(x+h) - c \cdot f(x)}{h} = \lim_{h\to 0} c \cdot \Big(\frac{f(x+h) - f(x)}{h}\Big)$$
$$= c \cdot \Big(\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}\Big) = cD_f(x) \qquad \text{(by continuity of scalar multiplication).}$$

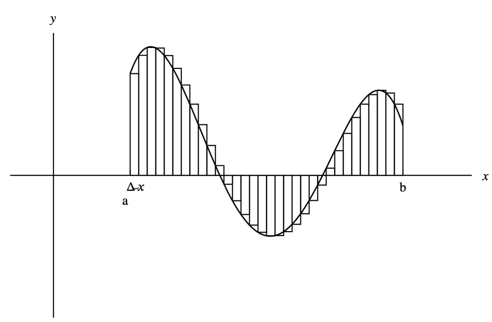
$$\begin{aligned}
D_{f+g}(x) &= \lim_{h\to 0}\frac{(f+g)(x+h)-(f+g)(x)}{h} = \lim_{h\to 0}\frac{f(x+h)+g(x+h)-f(x)+g(x)}{h} \\
&= \lim_{h\to 0}\Big(\frac{f(x+h)-f(x)}{h}+\frac{g(x+h)-g(x)}{h}\Big) \\
&= \Big(\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\Big)+\Big(\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\Big) \qquad \text{(by continuity of addition)} \\
&= D_f(x)+D_g(x).
\end{aligned}$$

$$\begin{aligned}
D_{fg}(x) &= \lim_{h\to 0}\frac{(fg)(x+h)-(fg)(x)}{h} = \lim_{h\to 0}\frac{(f(x+h)\cdot g(x+h)-f(x)\cdot g(x)}{h} \\
&= \lim_{h\to 0}\frac{(f(x+h)-f(x))(g(x+h)-g(x)+f(x+h)g(x)+f(x)g(x+h)-2f(x)g(x)}{h} \\
&= \lim_{h\to 0}\frac{(f(x+h)-f(x))(g(x+h)-g(x))+(f(x+h)-f(x))g(x)+f(x)(g(x+h)-g(x))}{h} \\
&= \lim_{h\to 0}\Big(h\frac{(f(x+h)-f(x))}{h}\frac{(g(x+h)-g(x))}{h}+\frac{f(x+h)-f(x)}{h}g(x)+f(x)\frac{g(x+h)-g(x)}{h}\Big) \\
&= \Big(\lim_{h\to 0}h\Big)\Big(\lim_{h\to 0}\frac{(f(x+h)-f(x))}{h}\Big)\Big(\lim_{h\to 0}\frac{(g(x+h)-g(x))}{h}\Big) \\
&\quad +\Big(\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}g(x)\Big)+\Big(\lim_{h\to 0}f(x)\frac{g(x+h)-g(x)}{h}\Big) \qquad \begin{pmatrix}\text{by continuity of}\\ \text{addition and}\\ \text{multiplication}\end{pmatrix} \\
&= 0\cdot D_f(x)D_g(x)+D_f(x)g(x)+f(x)D_g(x) \\
&= D_f(x)g(x)+f(x)D_g(x).
\end{aligned}$$

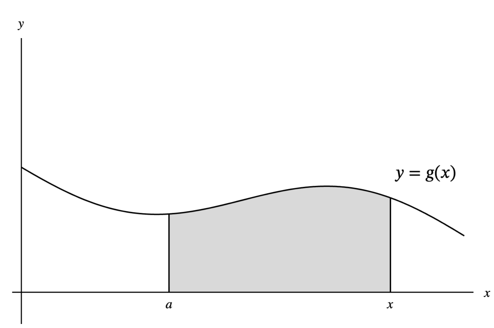
□

Example 3.34. (The fundamental theorem of measure) For $a, b\in\mathbb{R}$ with $p<a<b$ and a smooth continuous function $f\colon\mathbb{R}\to\mathbb{R}$ let

$$\int_a^b f\,dx = \lim_{h\to 0}\Big(\sum_{j=0}^{\lfloor\frac{b-a}{h}\rfloor} f(a+jh)h\Big)$$



If $f\colon\mathbb{R}\to\mathbb{R}_{\geq 0}$ then $\int_a^b f\,dx$ is the area under f between $x=a$ and $x=b$.



Let $p \in \mathbb{R}$ with $p < a < b$ and let $A\colon \mathbb{R}_{[p,b]} \to \mathbb{R}_{\geq 0}$ be the function given by

$$A(x) = \int_p^x f\, dx.$$

Prove that

$$\lim_{h\to 0} \frac{A(x+h) - A(x)}{h} = f(x) \qquad \text{and} \qquad A(b) - A(a) = \int_a^b f\, dx.$$

Proof.

$$\lim_{h\to 0} \frac{A(x+h)-A(x)}{h} = \lim_{h\to 0} \frac{\text{(area of last little box)}}{h} = \lim_{h\to 0} \frac{f(x)h}{h} = \lim_{h\to 0} f(x) = f(x)$$

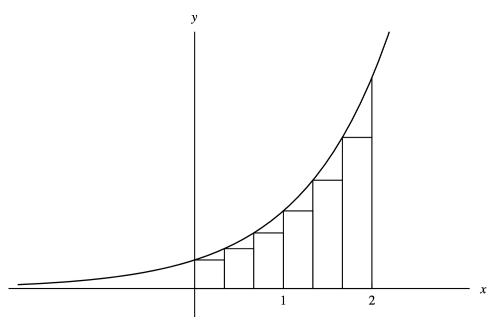
and

$$\begin{aligned} A(b) - A(a) &= (\text{area under } f(x) \text{ from } p \text{ to } b) - (\text{area under } f(x) \text{ from } p \text{ to } a) \\ &= (\text{area under } f(x) \text{ from } a \text{ to } b) \\ &= \int_a^b f\, dx. \end{aligned}$$

□

Example 3.35. Compute the limit $\int_0^2 e^x\, dx$ (without using the fundamental theorem of measure).

Proof.



Suppose $h = \frac{1}{3}$. Then

$$\begin{aligned} &e^0h + e^hh + e^{2h}h + e^{3h}h + e^{4h}h + \cdots + e^{2-h}h \\ &\quad = e^0\tfrac{1}{3} + e^{\frac{1}{3}}\tfrac{1}{3} + e^{\frac{2}{3}}\tfrac{1}{3} + e^{\frac{3}{3}}\tfrac{1}{3} + e^{\frac{4}{3}}\tfrac{1}{3} + e^{\frac{5}{3}}\tfrac{1}{3} \\ &\quad = \tfrac{1}{3}\big(1 + e^{\frac{1}{3}} + (e^{\frac{1}{3}})\big)^2 + \big(e^{\frac{1}{3}})\big)^3 + \big(e^{\frac{1}{3}})\big)^4 + \big(e^{\frac{1}{3}})\big)^5\big) \\ &\quad = \tfrac{1}{3}\Big(\frac{\big(e^{\frac{1}{3}})\big)^6 - 1}{e^{\frac{1}{3}} - 1}\Big) = \tfrac{1}{3}\Big(\frac{e^{\frac{6}{3}}) - 1}{e^{\frac{1}{3}} - 1}\Big) = (e^2 - 1)\Big(\frac{\frac{1}{3}}{e^{\frac{1}{3}} - 1}\Big). \end{aligned}$$

Suppose $h = \frac{1}{5}$. Then

$$\begin{aligned} &e^0h + e^hh + e^{2h}h + e^{3h}h + e^{4h}h + \cdots + e^{2-h}h \\ &\quad = e^0\tfrac{1}{5} + e^{\frac{1}{5}}\tfrac{1}{5} + e^{\frac{2}{5}}\tfrac{1}{5} + e^{\frac{3}{5}}\tfrac{1}{5} + \cdots + e^{\frac{9}{5}}\tfrac{1}{5} \\ &\quad = \tfrac{1}{5}\big(1 + e^{\frac{1}{5}} + (e^{\frac{1}{5}})\big)^2 + \big(e^{\frac{1}{5}})\big)^3 + \cdots + \big(e^{\frac{1}{3}})\big)^9\big) \\ &\quad = \tfrac{1}{5}\Big(\frac{\big(e^{\frac{1}{5}})\big)^{10} - 1}{e^{\frac{1}{5}} - 1}\Big) = \tfrac{1}{5}\Big(\frac{e^{\frac{10}{5}}) - 1}{e^{\frac{1}{5}} - 1}\Big) = (e^2 - 1)\Big(\frac{\frac{1}{5}}{e^{\frac{1}{5}} - 1}\Big). \end{aligned}$$

Suppose $h = \frac{1}{N}$. Then

$$\begin{aligned} &e^0 h + e^h h + e^{2h} h + e^{3h} h + e^{4h} h + \cdots + e^{2-h} h \\ &\quad = e^0 \tfrac{1}{N} + e^{\frac{1}{N}} \tfrac{1}{N} + e^{\frac{2}{N}} \tfrac{1}{N} + e^{\frac{3}{N}} \tfrac{1}{N} + \cdots + e^{2-\frac{1}{N}} \tfrac{1}{N} \\ &\quad = (e^2 - 1)\Big(\frac{\frac{1}{N}}{e^{\frac{1}{N}} - 1} \Big). \end{aligned}$$

So

$$\lim_{h\to 0} (e^0 h + e^h h + e^{2h} h + e^{3h} h + e^{4h} h + \cdots + e^{2-h} h) \quad = \lim_{h\to 0} (e^2 - 1)\Big(\frac{h}{e^h - 1} \Big) = (e^2 - 1)\cdot 1 = e^2 - 1.$$

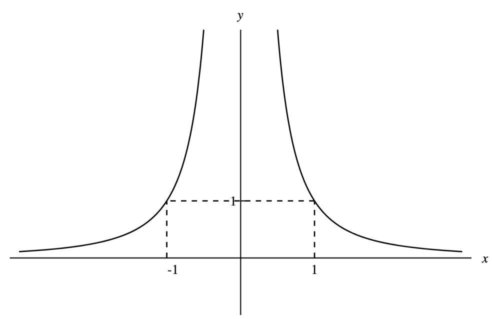
Note: If c is a constant then

$$(e^x + c)\big]_{x=0}^{x=2} = (e^2 + c) - (e^0 + c) = e^2 - 1 = \int_0^2 e^x\,dx.$$

□

Example 3.36. Compute the limit $\displaystyle\int_{-1}^{1} \frac{1}{x^2}\,dx$ (without using the fundamental theorem of measure).

Proof.



By adding up areas of little boxes:

$$\begin{aligned} \int_{-1}^{1} \frac{1}{x^2}\,dx &= \lim_{h\to 0} \Big(\frac{1}{(-1)^2} h + \frac{1}{(-1+h)^2} h + \frac{1}{(-1+2h)^2} h + \cdots + \frac{1}{(1-h)^2} h \Big) \\ &= \lim_{h\to 0} \left(\begin{matrix} \tfrac{1}{(-1)^2} h + \tfrac{1}{(-1+h)^2} h + \tfrac{1}{(-1+2h)^2} h + \cdots & + \tfrac{1}{0^2} h + & \cdots + \tfrac{1}{(1-h)^2} h \\ & \text{OOPS!!} & \end{matrix} \right), \end{aligned}$$

So $\int_{-1}^{1} \frac{1}{x^2}\,dx$ does NOT EXIST in $\mathbb{R}$.

Note: $\displaystyle\int \frac{1}{x^2}\,dx = \int x^{-2}\,dx = -x^{-1} + c$, where c si a constant and

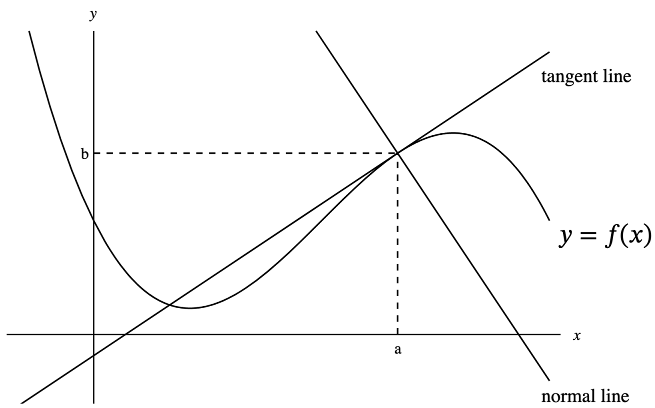
$$(-x^{-1} + c)\big]_{x=-1}^{x=1} = (-1^{-1} + c) - (-1(-1)^{-1} + c) = -1 + c - 1 - c = -2.$$

So this is an example when $\displaystyle\int_a^b \frac{df}{dx}\,dx \neq f(b) - f(a)$. □

3.4 Graphing: Tangent and normal lines

The *tangent line* to a curve $f(x)$ at the point (a,b) is the line through (a,b) with the same slope as $f(x)$ at the point (a,b).

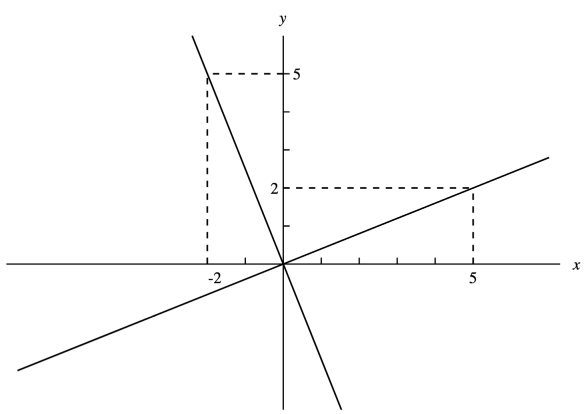
The *normal line* is the line through (a,b) which is perpendicular to the tangent line.



The slope of the tangent line at the point (a,b) is

$$\frac{df}{dx}\Big]_{x=a}.$$

If a line has slope $\frac{2}{5}$



then the perpendicular line has slope $\frac{5}{-2}$.

Example 3.37. Find the equations of the tangent and normal to the curve $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at the point where $x = 1$.

Proof. The slope of the tangent line at $x = 1$ is

$$\frac{df}{dx}\Big]_{x=1} = (4x^3 - 18x_2 + 26x - 10)\Big]_{x=1} = 4 = 18 + 26 - 10 = 2.$$

The tangent line goes through the point

$$x = 1,$$
$$y = 1 - 6 + 13 - 10 + 5 = 3.$$

The equation of a line is $y = mx + b$, where m is the slope. So, for our line

$$m = 2 \qquad \text{and} \qquad 3 = m \cdot 1 + b = 2 \cdot 1 + b.$$

So $b = 1$. So the tangent line is

$$y = 2x + 1.$$

The slope of the normal line is $\frac{1}{-2} = -\frac{1}{2}$. The equation of the normal line is $y = mx + b$ with $m = -\frac{1}{2}$ and $3 = m \cdot 1 + b = -\frac{1}{2} + b$. So

$$b = \tfrac{7}{2} \qquad \text{and} \qquad y = -\tfrac{1}{2}x + \tfrac{7}{2} \quad \text{is the normal line.}$$

□

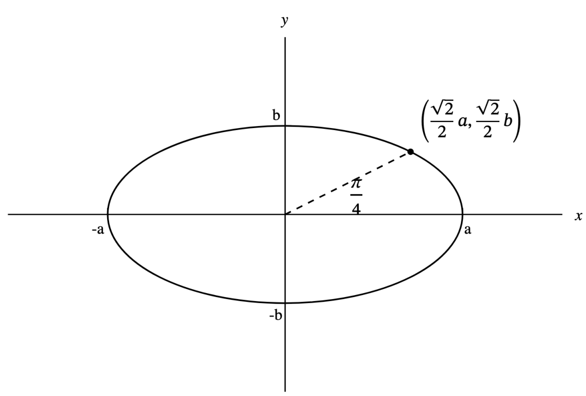
Example 3.38. Find the equation of the tangent and normal lines to the curve

$$x = a\cos(\theta), \qquad y = b\sin(\theta), \quad \text{at} \quad \theta = \frac{\pi}{4}.$$

Proof. First graph solutions of the equations. Let

$$\frac{x}{a} = \cos(\theta), \qquad \frac{y}{b} = \sin(\theta).$$

Then graph solutions of $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$.



When $\theta = \frac{\pi}{4}$,

$$x = a\cos(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}a,$$
$$y = b\sin(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}b.$$

The slope of the tangent line is

$$\frac{dy}{dx}\Big]_{\substack{x=\frac{\sqrt{2}}{2}a \\ y=\frac{\sqrt{2}}{2}b}} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}\Big]_{\theta=\frac{\pi}{4}} = \frac{\frac{db\sin(\theta)}{d\theta}}{\frac{da\cos(\theta)}{d\theta}}\Big]_{\theta=\frac{\pi}{4}} = \frac{b\cos(\theta)}{-a\sin(\theta)}\Big]_{\theta=\frac{\pi}{4}} = \frac{b\frac{\sqrt{2}}{2}}{-a\frac{\sqrt{2}}{2}} = -\frac{b}{a}.$$

So the equation of the tangent line is $y = mx + y_0$ with

$$m = -\frac{b}{a} \quad \text{and} \quad \frac{\sqrt{2}}{2}b = m\frac{\sqrt{2}}{2}a + y_0 = -\frac{b}{a}\frac{\sqrt{2}}{2}a + y_0.$$

So $y_0 = \frac{\sqrt{2}}{2}b + \frac{\sqrt{2}}{2}b = \sqrt{2}b$. So the equation of the tangent line is

$$y = -\frac{b}{a}x + \sqrt{2}b.$$

The equation of the normal line is $y = mx + y_0$ with

$$m = \frac{a}{b} \quad \text{and} \quad \frac{\sqrt{2}}{2}b = m\frac{\sqrt{2}}{2}a + y_0 = \frac{a}{b}\frac{\sqrt{2}}{2}a + y_0.$$

So

$$y_0 = \frac{\sqrt{2}}{2}b - \frac{\sqrt{2}}{2}\frac{a^2}{b} = \frac{\sqrt{2}}{2}\Big(\frac{b^2-a^2}{b}\Big).$$

So the equation of the normal line is

$$y = \frac{a}{b}x + \frac{\sqrt{2}}{2}\Big(\frac{b^2-a^2}{b}\Big).$$

□

Example 3.39. Find the equations of the normal line to $2x^2 - y^2 = 14$, parallel to the line $x+3y = 4$.

Proof. The line $x + 3y = 4$ is the same as

$$y = -\frac{1}{3}x + \frac{4}{3}. \qquad \text{So it has slope } -\tfrac{1}{3}.$$

So the slope of the normal line is $-\frac{1}{3}$. So the slope of the tangent line is 3. So

$$\frac{dy}{dx}\Big]_{x=a} = 3.$$

Now

$$4x - 2y\frac{dy}{dx} = 0. \qquad \text{So} \quad \frac{dy}{dx} = \frac{-4x}{-2y} = \frac{2x}{y}.$$

So we want $\frac{2x}{y} = 3$ and $2x^2 - y^2 = 14$. So

$$y = \frac{2}{3}x \qquad \text{and} \qquad 2x^2 - \Big(\frac{2}{3}x\Big)^2 = 14.$$

So

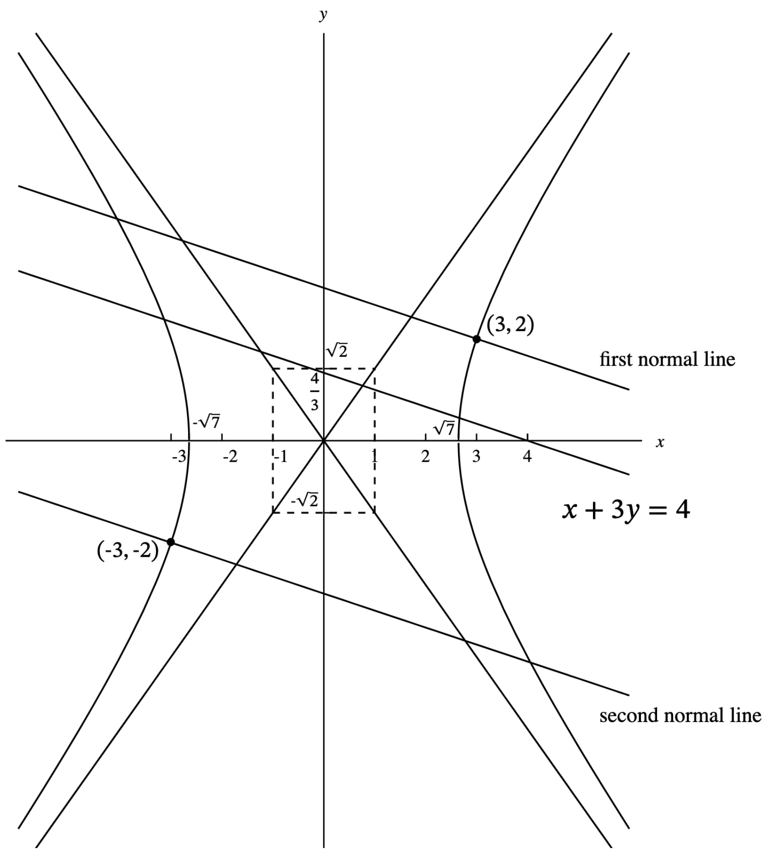
$$2x^2 - \frac{4}{9}x^2 = 14 \qquad \text{and} \qquad \frac{14}{9}x^2 = 14.$$

So

$$x^2 = 9 \qquad \text{and} \qquad x = \pm 3.$$

So $x = 3$ and $y = \frac{2}{3} \cdot 3 = 2$ or $x = -3$ and $y = \frac{2}{3}(-3) = -2$.
In the first case:
The normal has slope $-\frac{1}{3}$ and goes through $(3, 2)$.
So $m = -\frac{1}{3}$ and $2 = m \cdot 3 + y_0 = -\frac{1}{3} \cdot 3 + y_0$.
So $y_0 = 3$ and the equation of the normal line is $y = -\frac{1}{3}x + 3$.
In the second case:
The normal has slope $-\frac{1}{3}$ and goes through $(-3, -2)$.
So $m = -\frac{1}{3}$ and $-2 = m \cdot 3 + y_0 = -\frac{1}{3} \cdot (-3) + y_0$.

So $y_0 = -3$ and the equation of the normal line is $y = -\frac{1}{3}x - 3$.
The graph should explain how there can be *two* normal lines parallel to $x + 3y = 4$.



Notes:

(a) If $y = 0$ then $x = \pm\sqrt{7}$.

(b) $2 - \left(\frac{y}{x}\right)^2 = \frac{14}{x^2}$. So, as $x \to \infty$, this becomes $2 - \left(\frac{y}{x}\right)^2 = 0$ which means

$$\left(\frac{y}{x}\right)^2 = 2 \quad \text{and} \quad \left(\frac{y}{x}\right) = \pm\sqrt{2} \quad \text{and} \quad y = \pm\sqrt{2}x.$$

□

3.5 Areas and volumes

For computing areas and volumes:

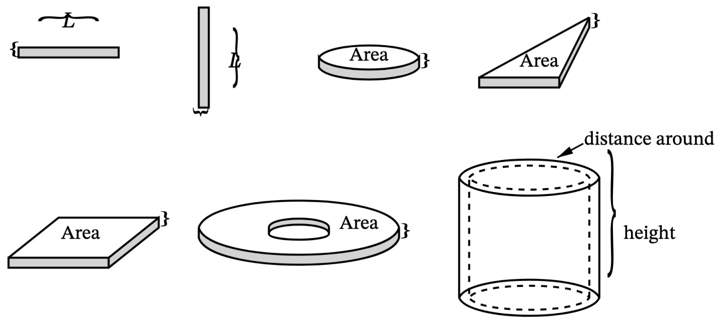
(1) Carefully draw the region.

(2) Slice it up, draw a typical slice.

(3) Find the volume of a slice.

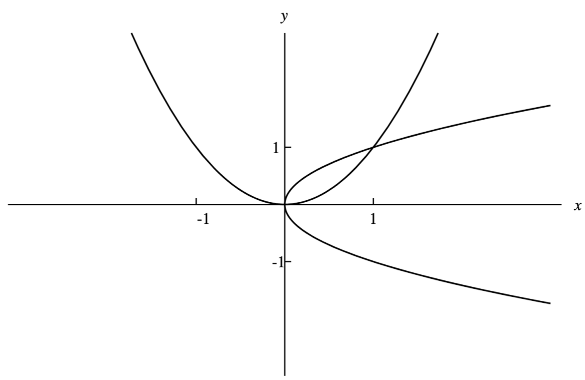
(4) Add up the volumes of the slices with an integral.

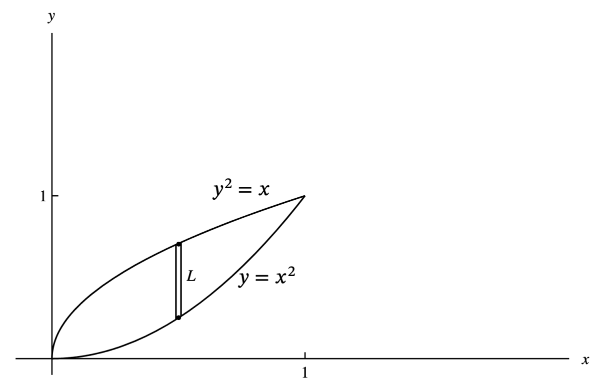
Typical slices might look like:



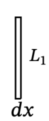
Example 3.40. Calculate the area of the region bounded by the parabolas $y = x^2$ and $y^2 = x$.

Proof.





Slice:



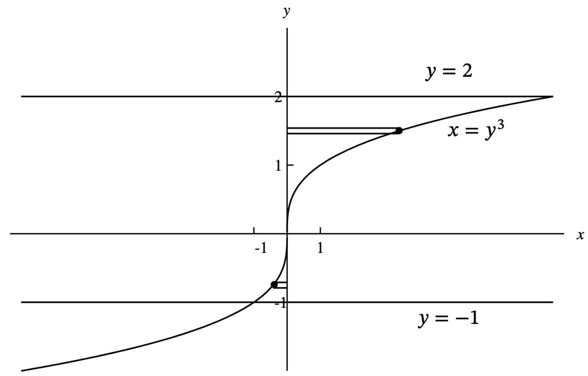
Area of slice $L\,dx$
Add slices from $x = 0$ to $x = 1$.

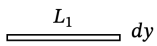
$$\int_{x-0}^{x=1} L\,dx = \int_{x-0}^{x=1} (y_{\text{upper}} - y_{\text{lower}})\,dx = \int_{x-0}^{x=1} (\sqrt{x} - x^2)\,dx$$
$$= \left(\frac{2}{3}x^{3/2} - \frac{x^3}{3}\right)\Big]_{x=0}^{x=1}$$
$$= \left(\frac{2}{3}1^{3/2} - \frac{1^3}{3}\right) - \left(\frac{2}{3}0^{3/2} - \frac{0^3}{3}\right) = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}.$$

□

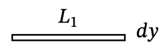
Example 3.41. Find the area of the region bounded by $y = -1$, $y = 2$, $x = y^3$ and $x = 0$.

Proof.



Type 1 slice: 

Area of slice: $L_1\,dy$
Add slices from $y = 0$ to $y = 2$.

Type 2 slice: 

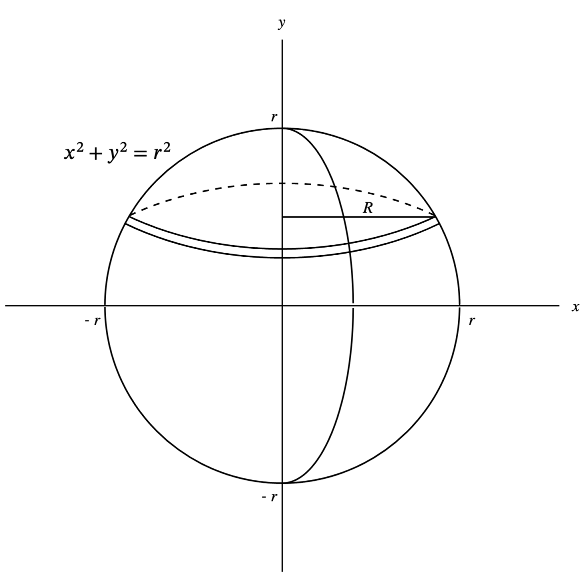
Area of slice: $L_2\,dy$
Add slices from $y = -1$ to $y = 0$.

$$\begin{aligned}\int_{y=0}^{y=2} L_1\,dy + \int_{y=-1}^{y=0} L_2\,dy &= \int_{y=0}^{y=2} x\,dy + \int_{y=-1}^{y=0} (-x)\,dy = \int_{y=0}^{y=2} y^3\,dy + \int_{y=-1}^{y=0} (-y^3)\,dy \\ &= \Big(\frac{y^4}{4}\Big)\Big]_{y=0}^{y=2} + \Big(\frac{-y^4}{4}\Big)\Big]_{y=-1}^{y=0} = \Big(\frac{2^4}{4} - \frac{0}{4}\Big) + \Big(-\frac{0}{4} - \frac{(-1)^4}{4}\Big) \\ &= 2^2 + \frac{1}{4} = 4\tfrac{1}{4}.\end{aligned}$$

□

Example 3.42. Find the volume of a sphere of radius r

Proof.



Slice: 

Volume of slice: $\pi R^2\,dy$
Add slices from $y = -r$ to $y = r$.

$$\begin{aligned}\text{Volume of sphere} &= \int_{y=-r}^{y=r} \pi R^2\,dy = \int_{y=-r}^{y=r} \pi x^2\,dy = \int_{y=-r}^{y=r} \pi(r^2-y^2)\,dy \\ &= \pi\Big(r^2y - \frac{y^3}{3}\Big)\Big]_{y=-r}^{y=r} \\ &= \pi\Big(r^2\cdot r - \frac{r^3}{3}\Big) - \pi\Big(r^2\cdot(-r) - \frac{(-r)^3}{3}\Big) \\ &= \pi\frac{2}{3}r^3 + \pi r^3 - \frac{\pi r^3}{3} = \frac{2}{3}\pi r^3 + \frac{2}{3}\pi r^3 = \frac{4}{3}\pi r^3.\end{aligned}$$

□

Example 3.43. Compute $\displaystyle\int_{-a}^{a} \sqrt{a^2-x^2}\,dx$.

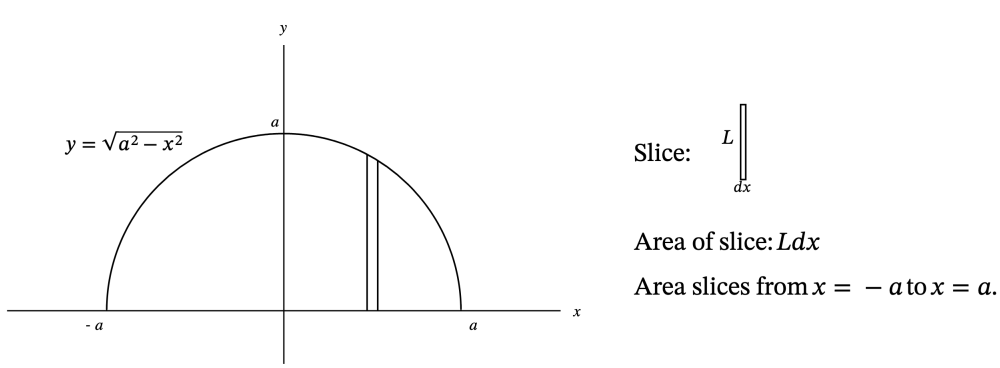
Proof. If $x = a\sin(\theta)$ then

$$\begin{aligned}\int_{-a}^{a}\sqrt{a^2-x^2}\,dx &= \int_{-a}^{a}\sqrt{a^2-a^2\sin(\theta)^2}\,dx = \int_{-a}^{a}\sqrt{a^2\cos(\theta)^2}\,dx = \int_{-a}^{a} a\cos(\theta)\,dx \\ &= \int_{-a}^{a} a\cos(\theta)\frac{dx}{d\theta}\,d\theta = \int_{-a}^{a} a\cos(\theta)\,a\cos(\theta)\,d\theta = \int_{-a}^{a} a^2\cos(\theta)^2\,d\theta \\ &= \int_{-a}^{a}\frac{1}{2}a^2(\cos(\theta)^2+\cos(\theta)^2)\,d\theta \\ &= \int_{-a}^{a}\frac{1}{2}a^2(\cos(\theta)^2+(1-\sin(\theta)^2))\,d\theta = \int_{-a}^{a}\frac{1}{2}a^2(\cos(\theta)^2-\sin(\theta)^2+1)\,d\theta \\ &= \int_{-a}^{a}\frac{1}{2}a^2(\cos(2\theta)+1)\,d\theta \\ &= \frac{1}{2}a^2\Big(\frac{\sin(2\theta)}{2}+\theta)\Big)\Big)\Big]_{x=-a}^{x=a} \\ &= \frac{1}{2}a^2\Big(\frac{\sin(2\theta)}{2}+\theta)\Big)\Big)\Big]_{\sin(\theta)=-1}^{\sin(\theta)=1} \\ &= \frac{1}{2}a^2\Big(\frac{\sin(2\theta)}{2}+\theta)\Big)\Big)\Big]_{\theta=-\pi/2}^{\theta=\pi/2} \\ &= \frac{1}{2}a^2\Big(\frac{\sin(\pi)}{2}+\frac{\pi}{2}\Big) - \frac{1}{2}a^2\Big(\frac{\sin(-\pi)}{2}-\frac{\pi}{2}\Big) = \frac{1}{2}a^2\frac{\pi}{2} - \frac{1}{2}a^2\Big(-\frac{\pi}{2}\Big) = \frac{\pi a^2}{4}.\end{aligned}$$

□

Example 3.44. Compute $\displaystyle\int_{x=-a}^{x=a} \sqrt{a^2-x^2}\,dx$.

Proof.

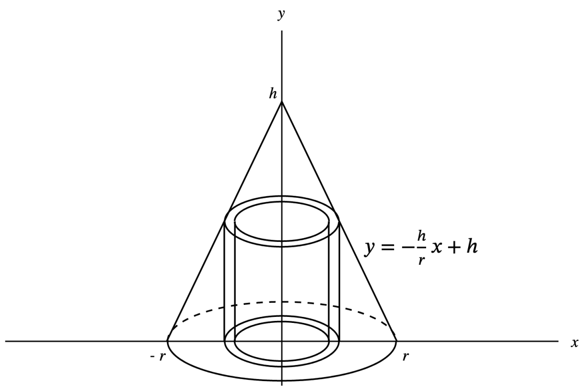


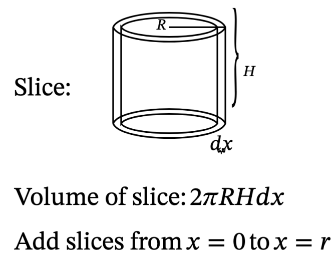
$$\frac{\pi a^2}{2} = \text{Area of semicircle} = \int_{x=-a}^{x=a} L\,dx = \int_{x=-a}^{x=a} y\,dx = \int_{x=-a}^{x=a} \sqrt{a^2-x^2}\,dx.$$

□

Example 3.45. Find the volume of a right circular cone of height h and radius r.

Proof.



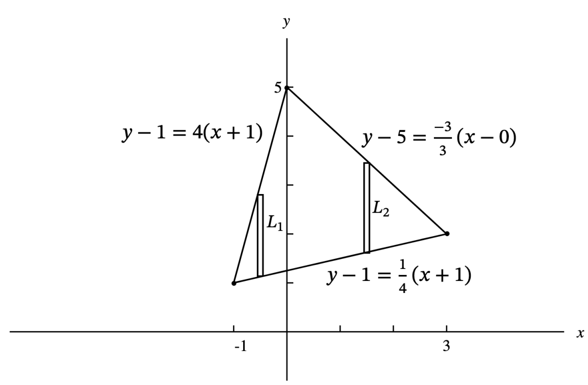


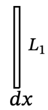
$$\begin{aligned}\int_{x=0}^{x=r} 2\pi RH\,dx &= \int_{x=0}^{x=r} 2\pi xy\,dx = \int_{x=0}^{x=r} 2\pi x\big(-\frac{h}{r}+h\big)\,dx \\ &= \int_{x=0}^{x=r} \big(-\frac{2\pi h}{r}x^2 + 2\pi hx\big)\,dx \\ &= \Big(-\frac{2\pi h}{r}\frac{x^3}{3} + \pi hx^2\Big)\Big]_{x=0}^{x=r} \\ &= \Big(-\frac{2\pi h}{r}\frac{r^3}{3} + \pi hr^2\Big) - (-0+0) \\ &= -\frac{2}{3}\pi r^2 h + \pi r^2 h = \frac{1}{3}\pi r^2 h.\end{aligned}$$

□

Example 3.46. Use integration to find the area of the triangle with vertices $(-1,1)$, $(0,5)$ and $(3,2)$.

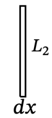
Proof.



Type 1 slice: 

Area of type 1 slice: $L_1 x$

Add slices from $x=-1$ to $x=0$

Type 2 slice: 

Area of type 2 slice: $L_2 x$

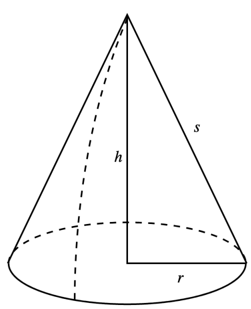
Add slices from $x=-1$ to $x=0$

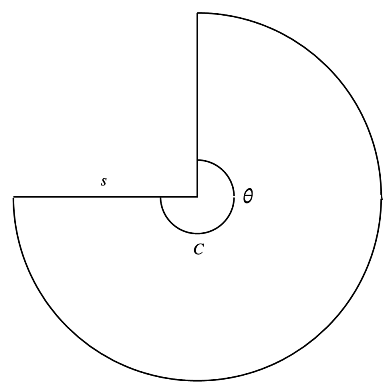
$$\begin{aligned}
\int_{x=-1}^{x=0} L_1\,dx + \int_{x=0}^{x=3} L_2\,dx &= \int_{x=-1}^{x=0} \big(y_{\text{top}\,1} - y_{\text{bottom}}\big)\,dx + \int_{x=0}^{x=3} \big(y_{\text{top}\,2} - y_{\text{bottom}}\big)\,dx \\
&= \int_{x=-1}^{x=0} \big((4(x+1)+1) - (\tfrac{1}{4}(x+1)+1)\big)\,dx + \int_{x=0}^{x=3} \big(-x+5-\frac{1}{4}x-\frac{1}{4}-1\big)\,dx \\
&= \int_{x=-1}^{x=0} (\frac{15}{4}x+\frac{15}{4})\,dx + \int_{x=0}^{x=3} \big(-\frac{5}{4}x+\frac{15}{4}\big)\,dx \\
&= \Big(\frac{15}{4}\frac{x^2}{2}+\frac{15}{4}x\Big)\Big]_{x=-1}^{x=0} + \Big(-\frac{5}{4}\frac{x^2}{2}+\frac{15}{4}x\Big)\Big]_{x=0}^{x=3} \\
&= \Big(\frac{15}{4}\cdot 0+\frac{15}{4}\cdot 0\Big) - \Big(\frac{15}{4}\frac{(-1)^2}{2}+\frac{15}{4}(-1)\Big) + \Big(-\frac{5}{4}\frac{3^2}{2}+\frac{15}{4}\cdot 3\Big) - (0+0) \\
&= 0+0-\frac{15}{8}+\frac{15}{4}-\frac{45}{8}+\frac{45}{4} = \frac{15}{8}+\frac{45}{8} = \frac{60}{8} = 7\tfrac{1}{2}.
\end{aligned}$$

□

Example 3.47. Find the curved surface area of a cone of radius r and height h (a right circular cone).

Proof.

Cut the cone 

open and lay it out to get 

The region C is a portion of a circle of radius s, where s is the slant height of the cone. The area of C is $\frac{1}{2}\theta s^2$. The arc length along the border is θs. This arc length is also the length around the circle at the base of the cone, which is $2\pi r$. So

$$\theta s = 2\pi r.$$

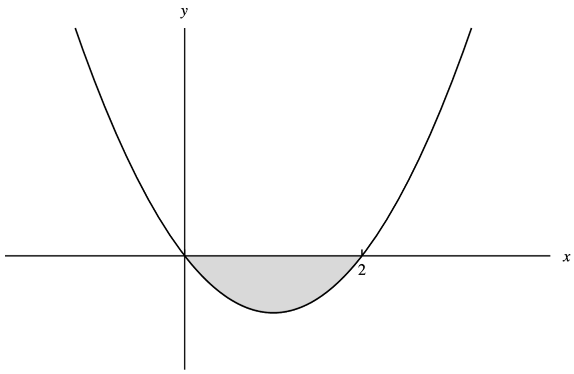
So

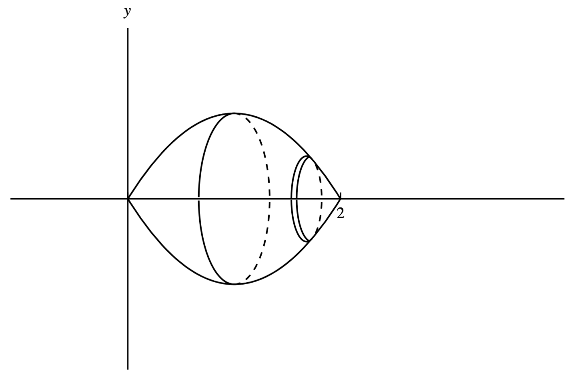
$$\text{curved surface area} = \frac{1}{2}\theta s^2 = \frac{1}{2}(\theta s)s = \frac{1}{2}(2\pi r)s = \pi r s = \pi r\sqrt{h^2+r^2}$$

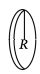
□

Example 3.48. Find the volume generated by the area bounded by $y = x^2 - 2x$ and $y = 0$ when it is rotated about the x-axis.

Proof.





Slice: 

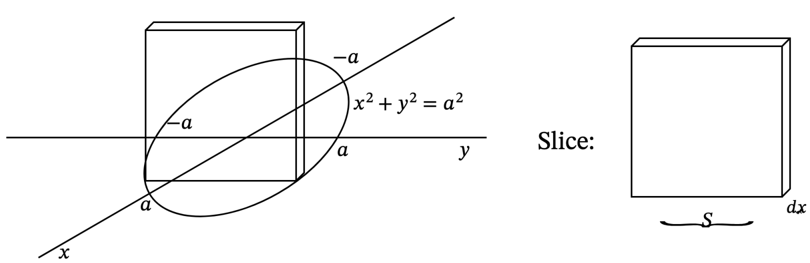
Volume of a slice: $\pi R^2\,dx$
Add slices from $x=0$ to $x=2$.

$$\begin{aligned}\int_{x=0}^{x=2} \pi R^2\,dx &= \int_{x=0}^{x=2} \pi(-y)^2\,dx = \int_{x=0}^{x=2} \pi y^2\,dx = \int_{x=0}^{x=2} \pi(x^2-2x)^2\,dx \\ &= \int_{x=0}^{x=2} \pi(x^4-4x^3+4x^2)\,dx = \pi\Big(\frac{x^5}{5}-\frac{4x^4}{4}+\frac{4x^3}{3}\Big)\Big]_{x=0}^{x=2} \\ &= \pi\Big(\frac{2^5}{5}-2^4+\frac{4}{3}2^3\Big)-\pi(0-0+0) \\ &= 2^3\pi\Big(\frac{2^2}{5}-2+\frac{4}{3}\Big) = 8\pi\Big(\frac{4}{5}-2+\frac{4}{3}\Big) \\ &= 8\pi\Big(-\frac{6}{5}+\frac{4}{3}\Big) = 8\pi\Big(-\frac{18}{15}+\frac{20}{15}\Big) = \frac{8\pi\cdot 2}{15} = \frac{16\pi}{15}.\end{aligned}$$

□

Example 3.49. The base of a solid is $x^2+y^2=a^2$. Each plane section, perpendicular to the x-axis, is a square, with one edge of the square in the base of the solid. Find the volume.

Proof.

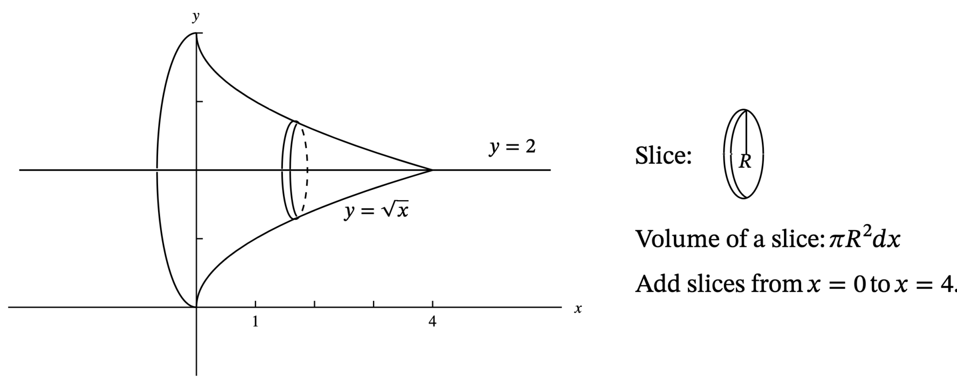


$$\begin{aligned}\int_{x=-a}^{x=a} S^2\,dx &= \int_{x=-a}^{x=a} (2y)^2\,dx = \int_{x=-a}^{x=a} 4y^2\,dx = \int_{x=-a}^{x=a} 4(a^2-x^2)\,dx \\ &= 4\Big(a^2x - \frac{x^3}{3}\Big)\Big]_{x=-a}^{x=a} = 4\Big(a^2\cdot a - \frac{a^3}{3}\Big) - 4\Big(a^2\cdot(-a) - \frac{(-a)^3}{3}\Big) \\ &= 4\Big(a^3 - \frac{1}{3}a^3\Big) - 4\Big(-a^3 + \frac{1}{3}a^3\Big) \\ &= 4\cdot\frac{2}{3}a^3 - 4\big(-\frac{2}{3}a^3\big) = 4\cdot\frac{4}{3}a^3 = \frac{16a^3}{3}.\end{aligned}$$

□

Example 3.50. Find the volume generated when the area bounded by $y=\sqrt{x}$, $y=2$ and $x=0$ is rotated about the line $y=2$.

Proof.



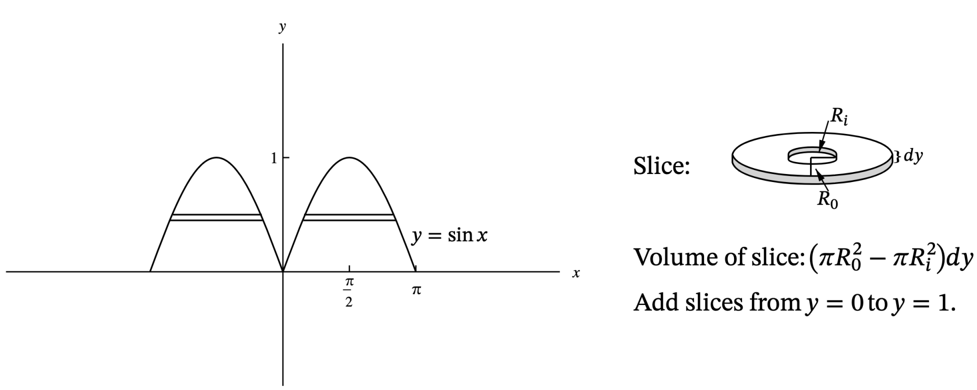
Add slices from $x=0$ to $x=4$.

$$\begin{aligned}\int_{x=0}^{x=4} \pi R^2\,dx &= \int_{x=0}^{x=4} \pi(2-y)^2\,dx = \int_{x=0}^{x=4} \pi(2-\sqrt{x})^2\,dx = \int_{x=0}^{x=4} \pi(4-4\sqrt{x}+x)\,dx \\ &= \pi\Big(4x - \frac{2\cdot 4}{3}x^{3/2} + \frac{x^2}{2}\Big)\Big]_{x=0}^{x=4} = \pi\Big(4\cdot 4 - \frac{2\cdot 4}{3}\cdot 4^{3/2} + \frac{4^2}{2}\Big) - \pi(0-0+0) \\ &= \pi\Big(16 - \frac{8}{3}\cdot 8 + \frac{16}{2}\Big) = 8\pi\Big(2 - \frac{8}{3} + 1\Big) = 8\pi\cdot\frac{1}{3} = \frac{8\pi}{3}.\end{aligned}$$

□

Example 3.51. Find the volume generated when the area bounded by $y=\sin(x)$ for $0\le x\le\pi$, and $y=0$ is rotated about the y-axis.

Proof.

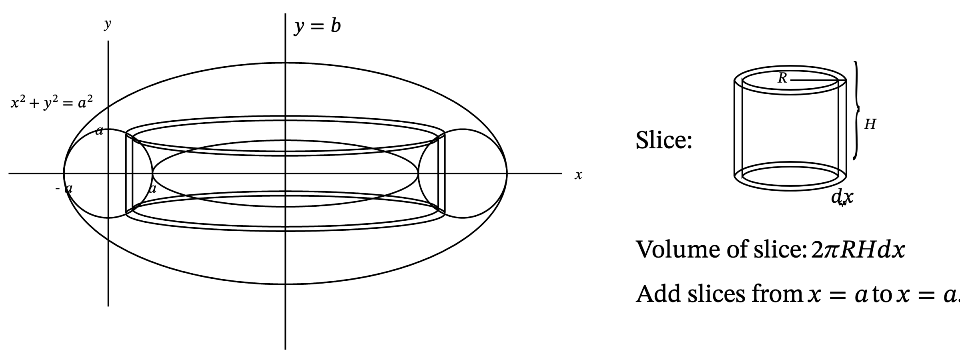


$$\begin{aligned}
\int_{y=0}^{y=1} (\pi R_0^2 - \pi R_1^2))\,dy &= \int_{y=0}^{y=1} \pi(x_{\mathrm{right}}^2 - x_{\mathrm{left}}^2))\,dy = \int_{y=0}^{y=1} \pi((\pi - x_{\mathrm{left}})^2 - x_{\mathrm{left}}^2))\,dy \\
&= \int_{y=0}^{y=1} \pi(\pi^2 - 2\pi x + x^2 - x^2)\,dy = \int_{y=0}^{y=1} \pi(\pi^2 - 2\pi x)\,\frac{dy}{dx}\,dx \\
&= \int_{x=0}^{x=\pi/2} \pi(\pi^2 - 2\pi x)\cos(x)\,dx = \int_{x=0}^{x=\pi/2} (\pi^3\cos(x) - 2\pi^2 x\cos(x))\,dx \\
&= \big(\pi^3\sin(x) - 2\pi^2(x\sin(x) + \cos(x))\big]_{x=0}^{x=\pi/2} \\
&= \big(\pi^3\sin(\pi/2) - 2\pi^2(\frac{\pi}{2}\sin(\pi/2) + \cos(\pi/2)\big) - (\pi^3\sin(0) - 2\pi^2(0 + \cos(0)) \\
&= \pi^3 - \frac{2\pi^2\cdot\pi}{2} + 2\pi^2 = 2\pi^2.
\end{aligned}$$

□

Example 3.52. Find the volume of a bagel produced by rotating the circle $x^2 + y^2 = a^2$ about the line $y = b$.

Proof.

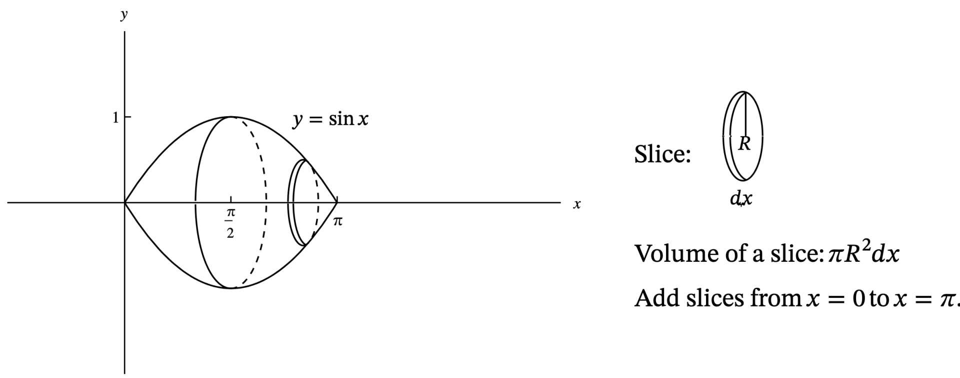


$$\begin{aligned}
\int_{x=-a}^{x=a} 2\pi RH\,dx &= \int_{x=-a}^{x=a} 2\pi(b-x)2y\,dx = \int_{x=-a}^{x=a} 2\pi(b-x)2\sqrt{a^2-x^2}\,dx \\
&= \int_{x=-a}^{x=a} \big(4\pi b\sqrt{a^2-x^2} - 4\pi x\sqrt{a^2-x^2}\big)\,dx \\
&= \int_{x=-a}^{x=a} 4\pi b\sqrt{a^2-x^2}\,dx - \frac{4\pi}{-2} - 2x\sqrt{a^2-x^2}\big)\,dx \\
&= 4\pi b(\text{area of a semicircle of radius } a) + \big(2\pi(a^2-x^2)^{3/2}\big)\Big]_{x=-a}^{x=a} \\
&= 4\pi b\frac{\pi a^2}{2} + 2\pi(0^{3/2}) - 2\pi(0^{3/2}) = \frac{4\pi^2 a^2 b}{2} = 2\pi^2 a^2 b.
\end{aligned}$$

□

Example 3.53. Find the volume generated when the area bounded by $y = \sin(x)$ for $0 \le x \le \pi$, and $y = 0$ is rotated about the x-axis.

Proof.

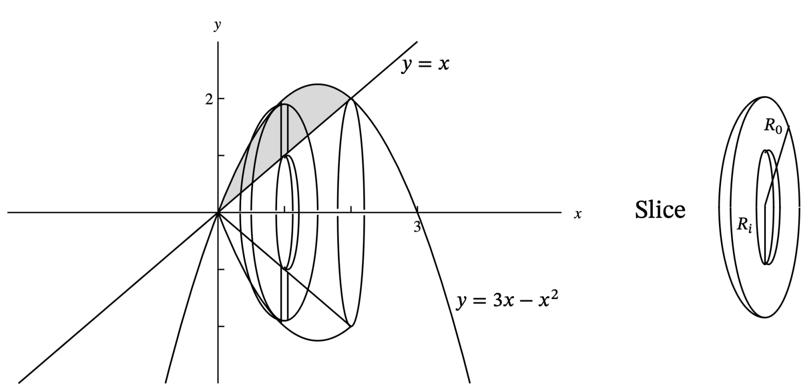


$$\begin{aligned}\int_{x=0}^{x=\pi} \pi R^2\,dx &= \int_{x=0}^{x=\pi} \pi y^2\,dx = \int_{x=0}^{x=\pi} \pi \sin(x)^2\,dx \\ &= \int_{x=0}^{x=\pi} \frac{\pi}{2}(\sin(x)^2 + \sin(x)^2)\,dx \\ &= \int_{x=0}^{x=\pi} \frac{\pi}{2}(\sin(x)^2 + 1 - \cos(x)^2)\,dx \\ &= \int_{x=0}^{x=\pi} \frac{\pi}{2}(1 - (\cos(x)^2 - \sin(x)^2))\,dx \\ &= \int_{x=0}^{x=\pi} \frac{\pi}{2}(1 - \cos(2x))\,dx = \frac{\pi}{2}\Big(x - \frac{\sin(2x)}{2}\Big)\Big]_{x=0}^{x=\pi} \\ &= \frac{\pi}{2}\Big(\pi - \frac{\sin(2\pi)}{2}\Big) - \frac{\pi}{2}\Big(0 - \frac{\sin(0)}{2}\Big) = \frac{\pi}{2}(\pi - 0) = \frac{\pi^2}{2}.\end{aligned}$$

□

Example 3.54. Find the volume generated by rotating the area bounded by the curves $y = 3x - x^2$ and $y = x$ about the x-axis.

Proof.

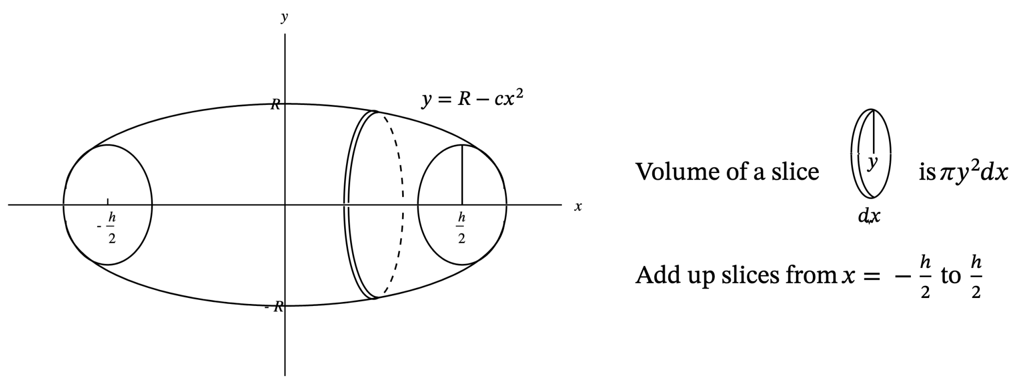


$$\begin{aligned}\int_{x=0}^{x=2} (\pi R_0^2 - \pi R_1^2)\,dx &= \int_{x=0}^{x=2} (\pi y_{\rm top}^2 - \pi y_{\rm bottom}^2)\,dx = \int_{x=0}^{x=2} (\pi(3x-x^2)^2 - \pi x^2)\,dx \\ &= \int_{x=0}^{x=2} \pi(9x^2 - 6x^3 + x^4 - x^2)\,dx \\ &= \int_{x=0}^{x=2} \pi(8x^2 - 6x^3 + x^4)\,dx \\ &= \pi\Big(\frac{8x^3}{3} - \frac{6x^4}{4} + \frac{x^5}{5}\Big)\Big]_{x=0}^{x=2} = \pi\Big(\frac{8\cdot 8}{3} - \frac{6\cdot 2^4}{4} + \frac{2^5}{5}\Big) - \pi(0-0+0) \\ &= \pi 2^5\big(\frac{2}{3} - \frac{3}{4} + \frac{1}{5}\big) = 32\pi\big(\frac{40}{60} - \frac{45}{60} + \frac{12}{60}\big) = \frac{32\pi\cdot 7}{60} = \frac{8\cdot 7\pi}{15} = \frac{56\pi}{15}.\end{aligned}$$

□

Example 3.55. A barrel of height h and maximum radius R is constructed by rotation of the parabola $y = R - cx^2$ for $-\frac{h}{2} \le x \le \frac{h}{2}$.

Proof.

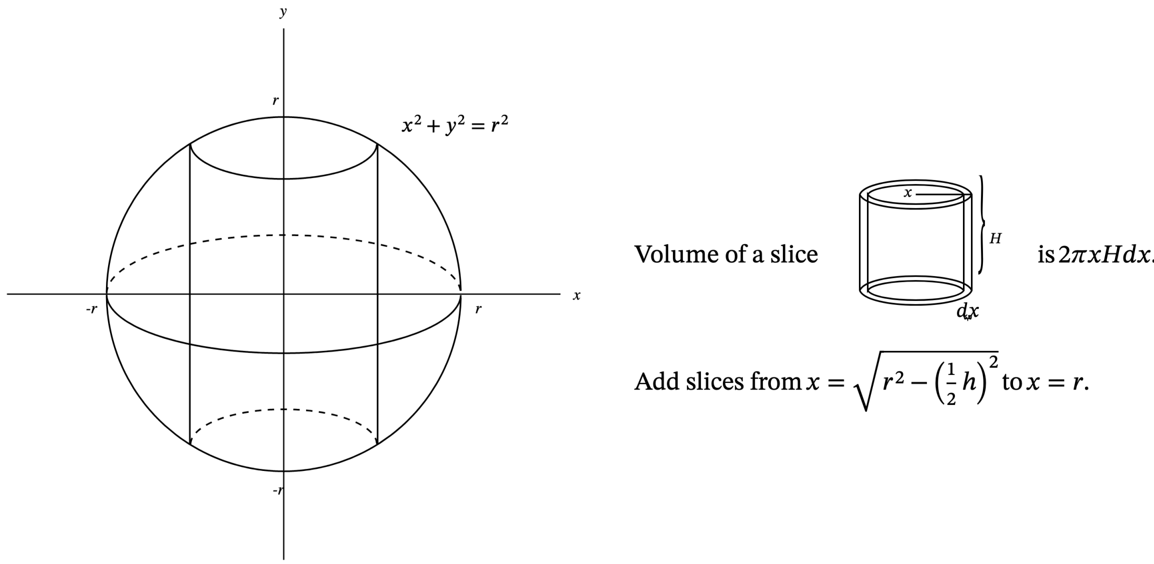


$$\begin{aligned}\int_{x=-\frac{h}{2}}^{x=\frac{h}{2}} \pi y^2\,dx &= \int_{x=-\frac{h}{2}}^{x=\frac{h}{2}} \pi(R - cx^2)^2\,dx = \int_{x=-\frac{h}{2}}^{x=\frac{h}{2}} \pi(R^2 - 2cRx^2 + c^2x^4)\,dx \\ &= \pi\Big(R^2x - 2cR\frac{x^3}{3} + c^2\frac{x^5}{5}\Big)\Big]_{x=-\frac{h}{2}}^{x=\frac{h}{2}} \\ &= \pi\Big(R^2\frac{h}{2} - 2cR\frac{h^3}{8} + \frac{c^2}{5}\frac{h^5}{2^5}\Big) - \pi\Big(R^2\frac{-h}{2} + 2cR\frac{h^3}{2^3} + \frac{(-c^2h^5)}{5\cdot 2^5}\Big) \\ &= \pi\Big(R^2h - \frac{2cRh^3}{3\cdot 8} + \frac{2c^2h^5}{5\cdot 2^5}\Big) = \pi\Big(R^2h - \frac{cRh^3}{6} + \frac{c^2h^5}{5\cdot 16}\Big).\end{aligned}$$

□

Example 3.56. You are given two spherical balls of wood, one of radius r and a second one of radius R. A circular hole is bored through each ball and the resulting napkin rings have height h. Which napkin ring contains more wood?

Proof.

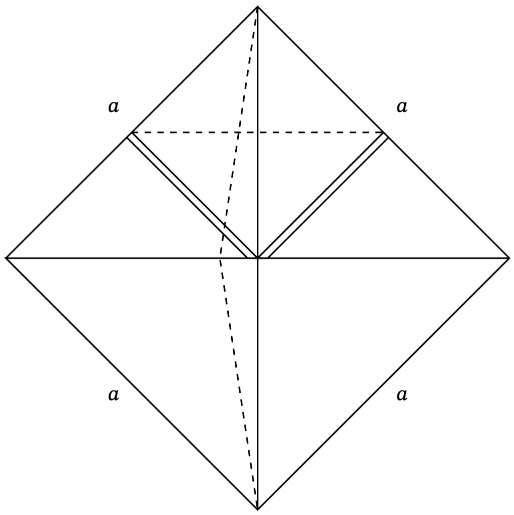


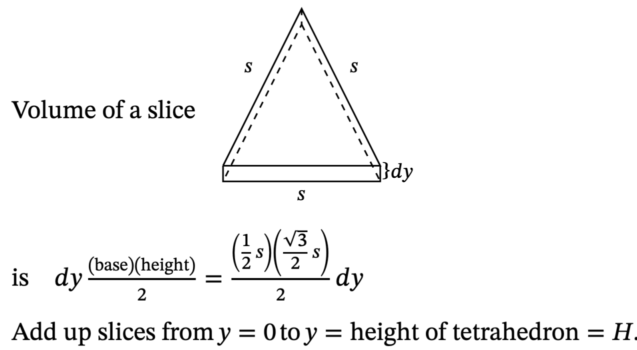
$$\begin{aligned}
\int_{x=\sqrt{r^2-\left(\frac{1}{2}h\right)^2}}^{x=r} 2\pi x 2y\,dx &= \int_{x=\sqrt{r^2-\left(\frac{1}{2}h\right)^2}}^{x=r} 4\pi x\sqrt{r^2-x^2}\,dx \\
&= \int_{x=\sqrt{r^2-\left(\frac{1}{2}h\right)^2}}^{x=r} (-2\pi)(-2x)(r^2-x^2)^{1/2}\,dx \\
&= (-2\pi)(r^2-x^2)^{3/2}\frac{2}{3}\Big]_{x=\sqrt{r^2-\left(\frac{1}{2}h\right)^2}}^{x=r} \\
&= (-2\pi)(r^2-r^2)^{3/2}\frac{2}{3} - (-2\pi)(r^2-\big(r^2-\frac{h^2}{4}\big))^{3/2}\frac{2}{3} \\
&= 0 + 2\pi\big(\frac{h^2}{4}\big)^{3/2}\frac{2}{3} = 2\pi\big(\frac{h}{2}\big)^3\frac{2}{3} = \frac{4\pi}{3}\frac{h^3}{8} = \frac{\pi h^3}{8}.
\end{aligned}$$

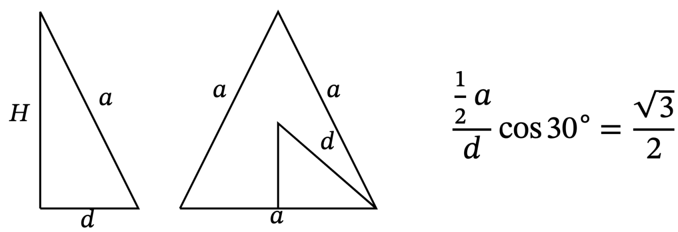
This **doesn't depend** on r!! So both napkin rings contain the same amount of wood. □

Example 3.57. Find the volume of a tetrahedron where each side of the tetrahedron is an equilateral triangle with side length a.

Proof.







So

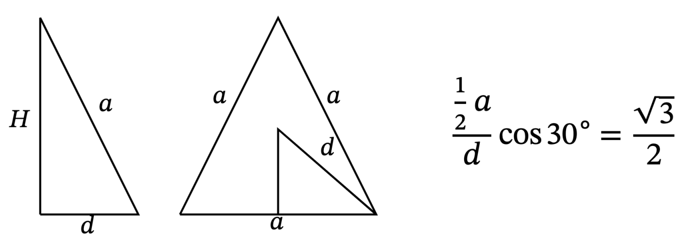
$$d = \frac{\frac{1}{2}a}{\frac{\sqrt{3}}{2}} = \frac{a}{\sqrt{3}}.$$

So

$$H = \sqrt{a^2 - \left(\frac{a}{\sqrt{3}}\right)^2} = \sqrt{a^2 - \frac{a^2}{3}} = \frac{2a^2}{3} = a\frac{\sqrt{2}}{\sqrt{3}}.$$

So we want

$$2\int_{y=0}^{y=H} \frac{H\left(\frac{1}{2}s\right)\left(\frac{\sqrt{3}}{2}s\right)}{2}\,dy = \int_{y=0}^{y=H} 2\cdot\frac{\sqrt{3}s^2}{8}\,dy.$$



But $H - y = \frac{\sqrt{2}}{\sqrt{3}}s$. So $y = H - \frac{\sqrt{2}}{\sqrt{3}}s$. So $\frac{dy}{ds} = -\frac{\sqrt{2}}{\sqrt{3}}$.

$$\begin{aligned}\int_{y=0}^{y=H} 2\cdot\frac{\sqrt{3}}{8}s^2\,dy &= \int_{y=0}^{y=H} 2\cdot\frac{\sqrt{3}}{8}s^2\frac{dy}{ds}\,ds = \int_{y=0}^{y=H} 2\cdot\frac{\sqrt{3}}{8}s^2\Big(-\frac{\sqrt{2}}{\sqrt{3}}\Big)\,ds \\ &= 2\cdot\frac{\sqrt{3}}{8}\frac{s^3}{3}\Big(-\frac{\sqrt{2}}{\sqrt{3}}\Big]_{y=0}^{y=H} = -\frac{\sqrt{2}}{8}\frac{2}{3}s^3\Big]_{s=a}^{s=0} \\ &= \Big(-\frac{\sqrt{2}\cdot 2}{8\cdot 3}0^3\Big) - \Big(-\frac{\sqrt{2}\cdot 2}{8\cdot 3}a^3\Big) = \frac{\sqrt{2}\cdot 2}{24}a^3 = \frac{\sqrt{2}}{12}a^3.\end{aligned}$$

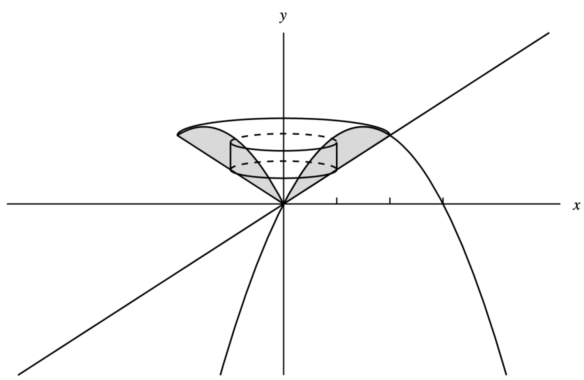
Since $H = \frac{a\sqrt{2}}{\sqrt{3}}$ we can also write this as

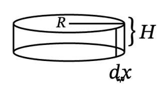
$$\frac{\sqrt{2}}{12}a^3 = \Big(\frac{\sqrt{2}}{\sqrt{3}}a\Big)\frac{\sqrt{3}}{12}a^2 = H = \frac{\sqrt{3}a^2}{12} = \frac{\sqrt{3}a^2H}{12}.$$

□

Example 3.58. Find the volume generated by rotating the area bounded by $y = 3x - x^2$ and $y = x$ about the y-axis.

Proof.



Volume of a slice 

is $2\pi RHdx$.

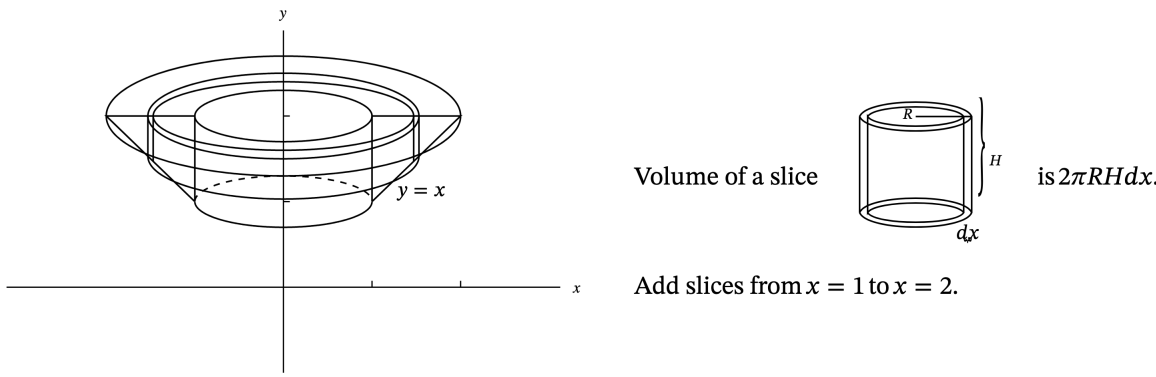
Add slices from $x = 0$ to $x = 2$.

$$\begin{aligned}\int_{x=0}^{x=2} 2\pi RH\,dx &= \int_{x=0}^{x=2} 2\pi x(y_{\text{upper}} - y_{\text{lower}})\,dx = \int_{x=0}^{x=2} 2\pi x(3x - x^2 - x)\,dx \\ &= \int_{x=0}^{x=2} 2\pi x(2x - x^2)\,dx = \int_{x=0}^{x=2} 2\pi(2x^2 - x^3)\,dx = 2\pi\Big(\frac{2x^3}{3} - \frac{x^4}{4}\Big)\Big]_{x=0}^{x=2} \\ &= 2\pi\Big(\frac{2\cdot 8}{3} - \frac{16}{4}\Big) - 2\pi(0-0) \\ &= 2\pi\Big(\frac{16}{3} - \frac{16}{4}\Big) = 2\pi\cdot 16\Big(\frac{1}{3} - \frac{1}{4}\Big) = \frac{32\pi}{12} = \frac{16\pi}{6} = \frac{8\pi}{3}.\end{aligned}$$

□

Example 3.59. Find the volume generated by revolving the triangle with verttices $(1,1)$, $(1,2)$ and $(2,2)$ about the y-axis.

Proof.

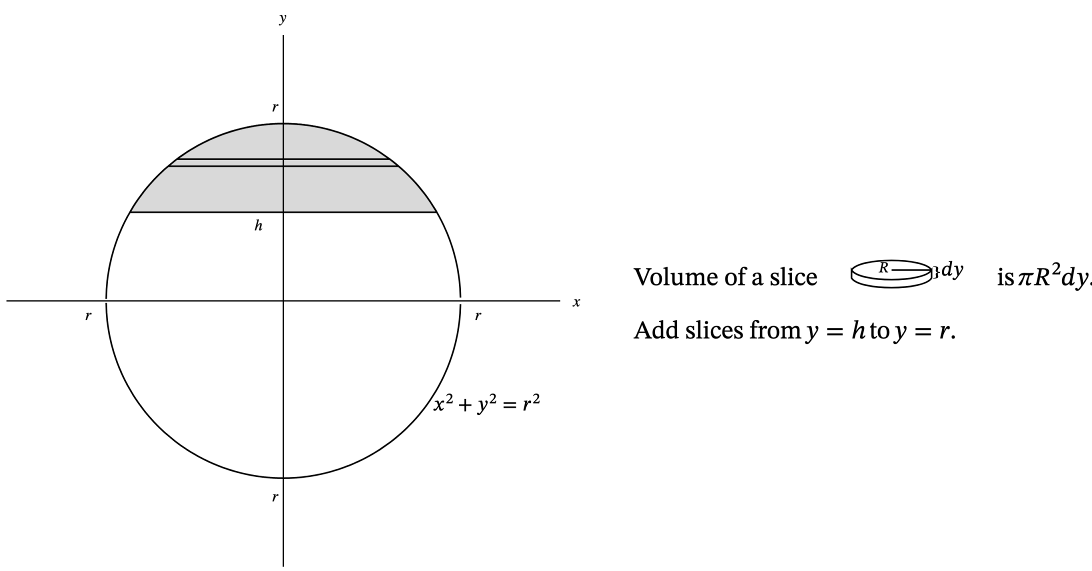


$$\int_{x=1}^{x=2} 2\pi RH\,dx = \int_{x=1}^{x=2} 2\pi x(2-x)\,dx = \int_{x=1}^{x=2} 2\pi(2x-x^2)\,dx$$
$$= 2\pi\Big(x^2 - \frac{x^3}{3}\Big)\Big]_{x=1}^{x=2} = 2\pi(4-\frac{8}{3}) - 2\pi(1-\frac{1}{3}) = 2\pi\frac{4}{3} - 2\pi\frac{2}{3} = \frac{4\pi}{3}.$$

□

Example 3.60. Find the volume of the chunk obtained by chopping off the end of a sphere of radius r, if the chunk has thickness h at its thickest point.

Proof.



$$\int_{y=h}^{y=r} \pi R^2\,dy = \int_{y=h}^{y=r} \pi x^2\,dy = \int_{y=h}^{y=r} \pi(r^2-y^2)\,dy = \pi\Big(r^2y - \frac{y^3}{3}\Big)\Big]_{y=h}^{y=r}$$
$$= \pi\Big(r^3 - \frac{r^3}{3}\Big) - \pi\Big(r^2h - \frac{h^3}{3}\Big) = \pi\frac{2}{3}r^3 - \pi r^2 h + \frac{\pi h^3}{3} = \frac{\pi}{3}(2r^3 - 3r^2h + h^3).$$

□

4 Rates, optimization, changes, growth, decay, mixing, motion

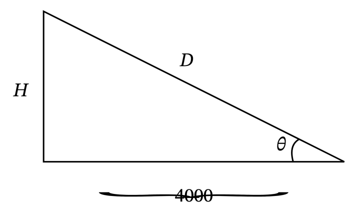
4.1 Motion models

If p is position and v is velocity and a is acceleration then

$$v = \frac{dp}{dt} \qquad \text{and} \qquad a = \frac{dv}{dt}.$$

Example 4.1. A TV camera is 4000 feet from the base of a launch pad. A rocket is launched and has a speed of 600 ft/s when it is 3000 ft high. How fast is the distance between the camera and the rocket changing?

Proof.



$H =$ (height of rocket),
$D =$ (distance between camera and rocket),
velocity $=$ (change in height as time changes) $= \frac{dH}{dt}$.

We know

$$\left.\frac{dH}{dt}\right]_{H=3000} = 600 \qquad \text{in ft/s.}$$

We want to determine

$$\left.\frac{dD}{dt}\right]_{H=3000} = 600.$$

From the picture $4000^2 + H^2 = D^2$. So

$$2h\frac{dH}{dt} = 2D\frac{2D}{dt}.$$

So

$$\left.\frac{dD}{dt}\right]_{H=3000} = \frac{3000}{\sqrt{3000^2+4000^2}}\left.\frac{dH}{dt}\right]_{H=3000} = \frac{3000}{\sqrt{5000^2}}\cdot 600 = \frac{3000\cdot 600}{5000} = 3\cdot\frac{600}{5} = 3\cdot 120 = 360,$$

in feet per second. How fast is the angle of the camera changing? We want $\left.\frac{d\theta}{dt}\right]_{H=3000}$.

$$\text{Since} \quad \tan(\theta) = \frac{H}{4000} \qquad \text{them} \quad \frac{d\theta}{dt}\sec(\theta)^2 = \frac{1}{4000}\frac{dH}{dt}.$$

$$\text{Since} \quad \sec(\theta) = \frac{1}{\cos(\theta)} = \frac{D}{4000} \qquad \text{them} \quad \frac{d\theta}{dt}\frac{D^2}{4000^2} = \frac{1}{4000}\frac{dH}{dt}.$$

So

$$\frac{d\theta}{dt} = \frac{4000^2}{D^2}\frac{1}{4000}\frac{dH}{dt} = \frac{4000}{D^2}\frac{dH}{dt}.$$

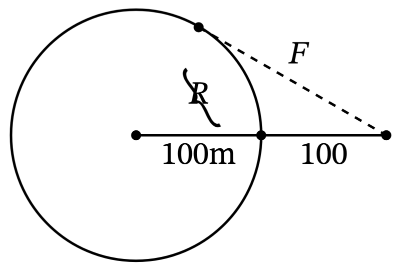
So

$$\left.\frac{d\theta}{dt}\right]_{H=3000} = \frac{4000}{(3000^2+4000^2)}\cdot\left.\frac{dH}{dt}\right]_{H=3000} = \frac{4000}{5000^2}\cdot 600 = \frac{4\cdot 600}{5\cdot 1000} = \frac{4\cdot 120}{1000} = \frac{24}{50} \text{ radians per second.}$$

□

Example 4.2. A runner runs around a circular track of radius 100m at a speed of 7 m/s. The runner's friend is standing 200m from the center. How fast is the distance between them changing when their distance is 200m?

Proof.



$$\text{velocity of runner} = \text{change in runner's distance w.r.t time} = \frac{dR}{dt}.$$

We want

$$\text{change in distance between friends w.r.t. time} = \frac{dF}{dt}.$$

Really we want $\frac{dF}{dt}\Big]_{F=200}$.

The arclength $R = 100 \cdot \theta$ if θ is the angle at the point (x, y) at which the runner is at.

$$F = \sqrt{(200-x)^2 + y^2} = \sqrt{200^2 - 400x + x^2 + y^2} = \sqrt{200^2 - 400x + 100^2}.$$

So

$$F^2 = 200^2 + 100^2 - 400x \qquad \text{and} \qquad 2F\frac{dF}{dt} = -400\frac{dx}{dt}.$$

So

$$\frac{dF}{dt} = -\frac{200}{F}\frac{dx}{dt}.$$

Now $x = 100\cos(\theta)$ and $7 = \frac{dR}{dt} = 100\frac{d\theta}{dt}$. So

$$\frac{dx}{dt} = -100\sin(\theta)\frac{d\theta}{dt} = -100\sin(\theta)\frac{7}{100} = -7\sin(\theta).$$

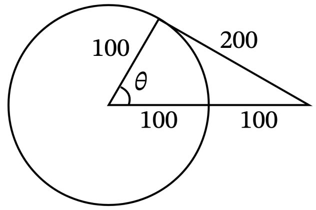
So

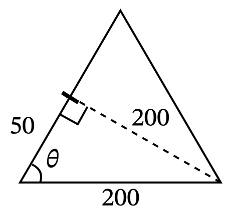
$$\frac{dF}{dt} = \frac{-(200)}{F}(-7\sin(\theta)) = \frac{1400\sin(\theta)}{F}.$$

So

$$\frac{dF}{dt}\Big]_{F=200} = \frac{1400\sin(\theta)}{F}\Big]_{200} = \frac{1400}{200}\sin(\theta)\Big]_{F=200} = 7\sin(\theta)\Big]_{F=200}.$$

When $F = 200$,





$$\text{so} \quad \sin(\theta) = \frac{\sqrt{200^2 - 50^2}}{200}.$$

So

$$\frac{dF}{dt}\Big]_{F=200} = \frac{7\sqrt{200^2 - 50^2}}{200} = \frac{7\sqrt{40000 - 2500}}{200} = \frac{7\sqrt{37500}}{200} = \frac{7}{2}\sqrt{375}$$

in meters per second. □

Example 4.3. A steel ball falls from the top of a tower and in the last second before it hits the ground it falls $\frac{9}{25}$ of the total height of the tower. Find the height of the tower.

Proof. To get control, find equations for the acceleration a, the velocity v and position p of the ball. The acceleration of the ball is $a = 9.8$ m/s^2. So

$$\text{So}\quad \frac{dv}{dt} = -9.8. \qquad\qquad \text{So}\quad \int \frac{dv}{dt}\,dt = \int -9.8\,dt.$$

So $\int dv = \int -9.8\,dt$ and

$$v = 9.8t + c, \qquad \text{where } c \text{ is a constant.}$$

At $t = 0$ the velocity v is 0.

$$\text{So}\quad 0 = -9.8\cdot 0 + c. \qquad\qquad \text{So } c = 0 \text{ and } v = -9.8t.$$

Since $v = \frac{dp}{dt}$ then

$$p = \int \frac{dp}{dt}\,dt = \int v\,dt = \int -9.8t\,dt = -9.8\frac{t^2}{2} + c_1,$$

where c_1 is a constant. At $t = 0$ the position is H. So

$$H = -\frac{9.8}{2}\cdot 0^2 + c_1. \qquad\qquad \text{So } c_1 = H.$$

So

$$p = -\frac{9.8}{2}t^2 + H.$$

The ball hits the ground when p is 0. When p is 0,

$$0 = -\frac{9.8}{2} + H \qquad \text{and} \qquad \frac{9.8}{2}t^2 = H \qquad \text{and} \qquad t^2 = \frac{2H}{9.8} = \frac{H}{4.9}.$$

So, when the ball hits the ground

$$t = \sqrt{\frac{H}{4.9}}.$$

On second before the ball hits the ground its height is $\frac{9}{25}H$. So

$$\text{when}\quad t = \sqrt{\frac{H}{4,9}} - 1 \qquad \text{then} \qquad p = \frac{9}{25}H.$$

So

$$\frac{9}{25}H = -\frac{9.8}{2}\Big(\sqrt{\frac{H}{4.9}} - 1\Big)^2 + H.$$

So

$$\frac{9}{25}H = -4.9\Big(\frac{H}{4.9} - \frac{2}{\sqrt{4.9}}\sqrt{H} + 1\Big) + H = -H + 2\sqrt{4.9}\sqrt{H} - 4.9 + H = 2\sqrt{4.9}\sqrt{H} - 4.9.$$

So

$$\frac{9}{25}H - 2\sqrt{4.9}\sqrt{H} + 4.9 = 0.$$

So

$$\sqrt{H} = \frac{2\sqrt{4.9} \pm \sqrt{(2\sqrt{4.9})^2 - \frac{4.9}{25}(4.9)}}{2\cdot\frac{9}{25}} = \frac{2\sqrt{4.9} \pm \sqrt{4\cdot 4.9 - 4\cdot 4.9\frac{9}{25}}}{\frac{18}{25}}$$

$$= \frac{2\sqrt{4.9} \pm 2\sqrt{4.9}\sqrt{\frac{16}{25}}}{\frac{18}{25}} = \frac{2\sqrt{4.9}\left(1\pm\frac{4}{5}\right)}{\frac{18}{25}} = \begin{cases} \frac{\sqrt{4.9}\frac{9}{25}}{\frac{9}{25}} \\ \text{or} \\ \frac{\sqrt{4.9}\frac{1}{5}}{\frac{9}{25}} \end{cases} = \begin{cases} 5\sqrt{4.9} \\ \text{or} \\ \frac{5}{9}\sqrt{4.9}. \end{cases}$$

So $H = 25\cdot 4.9$ or $H = \frac{25}{81}\cdot 4.9$. □

4.2 A cooling model

Example 4.4. (Cooling model) Let $k, T_s, T_0 \in \mathbb{R}$. Solve the equation

$$\frac{dT}{dt} = -k(T - T_s), \qquad \text{assuming } T(0) = T_0.$$

Proof. Since

$$\frac{1}{(T-T_s)}\frac{dT}{dt} = -k \qquad \text{then} \qquad \int \frac{1}{(T-T_s)}\frac{dT}{dt}\,dt = \int -kdt$$

and

$$\log(T - T_s) = -kt + C, \qquad \text{where } C \text{ is a constant.}$$

So

$$T - T_s = e^{-kt+C} = ce^{-kt}, \qquad \text{where } c \text{ is a constant.}$$

So

$$T = T_s + ce^{-kt}, \qquad \text{where } c \text{ is a constant.}$$

Then

$$T_0 = T(0) = T_s + ce^{-k\cdot 0} = T_s + c \qquad \text{so that} \qquad c = T_0 - T_s.$$

So

$$T = T_s + (T_0 - T_s)e^{-kt}, \qquad \text{where } c \text{ is a constant.}$$

□

Example 4.5. (Cooling model) A roast turkey is taken from an oven when its temperature reaches 85C and is placed on a table in a room where the temperature is 22C. It cools at a rate proportional to the difference between its current temperature and the room temperature.

(a) If the temperature of the turkey is 60C after half and hour what is the temperature after 45 minutes?

(b) When will the turkey have cooled to 40C?

Proof. Idea: The change in temperature is propotional to current temerpature − room temperature.

$$\frac{dT}{dt} = k(T - R), \qquad \text{where } k \text{ is the proporition.}$$

So

$$\frac{dt}{dt} = k \qquad \text{and} \qquad \int \frac{1}{T-R}\frac{dT}{dt}\,dt = \int k\,dt.$$

So
$$\log(T-R) = kt + c, \qquad \text{where } c \text{ is a constant.}$$
So
$$T - R = e^{kt+c} = e^c e^{kt} = Ce^{kt}, \qquad \text{where } C \text{ is a constant.}$$
So
$$T = Ce^{kt} + R.$$
If $t = 0$ then $T = 85 = Ce^{k\cdot 0} + 22 = C + 22.$. So
$$C = 85 - 22 = 63 \qquad \text{and} \qquad T = 63e^{kt} + 22.$$
If $t = \frac{1}{2}$ then $T = 63e^{k\frac{1}{2}} + 22 = 60$. So
$$e^{k\frac{1}{2}} = \frac{60-22}{63} = \frac{48}{63} \quad \text{and} \quad \tfrac{1}{2}k = \log\left(\tfrac{48}{63}\right) \quad \text{and} \quad k = 2\log\left(\tfrac{48}{63}\right).$$
So
$$T = 63e^{2\log(\frac{48}{63})t} + 22.$$
(a) If $t = \frac{3}{4}$ then
$$T = 63e^{2\log(\frac{48}{63})\frac{3}{4}} + 22 = 63e^{\frac{3}{2}\log(\frac{48}{63})} + 22 = 63\Big(e^{\log(\frac{48}{63})}\Big)^{\frac{3}{2}} + 22 = 63\Big(\frac{48}{63}\Big)^{\frac{3}{2}} + 22.$$
(b) If $T = 40$ then $63e^{2\log(\frac{48}{63})t} + 22 = 40$. So
$$e^{2\log(\frac{48}{63})t} = \frac{40-22}{63} = \frac{18}{63} \qquad \text{and} \qquad 2\log(\frac{48}{63})t = \log\big(\frac{18}{63}\big).$$
So
$$t = \frac{\log\left(\frac{18}{63}\right)}{2\log(\frac{48}{63})}.$$
□

Example 4.6. (Tree growth model) Let $a, b, h_0 \in \mathbb{R}$. Solve the equation
$$\frac{dh}{dt} = a(1 - bh), \qquad \text{assuming } h(0) = h_0.$$
Proof. Since
$$\frac{1}{(1-bh)}\frac{dh}{dt} = a \qquad \text{then} \qquad \int \frac{1}{1-bh}(-b)\frac{dh}{dt}dt = \int a(-b)dt$$
and
$$\log(1-bh) = -abt + C, \qquad \text{where } C \text{ is a constant.}$$
So
$$1 - bh = e^{-abt+C} = e^C e^{-abt} = ce^{-abt}, \qquad \text{where } c \text{ is a constant.}$$
So
$$h = \frac{1}{b}(1 - ce^{-abt}), \qquad \text{where } c \text{ is a constant.}$$
Then
$$h_0 = h(0) = \frac{1}{b}(1 - ce^0) = \frac{1-c}{b} \qquad \text{so that} \qquad c = 1 - bh_0.$$
So
$$h = \frac{1}{b}(1 - (1-bh_0)e^{-abt}) = \frac{1}{b} - \frac{(1-bh_0)}{b}e^{-abt}.$$
□

4.3 Interest and loans

Example 4.7. (Interest and loans) If you buy a \$1,000,000 home and put 5% down and take out a 30 year fixed rate mortgage at 5% per year compute how much your payment would be if you paid it all off in one big payment at the end of 30 years.

Proof. Idea: The change in the money is .05 of its current amount.

$$\frac{dM}{dt} = .05M.$$

So

$$\frac{1}{M}\frac{dM}{dt} = .05 \qquad \text{and} \qquad \int \frac{1}{M}\frac{dM}{dt}\,dt = \int .05\,dt.$$

So $\log(M) = 0.5t + c$, where c is a constant. So

$$M = e^{.05t+c} = e^c e^{.05t} = Ce^{.05t}, \qquad \text{where } C \text{ is a constant.}$$

At time $t = 0$ we owe $1,000,000 - 50,000 = 950,000$. So

$$950000 = Ce^{.05t} = C \qquad \text{and} \qquad M = 950000e^{.05t}.$$

After 30 years we owe

$$M = 950000e^{.05\cdot 30} = 950000e^{1.5} \quad \text{dollars.}$$

Note that $950,000e^{1.5} \approx 4,257,604.62$. □

4.4 Radioactive decay

Example 4.8. (Radioactive decay) The majority of naturally occurring rhenium is ${}^{187}_{75}\mathrm{Re}$, which is radioactive and has a half life of $7 \cdot 10^{10}$ years. In how many years will 5% of the earth's ${}^{187}_{75}\mathrm{Re}$ decompose?

Proof. Idea: The change in ${}^{187}_{75}\mathrm{Re}$ is proportional to the existing amount of ${}^{187}_{75}\mathrm{Re}$.

$$\frac{dR}{dt} = kR, \quad \text{where } k \text{ is the proportion.}$$

So

$$\frac{1}{R}\frac{dR}{dt} = k \qquad \text{and} \qquad \int \frac{1}{R}\frac{dR}{dt}\,dt = \int k\,dt.$$

So

$$\log(R) = kt + c \qquad \text{and} \qquad R = e^{kt+c} = e^c e^{kt} = Ce^{kt},$$

where C is a constant.

When $t = 0$ the amount is R_0. So

$$R_0 = Ce^{k\cdot 0} = C \qquad \text{and} \qquad R = R_0 e^{kt}.$$

When $t = 7 \cdot 10^{10}$ the amount is $\frac{1}{2}R_0$. So

$$\frac{1}{2}R_0 = R_0 e^{k\cdot 7\cdot 10^{10}} \qquad \text{and} \qquad \frac{1}{2} = e^{k\cdot 7\cdot 10^{10}}.$$

So

$$\log(\frac{1}{2}) = k \cdot 7 \cdot 10^{10} \qquad \text{and} \qquad k = \frac{\log(\frac{1}{2})}{7 \cdot 10^{10}}.$$

So

$$R = R_0 e^{\frac{log(\frac{1}{2})}{7 \cdot 10^{10}} t}.$$

We want to know when $R = .05R_0$.

$$.05R_0 = R_0 e^{\frac{log(\frac{1}{2})}{7 \cdot 10^{10}} t} \quad \text{then} \quad \frac{1}{20} = e^{\frac{log(\frac{1}{2})}{7 \cdot 10^{10}} t} \quad \text{and} \quad \log(\frac{1}{20}) = \frac{log(\frac{1}{2})}{7 \cdot 10^{10}} t.$$

So

$$t = \frac{7 \cdot 10^{10} \log(\frac{1}{20})}{\log(\frac{1}{2})}.$$

□

Example 4.9. (Radioactive decay - carbon dating) A sample of a wooden artifact from an Egyptian tomb has a ${}^{14}C/{}^{10}C$ ration which is 54.2% of that of freshly cut wood. In approximately what year was the old wood cut? The half life of ${}^{14}C$ is 5720 years.

Proof. Idea: The change in ${}^{14}C$ is proportional to the existing amount.

$$\frac{d^{14}C}{dt} = k^{14}C, \qquad \text{where } k \text{ is the proportion.}$$

So

$$\frac{1}{{}^{14}C} \frac{d^{14}C}{dt} = k \qquad \text{and} \qquad \int \frac{1}{{}^{14}C} \frac{d^{14}C}{dt}, dt = \int k \, dt.$$

So

$$\log({}^{14}C) = kt + c \qquad \text{and} \qquad {}^{14}C = e^{kt+c} = e^c e^{kt} = K e^{kt},$$

where K is a constant. Suppose that at $t = 0$ the amount of ${}^{1}4C$ is C_0. Then

$$C_0 = K e^{k \cdot 0} = K \qquad \text{and} \qquad {}^{14}C = C_0 e^{kt}.$$

The half life of ${}^{14}C$ is 5720 years. So,

$$\text{at } t = 5720, \qquad \frac{1}{2} C_0 = C_0 e^{kt} = C_0 e^{k \cdot 5720}.$$

So

$$\tfrac{1}{2} = e^{k \cdot 5720} \quad \text{and} \quad \log(\tfrac{1}{2}) = k \cdot 5720 \quad \text{and} \quad k = \frac{log(\frac{1}{2})}{5720}.$$

So

$${}^{14}C = C_0 e^{\frac{log(\frac{1}{2})}{5720} t}.$$

Now there is 54.2% of the original ${}^{14}C$. So

$$(.542) \cdot C_0 = C_0 e^{\frac{\log(\frac{1}{2}}{5720} t} \qquad \text{and} \qquad (.542) = e^{\frac{\log(\frac{1}{2}}{5720} t}.$$

So

$$\log(.542) = \frac{\log(\frac{1}{2})}{5720} t \qquad \text{and} \qquad t = \frac{\log(.542) \cdot 5720}{\log(\frac{1}{2})}.$$

□

4.5 Population models

Example 4.10. (Population model) Let $k, h \in \mathbb{C}$ with $k \neq 0$. The elements $p \in \mathbb{C}((t))$ which satisfy the equation

$$\frac{dp}{dt} = kp - h$$

are

$$p = \tfrac{1}{k}(e^{kc_1}e^{kt} + h) = Ce^{kt} + \tfrac{h}{k}, \qquad \text{where } C \text{ is a constant.}$$

Proof. Since

$$\frac{1}{(kp-h)}\frac{dp}{dt} = 1, \qquad \text{then} \quad \tfrac{1}{k}\log(kp-h) = t + c_1, \quad \text{where } c_1 \text{ is a constant,}$$

and

$$p = \tfrac{1}{k}(e^{kc_1}e^{kt} + h) = Ce^{kt} + \tfrac{h}{k}, \qquad \text{where } C \text{ is a constant.}$$

Then $p_0 = C + \frac{h}{k}$ and

$$p = \tfrac{1}{k}(e^{kc_1}e^{kt} + h) = \big(p_0 - \tfrac{h}{k}\big)e^{kt} + \tfrac{h}{k}, \text{where } p_0 = p(0).$$

PHASE PLOT and SOLUTIONPLOT

□

Example 4.11. (Population model) Let $a, k, h \in \mathbb{C}$ with $k \neq 0$ and $a \neq 0$. The elements $p \in \mathbb{C}((t))$ which satisfy the equation

$$\frac{dp}{dt} = kp(1 - \tfrac{1}{a}p) - h$$

are

$$p = -\beta + \frac{\alpha + \beta}{(1 - Ce^{\frac{k}{a}(\alpha-\beta)t})}, \qquad \text{where } C \text{ is a constant,}$$

and where $p^2 - ap + \frac{ha}{k} = (p-\alpha)(p-\beta)$ with

$$\alpha = \frac{a + \sqrt{a^2 - 4ha/k}}{2} \quad \text{and} \quad \beta = \frac{a - \sqrt{a^2 - 4ha/k}}{2}. \tag{4.1}$$

Proof. Let α and β be as in (4.1).

$$\frac{dp}{dt} = \tfrac{k}{a}p^2 + kp - h = \frac{k}{a}(p^2 + ap - \tfrac{ha}{k})$$

and

$$\frac{1}{(p-\alpha)(p-\beta)}\frac{dp}{dt} = \tfrac{k}{a} \qquad \text{gives} \qquad \frac{1}{\alpha-\beta}\Big(\frac{1}{p-\alpha} - \frac{1}{p-\beta}\Big)\frac{dp}{dt} = \tfrac{k}{a}$$

so that

$$\log(p-\alpha) - \log(p-\beta) = \tfrac{k}{a}(\alpha-\beta)t + c_1 \qquad \text{and} \qquad \frac{p-\alpha}{p-\beta} = Ce^{\frac{k}{a}(\alpha-\beta)t}, \tag{4.2}$$

where c_1 and C are constants. Then

$$p(1 - Ce^{\frac{k}{a}(\alpha-\beta)t}) = \alpha + \beta Ce^{\frac{k}{a}(\alpha-\beta)t} \qquad \text{and} \qquad p = \frac{\alpha + \beta Ce^{\frac{k}{a}(\alpha-\beta)t}}{(1 - Ce^{\frac{k}{a}(\alpha-\beta)t})}.$$

So

$$p = -\beta + \frac{\alpha+\beta}{(1 - Ce^{\frac{k}{a}(\alpha-\beta)t})}, \qquad \text{where } C \text{ is a constant.}$$

From (4.2)

$$C = \frac{p_0 - \alpha}{p_0 - \beta}.$$

PHASE PLOT and SOLUTIONPLOT

□

Example 4.12. (Population model) If the bacteria in a culture increase continuously at a rate proportional to the number present, and the initial number is N_0 find the number at time t.

Proof. Idea: The change in bacteria is proportional to the amount of bacteria.

$$\frac{dB}{dt} = kB, \qquad \text{where } k \in \mathbb{R}, \text{ is the proportion.}$$

What could B be?

$$\frac{1}{B}\frac{db}{dt} = k. \qquad \text{So} \quad \int \frac{1}{B}\frac{dB}{dt}dt = \int kdt.$$

So $\log(B) = kt + c$, where c is a constant. So

$$B = e^{kt+c} = e^c e^{kt} = Ce^{kt}, \quad \text{where } C \text{ is a constant.}$$

At time $t = 0$,

$$B = N_0 = Ce^{k\cdot 0} = C.$$

So $C = N_0$ and

$$B = N_0 e^{kt}.$$

□

Example 4.13. (Population model) A pharmaceutical company grows engineered yeast to produce a drug. The yeast is continuously harvested to collect the drug.

The population p (in millions of yeast cells) at time t days is described by

$$\frac{dp}{dt} = 3p - 2, \qquad \text{for } p \in \mathbb{R}_{\geq 0} \text{ and } t \in \mathbb{R}_{\geq 0}.$$

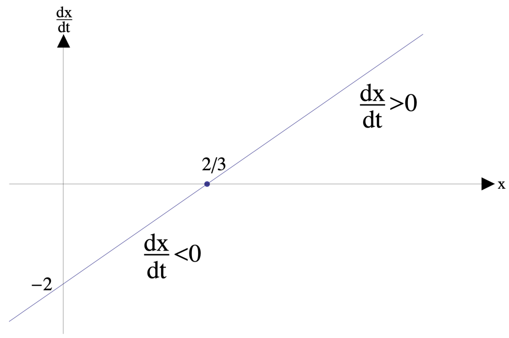
(a) For what initial polulation sizes $p(0)$ will the yeast population eventually die out?
(b) Find the time taken for the population to die out, if the initial population size is $p(0) = \frac{1}{2}$.

(0) Find all equilibrium solutions.
(a) Draw a phase plot,
(b) Sketch the family of solutions of the ODE, including any equilibria.
(c) Describe the long term behaviour of solutions with initial conditions

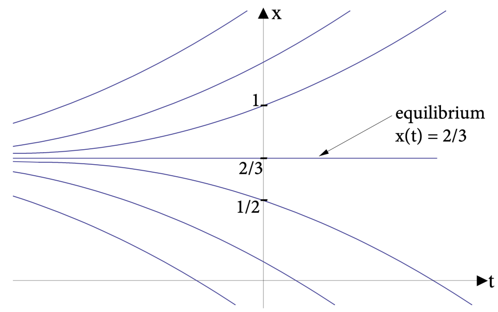
   (i) $p(0) = \frac{1}{2}$;
   (ii) $p(0) = 1$;

(d) Determine the stability of the equilibrium.

Proof. The phase plot determines the solution plot.



Phase plot: Graph of $\frac{dp}{dt}$ as a function of p



Solution plot: Solutions of $\frac{dp}{dt} = 3p - 2$

and

(A) If the initial population is less than $\frac{2}{3}$ then the population decreases and eventually dies out as times passes. In particular, if the initial population is $\frac{1}{2}$ then the population decreases and eventually dies out.

(B) If the initial population is greater than $\frac{2}{3}$ then the population gets larger and larger as time passes. In particular, if the initial population is 1 then the population gets larger and larger as time passes.

(C) If the initial population is $\frac{2}{3}$ then the population will stay $\frac{2}{3}$ forever, but this equilibrium is unstable as any chance aberration will cause the population to start to increase and grow forever, or to start to decrease and then eventualy die out.

Since

$$\frac{1}{(3p-2)}\frac{dp}{dt} = 1 \qquad \text{then} \qquad \int \frac{1}{(3p-2)}\frac{dp}{dt}\,dt = \int 1\,dt.$$

Since $\frac{d}{dp}(\frac{1}{3}\log(3p-2)) = \frac{1}{3p-2}$ and $\int 1\,dt = t + c$ then

$$\tfrac{1}{3}\log(3p-2) = t + c, \quad \text{where } c \text{ is a constant,}$$

Then

$$3p - 2 = e^{3c}e^{3t} = Ce^{3t} \quad \text{and} \quad p = \tfrac{1}{3}Ce^{3t} + \tfrac{2}{3}, \quad \text{where } C \text{ is a constant.}$$

(b) Assume that the initial population is $\frac{1}{2}$. Then $p(0) = \frac{1}{2}$ and $\frac{1}{2} = \frac{1}{3}C + \frac{2}{3}$ so that $C = -\frac{1}{2}$. So

$$p(t) = -\tfrac{1}{6}e^{3t} + \tfrac{2}{3}.$$

Then

$$p(t) = 0 \qquad \text{when} \qquad 3t = \log(-\tfrac{2}{3}\cdot(-6)) = \log(4).$$

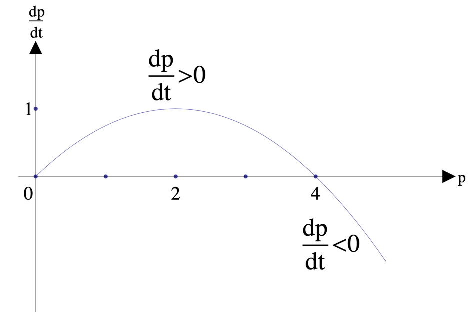
So the population dies out when $t = \frac{1}{3}\log(4)$. □

Example 4.14. A population is modelled by the logistic model

$$\frac{dp}{dt} = p(1 - \tfrac{1}{4}p),$$

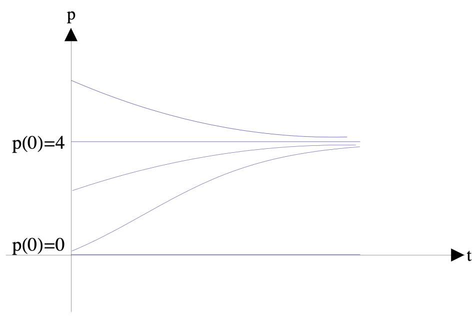
find the equilibrium solutions, determine their stability and sketch the family of solutions for the ODE.

Proof. The phase plot is



Graph of $\frac{dp}{dt}$ as a function of p

and the phase plot indicates that the solution plot is



Solutions of $\frac{dx}{dt} = p - \frac{1}{4}p^2$

The equilibrium solutions are $p = 0$ and $p = 4$ since these make $p(1 - \frac{1}{4}p) = \frac{dp}{dt} = 0$.

(A) If the initial population is greater than 4 then the population decreases to 3 as $t \to \infty$,

(B) If the initial population is less than 4 then the population decreases to 3 as $t \to \infty$,

(C) If the population starts out at 3 then it stays 3 forever. This equilibrium is stable and is not disturbed by small aberrations. After any emergency it will naturally return to the stable value of 3 as time passes.

□

Example 4.15. (Population model) For a population described by the logistic model with harvesting

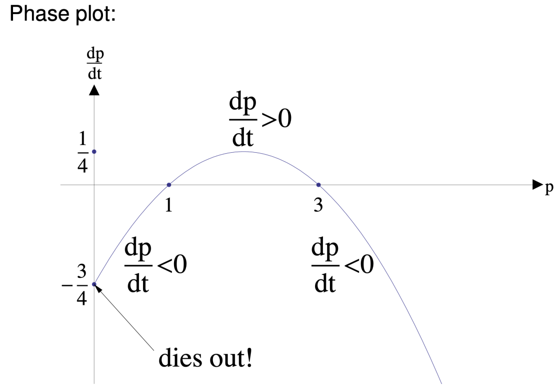
$$\frac{dp}{dt} = p(1 - \tfrac{1}{4}p) - \tfrac{3}{4},$$

determine the long term consequences for the population predicted by the model.

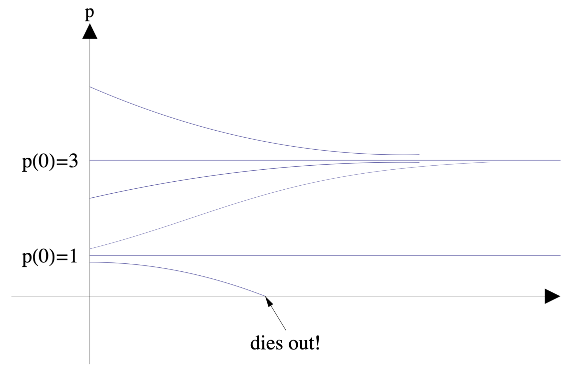
Proof. The equation is

$$\frac{dp}{dt} = p(1 - \tfrac{1}{4}p) - \tfrac{3}{4} = -\tfrac{1}{4}p^2 + p - \tfrac{3}{4} = -\tfrac{1}{4}(p^2 - 4p + 3) = -\tfrac{1}{4}(p - 1)(p - 3).$$

Thus the phase plot (the plot of $\frac{dp}{dt}$ versus p) and the solution plot (the plot of p versus t) are



Graph of $\frac{dp}{dt}$ as a function of p



Graph of p as a function of t

(a) If the initial population is greater than 1 then the system is stable and the population approaches 3 in the long term.

(b) If the iniial population is less than 1 then the population will die out in the long term.

(c) If the initial population is 1 then the population will stay constant at 1, but this is not a stable situation, any small deviation will cause the population to either start to die out, or start to increase to 3.

□

4.6 Mixing

Example 4.16. Effluent (pollutant concentration $2\text{g}/\text{m}^3$) flows into a pond (volume 1000m^3, initially 100g pollutant) at a rate of $10\text{m}^3/\text{min}$. The pollutant mixes quickly and uniformly with pond water and flows out of the pond at a rate of $10\ \text{m}^3/\text{min}$.

(a) Find the concentration of pollutant in the pond at any time, and intepret the long term behaviour of the system.

(b) Derive an ODE describing the amount x of pollutant in the lake at time t (minutes), if the input flow rate is decreased to $5\ \text{m}^3/\text{min}$.

Proof. (a) The volume of water in the pond is

$$V = 1000.$$

Let p be the amount of pollutant in the pond. Then

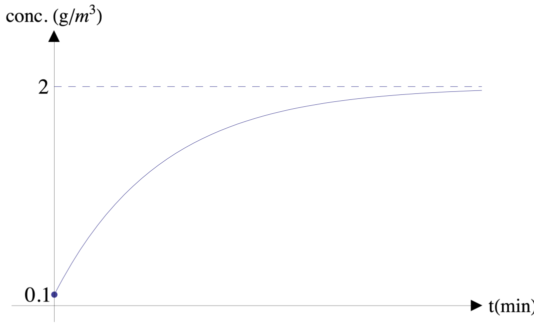
$$\frac{dp}{dt} = 2\cdot 10 - \frac{p}{1000}\cdot 10 = 20 - \frac{p}{100} = -\frac{1}{100}(p-2000), \qquad \text{with } p(0) = 100.$$

So

$$\log(p-2000) = -\frac{1}{100}t + c, \qquad \text{and} \qquad p = 2000 + Ce^{-\frac{1}{100}t} \qquad \text{with } C = e^c == -1900.$$

The concentration of pollutant in the pond is

$$K = \frac{p}{1000} = 2 - 1.9e^{-\frac{1}{100}t}.$$



Graph of K as a function of t

(b) The volume of water in the pond at time t is

$$V = 1000 - 5t.$$

Let p be the amount of pollutant in the pond. Then

$$\frac{dp}{dt} = 2 \cdot 5 - \frac{p}{V} \cdot 10 = 10 - \frac{10p}{1000 - 5t}.$$

At time $t = 200$ the pond is empty. The equation is

$$\frac{dp}{dt} + \Big(\frac{10}{1000 - 5t}\Big)p = 10,$$

which can probably be solved with an integrating factor (product rule). □

4.7 Oscillating motion: swings and springs

Example 4.17. A $\frac{40}{49}$kg mass streches a spring hanging from a fixed support by 0.2m. The mass is released from the equilibrium position with a downward velocity of 3m/s. Find the position of the mass y below the equilibrium at any time t, if the damping constant β is

(a) 0,

(b) $\frac{160}{49}$,

(c) $\frac{80}{7}$,

(d) $\frac{2000}{49}$.

Proof. Let $y(t)$ be the position of the mass at time t, let $a(t)$ be the acceleration of the mass at time t and let F denote the force on the mass.

$$\frac{d^2y}{dt^2} = a \qquad \text{and} \qquad F = ma = my''.$$

Let

$g = 9.8$m/s be the acceleration due to gravity,

k be the spring constant,

β be the damping constant,

s be the position of the mass at rest.

Then

$$\begin{aligned} my'' = F &= (\text{gravitational force}) + (\text{restoring force}) + (\text{damping force}) \\ &= mg + (-k)(s+y) + (-\beta)y' = -ky - \beta y'. \end{aligned}$$

When the mass is at rest then $y'' = 0$ and $y' = 0$ and $y = 0$ so that

$$0 = mg + (-k)(s+0) - \beta \cdot 0 \qquad \text{and} \qquad s = \frac{mg}{k}.$$

In our case,

$$m = \tfrac{40}{49} \quad \text{and} \quad k = \frac{mg}{s} = \frac{\frac{40}{49}\cdot 9.8}{0.2} = 40 \qquad \text{so that} \qquad \tfrac{40}{49}y'' + \beta y' + 40y = 0.$$

In our case the mass is released from the equilibrium (rest) position with a downward velocity of 3 so that

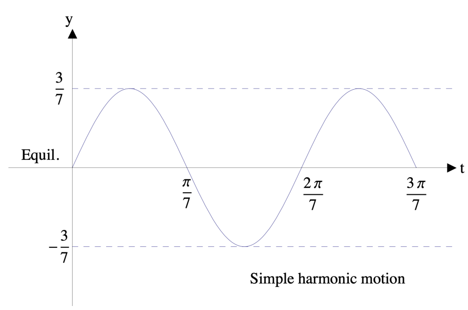
$$y(0) = 0 \qquad \text{and} \qquad y'(0) = 3.$$

(a) If $\beta = 0$ then the equation is $y'' + 49y = 0$. Let $D = \frac{d}{dt}$. Then the equation is

$$(D - 7i)(D + 7i)y = 0 \qquad \text{which has solutions} \qquad y = c_1 e^{7it} + c_2 e^{-7it},$$

where $c_1, c_2 \in \mathbb{C}$. Since $y(0) = 0$ then $c_1 + c_2 = 0$ and since $y'(0) = 3$ then $3 = 7ic_1 - 7ic_2$. So

$$y = \frac{3}{14i}e^{7it} - \frac{3}{14i}e^{-7it} = \frac{3}{7}\sin(7t)$$



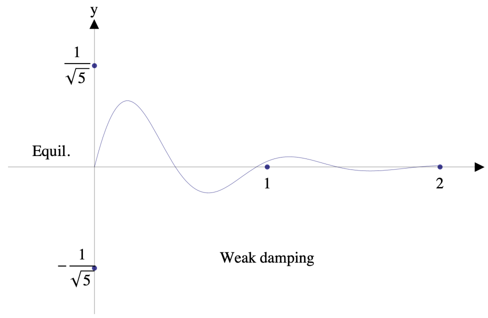
Graph of $y = \frac{3}{7}\sin(7t)$

(b) if $\beta = \frac{160}{49}$ then the equation is $y'' + 4y' + 49y = 0$. Let $D = \frac{d}{dt}$. Then the equation is

$$(D - (-2+3i))(D - (-2+3i)) = 0 \qquad \text{which has solutions} \qquad y = c_1 e^{(-2+3\sqrt{5}i)t} + c_2 e^{(-2-3\sqrt{5}i)t},$$

where $c_1, c_2 \in \mathbb{C}$. Since $y(0) = 0$ then $c_1 + c_2 = 0$ and since $y'(0) = 3$ then $3 = (-2 + 3\sqrt{5}i)c_1 + (-2 - 3\sqrt{5}i)c_2$. So

$$y = \frac{3}{6\sqrt{5}i}e^{-2t}e^{3\sqrt{5}it} - \frac{3}{6\sqrt{5}i}e^{-2t}e^{-3\sqrt{5}it} = \frac{1}{\sqrt{5}}e^{-2t}\sin(3\sqrt{5}t).$$



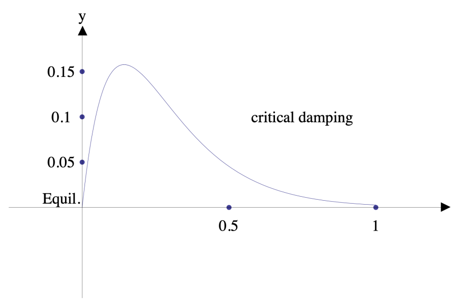
Graph of $y = \frac{1}{\sqrt{5}} e^{-2t} \sin(3\sqrt{5}t)$

(c) if $\beta = \frac{80}{7}$ then the equation is $y'' + 14y' + 49y = 0$. Let $D = \frac{d}{dt}$. Then the equation is

$$(D+7)(D+7) = 0 \qquad \text{which has solutions} \qquad y = c_1 e^{-7t} + c_2 t e^{-7t},$$

where $c_1, c_2 \in \mathbb{C}$. Since $y(0) = 0$ then $c_1 = 0$ and since $y'(0) = 3$ then $3 = -7c_1 + c_2$. So

$$y = 3te^{-7t}.$$



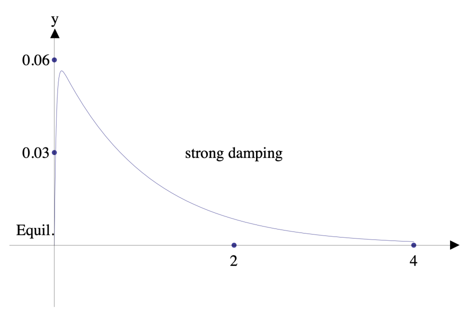
Graph of $y = 3te^{-7t}$

(d) if $\beta = \frac{2000}{49}$ then the equation is $y'' + 50y' + 49y = 0$. Let $D = \frac{d}{dt}$. Then the equation is

$$(D+1)(D+49) = 0 \qquad \text{which has solutions} \qquad y = c_1 e^{-t} + c_2 e^{-49t},$$

where $c_1, c_2 \in \mathbb{C}$. Since $y(0) = 0$ then $0 = c_1 + c_2$ and since $y'(0) = 3$ then $3 = -c_1 - 49c_2$. So

$$y = \frac{3}{48} e^{-t} - \frac{3}{48} e^{-49t} = \frac{1}{16} e^{-t} - \frac{1}{16} e^{-49t}.$$



Graph of $y = \frac{1}{16} e^{-t} - \frac{1}{16} e^{-49t}$

$\square$

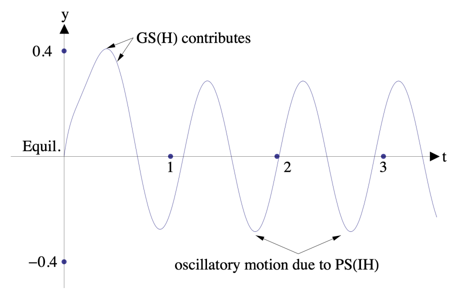
Example 4.18. Apply an external downwards force $f(t) = \frac{160}{7} \sin(7t)$ in Example 4.17 (a) and (c).

Proof. (c) The position y of the spring satisfies the equation $y'' + 14y' + 49y = \frac{160}{7}\sin(7t)$. This has general solution

$$y = Ae^{-7t} + Bte^{-7t} - \tfrac{2}{7}\cos(7t), \quad \text{where } A \text{ and } B \text{ are constants.}$$

The initial conditions $y(0) = 0$ and $y'(0) = 3$ give that $A = \frac{2}{7}$ and $B = 5$. So the position of the mass on the spring at time t is

$$y = \tfrac{2}{7}e^{-t} + 5te^{-7t} - \tfrac{2}{7}\cos(7t) = \left(\tfrac{2}{7} + 5t\right)e^{-7t} - \tfrac{2}{7}\cos(7t).$$



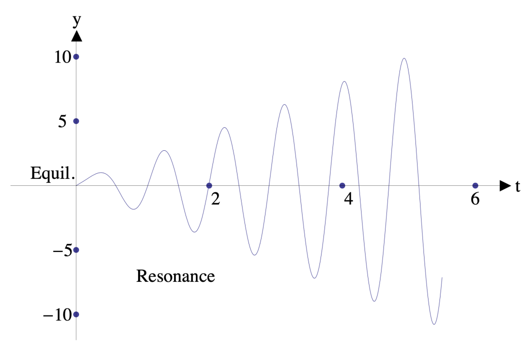
Graph of $y = \left(\frac{2}{7} + 5t\right)e^{-7t} - \frac{2}{7}\cos(7t)$

(a) The position y of the spring satisfies the equation $y'' + 49y = \frac{160}{7}\sin(7t)$. This has general solution

$$y = A\cos(7t) + B\sin(7t) - 2t\cos(7t), \qquad \text{where } A \text{ and } B \text{ are constants.}$$

The initial conditions give $A = 0$ and $b = \frac{5}{7}$ so that the position of the mass on the spring at time t is

$$y = \tfrac{5}{7}\sin(7t) - 2t\cos(7t).$$



Graph of $y = \frac{5}{7}\sin(7t) - 2t\cos(7t)$

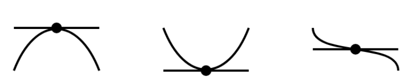
□

4.8 Optimization

Example 4.19. Find the local maxima and minima of $f(x) = 2x^3 - 24x + 107$ in the interval $\mathbb{R}_{[1,3]}$.

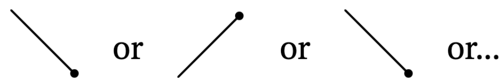
Proof. The critical points are

(a) points where $\frac{df}{dx}$ is 0;



(b) points where $f(x)$ is not continuous or not differentiable;

(c) points on the boundary of where $f(x)$ is defined.



If $f(x) = 2x^3 - 24x + 107$ in the interval $\mathbb{R}_{[1,3]}$ then $x = 1$ and $x = 3$ are critical points of type (c), and

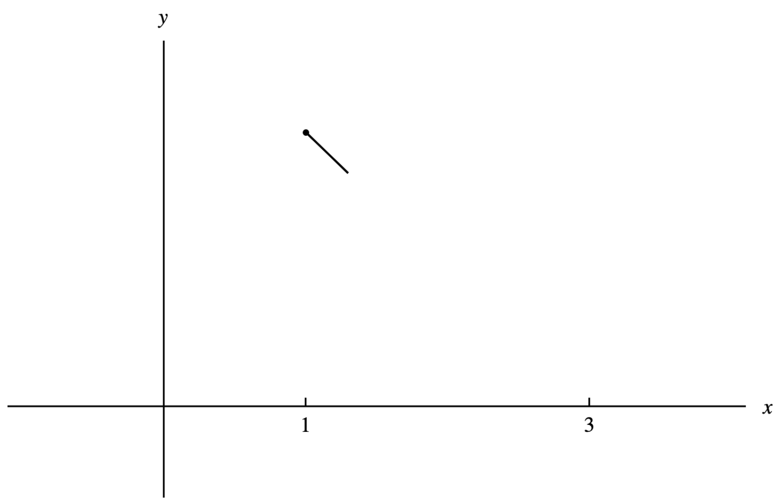
$$\frac{df}{dx} = 6xx^2 - 24 \quad \text{and} \quad 6x^2 - 24 = 0 \text{ when } x^2 = \tfrac{24}{6} = 4.$$

So $x \in \{-2, 2\}$ when $\frac{df}{dx}$ is 0. So $x = 2$ is a critical point in $\mathbb{R}_{[1,3]}$.

Critical point $x = 1$:

$$\frac{df}{dx}\Big]_{x=1} = (6x^2 - 24)\Big]_{x=1} = 6 - 24 < 0.$$

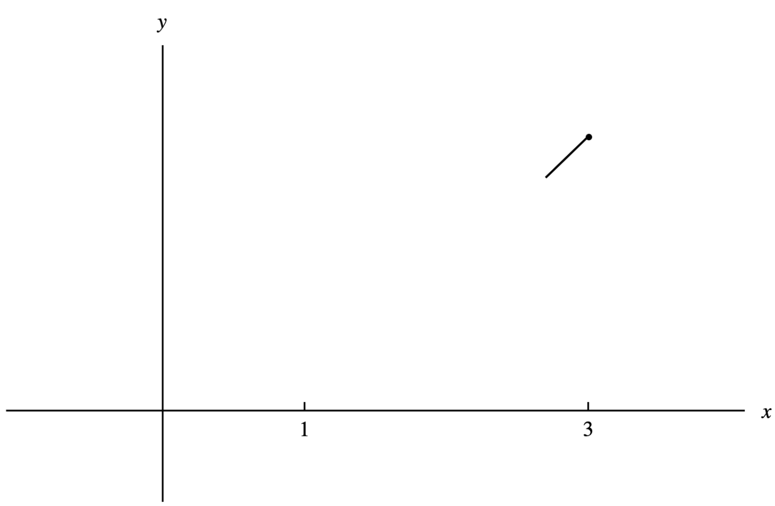
So $f(x)$ is decreasing at $x = 1$. So (from the picture) $x = 1$ is a *maximum*.



Critical point $x = 3$:

$$\frac{df}{dx}\Big]_{x=3} = (6x^2 - 24)\Big]_{x=3} = 6 \cdot 3^2 - 24 = 30 > 0.$$

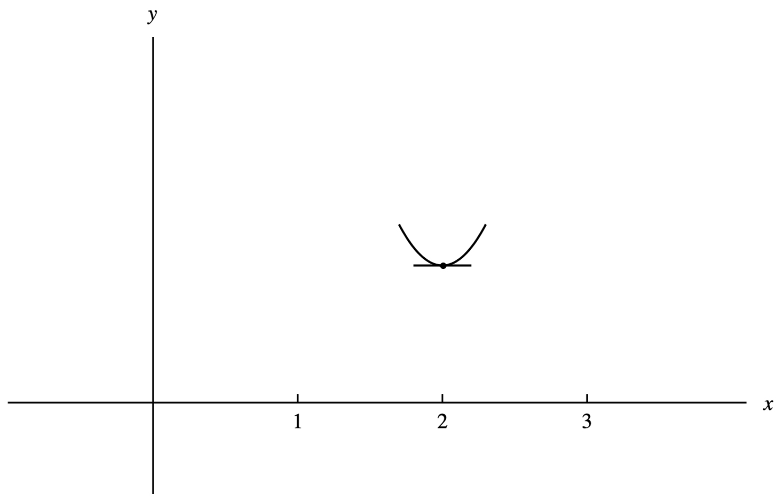
So $f(x)$ is increasing at $x = 3$. So (from the picture) $x = 3$ is a *maximum*.



Critical point $x = 2$:

$$\frac{df}{dx}\Big]_{x=2} = 0 \qquad \text{and} \qquad \frac{d^2f}{dx^2}\Big]_{x=2} = 12x\Big]_{x=2} = 24 > 0.$$

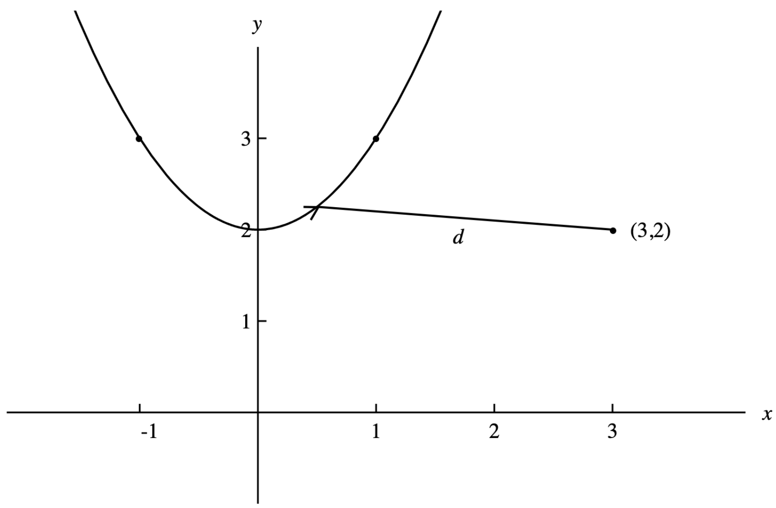
So $f(x)$ is slope zero and concave up at $x = 2$. So $x = 2$ is a minimum.



□

Example 4.20. An enemy jet is flying along the curve $y = x^2 + 2$. A soldier is placed at the poit (3.2). At what point will the jet be at when the soldier and the jet will be the closest?

Proof.



If the jet is at the point (p, q) then the distance between them is

$$d = \sqrt{(p-3)^2 + (q-2)^2}.$$

The point (p, q) is on the curve $y = x^2 + 1$ so $q = p^2 + 2$.
So $d = \sqrt{(p-3)^2 + (p^2 + 2 - 2)^2}$.
We want to minimize d (as the jet moves, i.e. as p changes).

The distance d will be minimum at the same time that d^2 will be minimum.
So we can minimize d^2.

$$d^2 = (p-3)^2 + (p^2)^2 = (p-3)^2 + p^4.$$

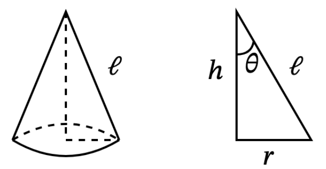
Find a critical point. When is

$$\frac{dd^2}{dp} = 2(p-3) + 4p^3 = 4p^3 + 2p - 6 = (p-1)(4p^2+4p+6) \qquad \text{equal to 0?}$$

Since $\frac{d(d^2)}{dp}\big]_{p=1} = 0$ then $p = 1$ is a critical point. The picture helps confirm that when the jet is at $(1,3)$ (i.e. $p = 1$ and $q = 3$) then the distance to the soldier is minimum. □

Example 4.21. Maximize the volume of a cone with a given slant hieght. Show that the angle of inclination is $\tan^{-1}(\sqrt{2})$.

Proof.



ℓ =slant height,
θ = angle of inclination,
$\frac{r}{\ell} = \sin(\theta)$,
$\frac{h}{\ell} = \cos(\theta)$.

The volume of a cone is

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(\ell\sin(\theta))^2 h = \frac{1}{3}\pi(\ell\sin(\theta))^2 \ell\cos(\theta).$$

The value of ℓ (the slant height) is fixed. We want to maximize V as θ changes.

$$\begin{aligned}
\frac{dV}{d\theta} &= \frac{d\frac{1}{3}\pi\ ell l^3 \sin(\theta)^2\cos(\theta)}{d\theta} \\
&= \frac{1}{3}\pi\ell^3(2\sin(\theta)\cos(\theta)^2 - \sin(\theta)^2\sin(\theta)) \\
&= \frac{1}{3}\pi\ell^3\sin(\theta)(2\cos(\theta)^2 - \sin(\theta)^2),
\end{aligned}$$

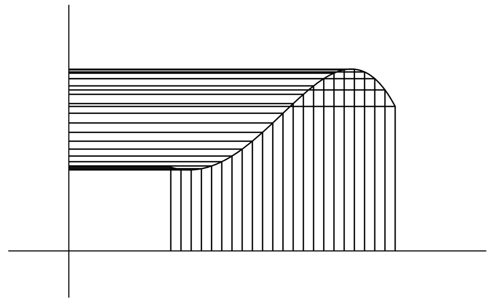
A critical point is when $\frac{dV}{d\theta}$ is zero: where $2\cos(\theta)^2 - \sin(\theta)^2 = 0$ or $\sin(\theta) = 0$.

So $2 = \tan(\theta)^2$ or $\theta = 0$.
So $\sqrt{2} = \tan(\theta)$ or $\theta = 0$.
So $\theta = \tan^{-1}(\sqrt{2})$ or $\theta = 0$.

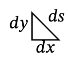
When $\theta = 0$ the cone is infinitely thin which does not have maximum volume.
So $\theta = \tan^{-1}(\sqrt{2})$ maximizes volume. □

4.9 Lengths and surface area

Idea: Use the grid to slice up the curve into little pieces.



Each little piece

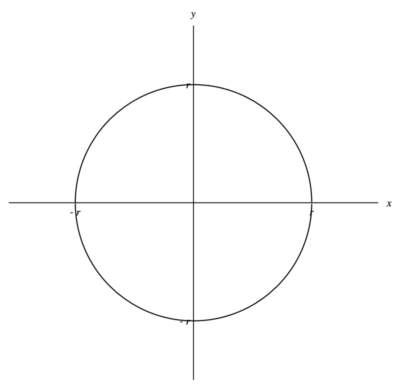


has $ds = \sqrt{(dx)^2 + (dy)^2}$.

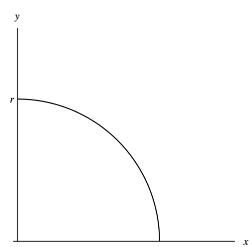
Add up the lengths of the little pieces with an integral.

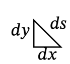
Example 4.22. Use integration to find the length of a circle of radius r.

Proof.



The length of the circle is 4 times the length of



(a) Divide this part of the curve into little pieces  .

(b) Each little piece has length $ds = \sqrt{(dx)^2 + (dy)^2}$.

(c) Add up the lengths of the little pieces with an integral.

$$\begin{aligned}\int_{x=0}^{x=r} ds &= \int_{x=0}^{x=r} \sqrt{(dx)^2+(dy)^2} \\ &= \int_{x=0}^{x=r} \frac{\sqrt{(dx)^2+(dy)^2}}{dx}\,dx \\ &= \int_{x=0}^{x=r} \sqrt{1+\big(\frac{dy}{dx}\big)^2}\,dx \\ &= \int_{x=0}^{x=r} \sqrt{1+\big(-\frac{2x}{2y}\big)^2}\,dx,\end{aligned}$$

since

$$x^2+y^2=r^2 \qquad \text{gives} \qquad 2x+2y\frac{dy}{dx}=0 \qquad \text{which gives} \quad \frac{dy}{dx}[=-\frac{2x}{2y}.$$

So

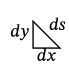
$$\begin{aligned}\int_{x=0}^{x=r} ds &= \int_{x=0}^{x=r} \sqrt{1+\frac{x^2}{y^2}}\,dx \\ &= \int_{x=0}^{x=r} \sqrt{\frac{y^2+x^2}{y^2}}\,dx \\ &= \int_{x=0}^{x=r} \sqrt{\frac{r^2}{r^2-x^2}}\,dx \\ &= \int_{x=0}^{x=r} \sqrt{\frac{1}{1-\big(\frac{x}{r}\big)^2}}\,dx \\ &= \int_{x=0}^{x=r} \frac{r\cdot\frac{1}{r}}{\sqrt{1-\big(\frac{x}{r}\big)^2}}\,dx \\ &= r\sin^{-1}\big(\frac{x}{r}\big)\Big]_{x=0}^{x=r} \\ &= r\sin^{-1}(1)-r\sin^{-1}(0) \\ &= r\frac{\pi}{2}-0.\end{aligned}$$

So the total length of the circle is

$$4\Big(\frac{r\pi}{2}\Big)=2\pi r.$$

□

Example 4.23. Find the length of the curve $x=t-\sin(t)$, $y=1-\cos(t)$, where $t\in\mathbb{R}_{[0,2\pi]}$.

Proof. (a) Divide the curve into little pieces 

.

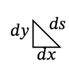
(b) Each little piece has length $ds=\sqrt{(dx)^2+(dy)^2}$.

(c) Add up the lengths of the little pieces.

$$\begin{aligned}
\int_{t=0}^{t=2\pi} ds &= \int_{t=0}^{t=2\pi} \sqrt{(dx)^2+(dy)^2} \\
&= \int_{t=0}^{t=2\pi} \frac{\sqrt{(dx)^2+(dy)^2}}{dt}\, dt \\
&= \int_{t=0}^{t=2\pi} \sqrt{\Big(\frac{dx}{dt}\Big)^2+\Big(\frac{dy}{dt}\Big)^2}\, dt \\
&= \int_{t=0}^{t=2\pi} \sqrt{(1-\cos(t))^2+\sin(t)^2}\, dt \\
&= \int_{t=0}^{t=2\pi} \sqrt{1-2\cos(t)+\cos(t)^2+\sin(t)^2}\, dt \\
&= \int_{t=0}^{t=2\pi} \sqrt{1-2\cos(t)+1}\, dt \\
&= \int_{t=0}^{t=2\pi} \sqrt{2-2\cos(t)}\, dt \\
&= \int_{t=0}^{t=2\pi} \sqrt{2-2\cos\Big(\frac{t}{2}+\frac{t}{2}\Big)}\, dt \\
&= \int_{t=0}^{t=2\pi} \sqrt{2-2\Big(\cos\Big(\frac{t}{2}\Big)^2-\sin\Big(\frac{t}{2}\Big)^2\Big)}\, dt \\
&= \int_{t=0}^{t=2\pi} \sqrt{2}\sqrt{1-\cos\Big(\frac{t}{2}\Big)^2+\sin\Big(\frac{t}{2}\Big)^2}\, dt \\
&= \int_{t=0}^{t=2\pi} \sqrt{2}\sqrt{\sin\Big(\frac{t}{2}\Big)^2+\sin\Big(\frac{t}{2}\Big)^2}\, dt \\
&= \int_{t=0}^{t=2\pi} \sqrt{2}\sqrt{2\sin\Big(\frac{t}{2}\Big)^2}\, dt \\
&= \int_{t=0}^{t=2\pi} \sqrt{2}\sqrt{2}\sin\Big(\frac{t}{2}\Big)\, dt \\
&= 2\Big(\cos\Big(\frac{t}{2}\Big)\Big)\cdot 2\Big]_{t=0}^{t=2\pi} \\
&= -4\cos\Big(\frac{2\pi}{2}\Big)-(-4\cos(0)) = (-4)(-1)+4\cdot 1 = 4+4 = 8.
\end{aligned}$$

□

Example 4.24. Find the length of the curve$x=\frac{3}{5}y^{\frac{5}{3}}-\frac{3}{4}y^{\frac{1}{3}}$ from $y=0$ to $y=1$.

Proof. (a) Divide the curve into little pieces 

.

(b) Each little piece has length $ds=\sqrt{(dx)^2+(dy)^2}$.

(c) Add up the lengths of the little pieces.

$$\begin{aligned}
\int_{y=0}^{y=1} ds &= \int_{y=0}^{y=1} \sqrt{(dx)^2+(dy)^2} \\
&= \int_{y=0}^{y=1} \frac{\sqrt{(dx)^2+(dy)^2}}{dy}\, dy \\
&= \int_{y=0}^{y=1} \sqrt{\frac{(dx)^2+(dy)^2}{(dy)^2}}\, dy \\
&= \int_{y=0}^{y=1} \sqrt{\big(\frac{dx}{dy}\big)^2+1}\, dy \\
&= \int_{y=0}^{y=1} \sqrt{\big(y^{\frac{2}{3}}-\frac{1}{4}y^{-\frac{2}{3}}\big)^2+1}\, dy \\
&= \int_{y=0}^{y=1} \sqrt{y^{\frac{4}{3}}-\frac{1}{2}+\frac{1}{16}y^{-\frac{4}{3}}+1}\, dy \\
&= \int_{y=0}^{y=1} \sqrt{y^{\frac{4}{3}}+\frac{1}{2}+\frac{1}{16}y^{-\frac{4}{3}}}\, dy \\
&= \int_{y=0}^{y=1} \sqrt{\big(y^{\frac{2}{3}}+\frac{1}{4}y^{-\frac{2}{3}}\big)^2}\, dy \\
&= \int_{y=0}^{y=1} \big(y^{\frac{2}{3}}+\frac{1}{4}y^{-\frac{2}{3}}\big)\, dy \\
&= \big(\frac{3}{5}y^{\frac{5}{3}}+\frac{1}{4}\cdot 3y^{\frac{1}{3}}\big)\Big]_{y=0}^{y=1} = \frac{3}{5}+\frac{3}{4}-(0+0) = \frac{12}{20}+\frac{15}{20} = \frac{27}{20}.
\end{aligned}$$

□

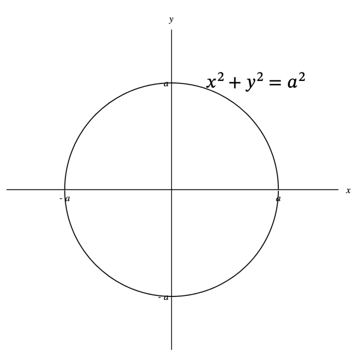
Example 4.25. Let $a \in \mathbb{R}_{>0}$. Find the surface area obtained by rotating the curve determined by $x = a\cos(\theta)^3$, $y = a\sin(\theta)^3$ about the x-axis.

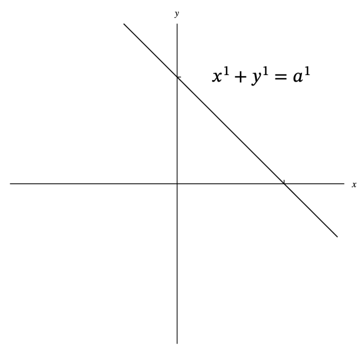
Proof. To graph the curve:

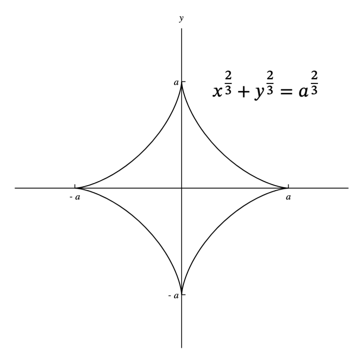
$$\cos(\theta)^3 = \frac{x}{a} \text{ and } \sin(\theta)^3 = \frac{y}{a}. \qquad \text{So} \quad \cos(\theta) = \big(\frac{x}{a}\big)^{\frac{1}{3}} \quad \text{and} \quad \sin(\theta) = \big(\frac{y}{a}\big)^{\frac{1}{3}}.$$

Since $\cos(\theta)^2 + \sin(\theta)^2 = 1$ then

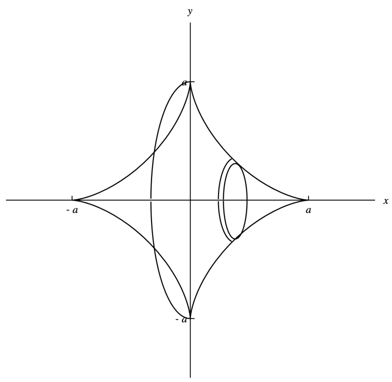
$$\big(\frac{x}{a}\big)^{\frac{2}{3}} + \big(\frac{y}{a}\big)^{\frac{2}{3}} = 1, \qquad \text{which is} \qquad x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}.$$

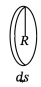






So when this is rotated about the x-axis



Slice: 

Surface area of a slice: $\pi R^2\, ds$.
Add up slices from $x = 0$ to $x = a$ and then multiply by 2.

$$\begin{aligned}
\text{Surface area} &= 2\int_{x=0}^{x=a} \pi R^2\, ds \\
&= 2\int_{x=0}^{x=a} \pi y^2 \sqrt{(dx)^2+(dy)^2} \\
&= 2\int_{x=0}^{x=a} \pi y^2 \frac{\sqrt{(dx)^2+(dy)^2}}{d\theta}\, d\theta \\
&= 2\int_{x=0}^{x=a} \pi y^2 \sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\, d\theta
\end{aligned}$$

Since $x = a\cos(\theta)^3$ and $y = a\sin(\theta)^3$ then

$$\frac{dx}{d\theta} = -3a\cos(\theta)^2\sin(\theta) \qquad \text{and} \qquad \frac{dy}{d\theta} = -3a\sin(\theta)^2\cos(\theta).$$

So

$$\begin{aligned}
\text{Surface area} &= 2\int_{x=0}^{x=a} \pi y^2 \sqrt{(-3a\cos(\theta)^2\sin(\theta))^2+(3a\sin(\theta)^2\cos(\theta))^2}\, d\theta \\
&= 2\int_{x=0}^{x=a} \pi a^2\sin(\theta)^6 \sqrt{9a^2\cos(\theta)^4\sin(\theta)^2+9a^2\sin(\theta)^4\cos(\theta)^2}\, d\theta \\
&= 2\int_{x=0}^{x=a} \pi a^2\sin(\theta)^6 3a\sin(\theta)\cos(\theta)\, d\theta \\
&= 2\int_{x=0}^{x=a} 3\pi a^3\sin(\theta)^7\cos(\theta)\, d\theta \\
&= 6\pi a^3 \frac{\sin(\theta)^8}{8}\Big]_{x=0}^{x=a} \\
&= \frac{6\pi a^3}{8}\sin(\theta)^8\Big]_{a\cos(\theta)^3=0}^{a\cos(\theta)^3=a} \\
&= \frac{3\pi a^3}{4}\sin(\theta)^8\Big]_{\cos(\theta)=0}^{\cos(\theta)=1} \\
&= \frac{3\pi a^3}{4}\sin(\theta)^8\Big]_{\theta=\frac{\pi}{2}}^{\theta=0} \\
&= \frac{3\pi a^3}{4}\sin(0)^8 - \frac{3\pi a^3}{4}\sin\left(\frac{\pi}{2}\right)^8 = -\frac{3\pi a^3}{4}\cdot 1^8 = -\frac{3\pi a^3}{4}.
\end{aligned}$$

So the surface area is $\frac{3\pi a^3}{4}$. □

4.10 Averages

Average of a bunch of numbers:

(a) Add up the numbers

(b) Divide by the number of values

Example 4.26. Compute the average of $1, 2, 3, \ldots, 100$.

Proof.

$$\begin{array}{ccccccccccccccccc} 1 & + & 2 & + & 3 & = & 4 & = & \cdots & + & 97 & + & 98 & + & 99 & + & 100 \\ 100 & + & 99 & + & 98 & + & 97 & + & \cdots & + & 4 & + & 3 & + & 2 & + & 1 \\ \hline 101 & + & 101 & + & 101 & + & 101 & + & \cdots & + & 101 & + & 101 & + & 101 & + & 101 & = & 10100 \end{array}$$

So

$$1 + 2 + 3 + \cdots + 100 = \frac{10100}{2} = 5050.$$

So the average is $\frac{1}{100}(1 + 2 + 3 + \cdots + 100) = \frac{5050}{100} = 50.5$. □

Example 4.27. Compute the average of $1,\ \frac{1}{3},\ \frac{1}{3^3},\ \frac{1}{3^3},\ \ldots,\ \frac{1}{3^{50}}$.

Proof. Since

$$\begin{aligned}(1 + x + x^2 + \cdots + x^{50})(1-x) &= 1 + x + x^2 + x^3 + \cdots + x^{50} \\ &\qquad - x - x^2 - x^3 - \cdots - x^{50} - x^{51} \\ &= 1 - x^{51}\end{aligned}$$

then

$$1 + x + x^2 + \cdots + x^{50} = \frac{1 - x^{51}}{1-x}.$$

So

$$1 + \frac{1}{3} + \frac{1}{3^2} + \cdots + \left(\frac{1}{3}\right)^{50} = \frac{1 - \left(\frac{1}{3}\right)^{51}}{1 - \frac{1}{3}} = \frac{1 - \frac{1}{3^{51}}}{\frac{2}{3}} = \frac{3 - \frac{1}{3^{50}}}{2}.$$

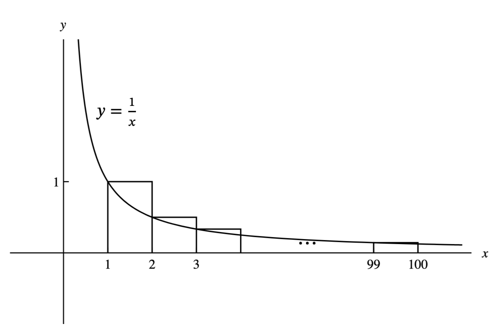
So the average is

$$\frac{1}{51}\left(1 + \frac{1}{3} + \frac{1}{3^2} + \cdots + \left(\frac{1}{3}\right)^{50}\right) = \frac{1}{51} \cdot \frac{3 - \frac{1}{3^{50}}}{2} = \frac{3 - \frac{1}{3^{50}}}{102} \approx \frac{3}{100} = .03.$$

□

Example 4.28. Estimate the average of $1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{100}$.

Proof. Since $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{99}$ is the area of the boxes in the picture



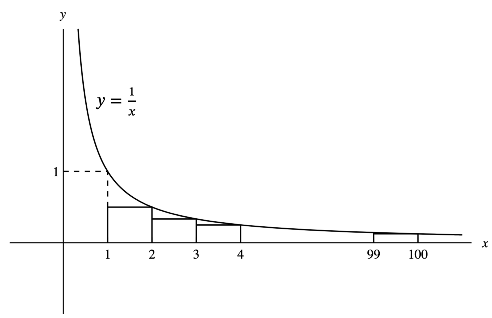
then

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{99}\geq\int_1^{100}\frac{1}{x}\,dx=\log(x)\big]_{x=1}^{x=100}=\log(100).$$

So

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{99}+\frac{1}{100}\geq\log(100)+\frac{1}{100}.$$

Since $\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{99}+\frac{1}{100}$ is the area of the boxes in the picture



then

$$\frac{1}{2}+\frac{13}{+}\cdots+\frac{1}{99}\leq\int_1^{100}\frac{1}{x}\,dx=\log(x)\big]_{x=1}^{x=100}=\log(100).$$

So

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{99}+\frac{1}{100}\leq 1+\log(100).$$

Since

$$\frac{1}{100}+\log(100)\leq 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{99}+\frac{1}{100}\leq 1+\log(100)$$

then

$$\frac{1}{101}\big(\frac{1}{100}+\log(100)\big)\leq\frac{1}{101}(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{99}+\frac{1}{100})\leq\frac{1}{101}(1+\log(100)).$$

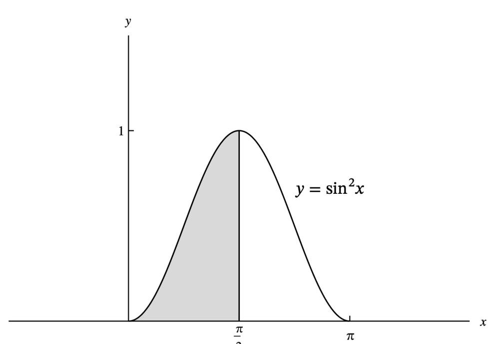
So the average of $1,\frac{1}{2},\frac{1}{3},\ldots,\frac{1}{100}$ is

$$\text{between}\quad\frac{\log(100)}{101}+\frac{1}{10100}\quad\text{and}\quad\frac{\log(100)}{101}+\frac{1}{101}.$$

□

Example 4.29. Find the average value of $f\colon\mathbb{R}_{[0,\frac{\pi}{2}]}\to\mathbb{R}$ given by $f(x)=\sin(x)^2$.

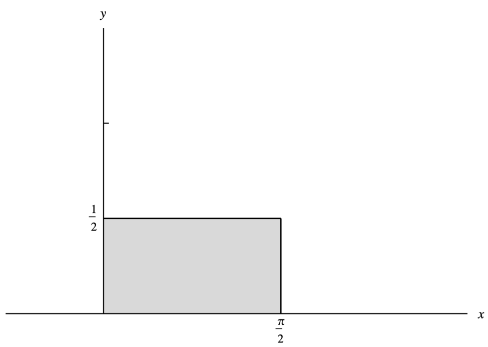
Proof.



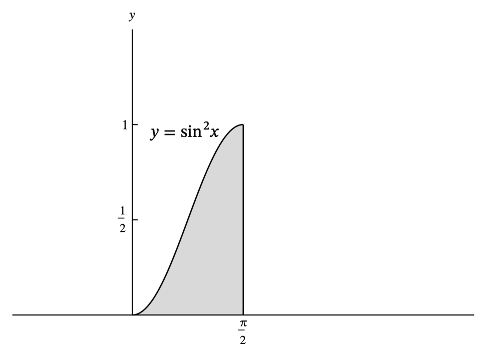
The average value is

$$\begin{aligned}
\frac{\int_0^{\frac{\pi}{2}} f(x)\,dx}{\frac{\pi}{2}-0} &= \frac{\int_0^{\frac{\pi}{2}} \sin(x)^2\,dx}{\frac{\pi}{2}-0} \\
&= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{1}{2}(\sin(x)^2+\sin(x)^2)\,dx \\
&= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{1}{2}(1-\cos(x)^2+\sin(x)^2)\,dx \\
&= \frac{1}{\pi}\int_0^{\frac{\pi}{2}} (1-(\cos(x)^2-\sin(x)^2))\,dx \\
&= \frac{1}{\pi}\int_0^{\frac{\pi}{2}} (1-\cos(2x))\,dx \\
&= \frac{1}{\pi}\big(x-\frac{\sin(2x)}{2}\big)\Big]_0^{\frac{\pi}{2}} \\
&= \frac{1}{\pi}\big(\frac{\pi}{2}-\frac{\sin(2\frac{\pi}{2})}{2}\big) - \frac{1}{\pi}\big(0-\frac{\sin(0)}{2}\big) = \frac{1}{\pi}\big(\frac{\pi}{2}-0\big) = \frac{1}{2}.
\end{aligned}$$

So



is the same area as



.

□

4.11 Center of mass

A *moment* and a *center of mass* are the *same* thing.

The *center of mass* is the average position of the mass in an object.

$$\begin{aligned}
\text{Center of mass} &= \frac{(\text{position of mass})\cdot\text{mass}}{\text{mass}} \\
&= \frac{\int (\text{position of a slice})\cdot\text{mass of a slice}}{\int \text{mass of a slice}}.
\end{aligned}$$

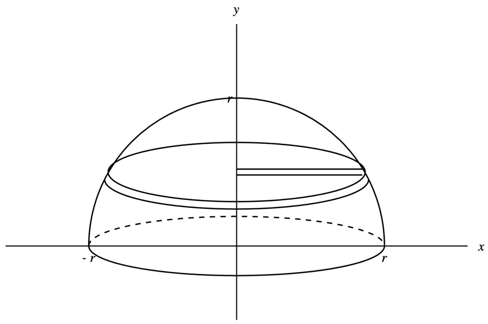
Note:

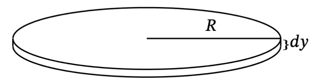
$$\text{mass of slice} = (\text{volume of slice})\cdot(\text{density of the slice}).$$

Center of mass and *center of gravity* are the *same* thing.

Example 4.30. Find the center of mass of a solid hemisphere of radui r if its density at apoint P is proportional to the distance between P and the base of the hemisphere.

Proof.



Slice: 

(a) Volume of a slice: $\pi R^2\, dy$;

(b) Density of a slice = Height of the slice;

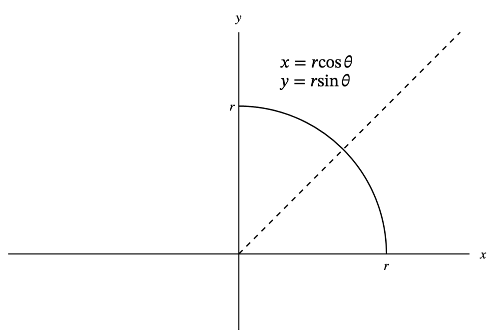
(c) Mass of a slice = $\pi R^2\, dy$(height of slice)

(d) Add slices from $y = 0$ to $y = r$.

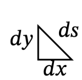
$$
\begin{aligned}
\text{Center of mass} &= \frac{\displaystyle\int (\text{center of mass of slice})\cdot(\text{mass of slice})}{\displaystyle\int (\text{mass of slice})} \\
&= \frac{\displaystyle\int_{y=0}^{y=r} y\pi R^2\,dy(\text{height of slice})}{\displaystyle\int_{y=0}^{y=r} \pi R^2\,dy(\text{height of slice})} \\
&= \frac{\displaystyle\int_{y=0}^{y=r} y\pi x^2\,dy\cdot y}{\displaystyle\int_{y=0}^{y=r} \pi x^2\,dy\cdot y} \\
&= \frac{\displaystyle\int_{y=0}^{y=r} \pi(r^2-y^2)y^2\,dy}{\displaystyle\int_{y=0}^{y=r} \pi(r^2-y^2)y\,dy} \\
&= \frac{\displaystyle\int_{y=0}^{y=r} (\pi r^2y^2-\pi y^4\,dy}{\displaystyle\int_{y=0}^{y=r} (\pi r^2y-\pi y^3)\,dy} \\
&= \frac{\left(\frac{\pi r^2y^3}{3}-\frac{\pi y^5}{5}\right)\Big]_{y=0}^{y=r}}{\displaystyle\int_{y=0}^{y=r} \left(\frac{\pi r^2y^2}{2}-\frac{\pi y^4}{4}\right)\Big]_{y=0}^{y=r}} \\
&= \frac{\frac{\pi r^5}{3}-\frac{\pi r^5}{5}-(0-0)}{\frac{\pi r^4}{2}-\frac{\pi r^4}{4}-(0-0)} = \frac{\pi r^5\left(\frac{1}{3}-\frac{1}{5}\right)}{\pi r^4\left(\frac{1}{2}-\frac{1}{4}\right)} = r\frac{\left(\frac{2}{15}\right)}{\left(\frac{1}{4}\right)} = r\frac{2}{15}\cdot 4 = \frac{8r}{15}.
\end{aligned}
$$

So the center of mass is at $\left(0, \frac{8r}{15}\right)$. □

Example 4.31. Find the center of gravity of the arc length of one quadrant of the circle.



Proof. The center of mass will be on the line $y = x$ so its x-coordinate and its y-coordinate will be the same.

Chop up the curve into little pieces 

(a) The mass of the little piece is (density) $\cdot\, ds$.

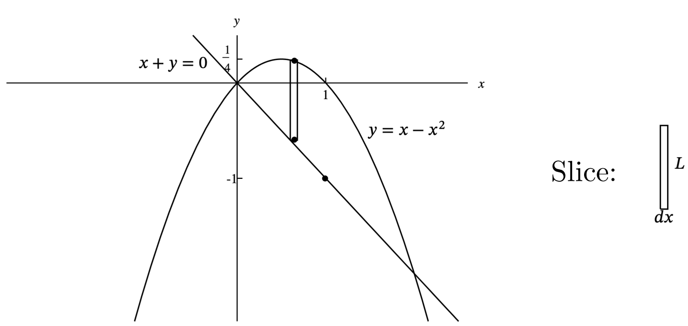
(b) Add up the little pieces from $\theta = 0$ to $\theta = \frac{\pi}{2}$.

$$\begin{aligned}
x\text{-coordinate of center of mass} &= \frac{\int_{\theta=0}^{\theta=\frac{\pi}{2}} x\delta\, ds}{\int_{\theta=0}^{\theta=\frac{\pi}{2}} \delta\, ds} \qquad \text{(where } \delta \text{ is the density)} \\
&= \frac{\int_{\theta=0}^{\theta=\frac{\pi}{2}} x\delta\sqrt{(dx)^2+(dy)^2}}{\int_{\theta=0}^{\theta=\frac{\pi}{2}} \delta\sqrt{(dx)^2+(dy)^2}} \\
&= \frac{\int_{\theta=0}^{\theta=\frac{\pi}{2}} x\delta\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}d\theta}{\int_{\theta=0}^{\theta=\frac{\pi}{2}} \delta\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\,d\theta} \\
&= \frac{\int_{\theta=0}^{\theta=\frac{\pi}{2}} \delta r\cos(\theta)\sqrt{(-r\sin(\theta))^2+(r\cos(\theta))^2}d\theta}{\int_{\theta=0}^{\theta=\frac{\pi}{2}} \delta\sqrt{(-r\sin(\theta))^2+(r\cos(\theta))^2}\,d\theta} \\
&= \frac{\int_{\theta=0}^{\theta=\frac{\pi}{2}} \delta r\cos(\theta)\sqrt{r^2\sin(\theta)^2+r^2\cos(\theta)^2}d\theta}{\int_{\theta=0}^{\theta=\frac{\pi}{2}} \delta\sqrt{r^2\sin(\theta)^2+r^2\cos(\theta)^2}\,d\theta} \\
&= \frac{\int_{\theta=0}^{\theta=\frac{\pi}{2}} \delta r\cos(\theta) r d\theta}{\int_{\theta=0}^{\theta=\frac{\pi}{2}} \delta r\, d\theta} \\
&= \frac{\delta r^2\sin(\theta)\big]_{\theta=0}^{\theta=\frac{\pi}{2}}}{\delta r\theta\big]_{\theta=0}^{\theta=\frac{\pi}{2}}} \\
&= \frac{\delta r^2\sin(\frac{\pi}{2}) - \delta r^2\cdot 0}{\delta r\frac{\pi}{2} - \delta r\cdot 0} \\
&= \frac{\delta r^2}{\delta r\frac{\pi}{2}} = \frac{2r}{\pi}.
\end{aligned}$$

□

Example 4.32. Find the center of gravity of the area bounded by the curve $y = x - x^2$ and the line $x + y = 0$.

Proof.



(a) mass of slice: (density) $L\,dx$;

(b) y-coordinate of center of mass of slice is at halfway between top and bottom;

(c) x-coordinate of center of mass of slice is at x-position of slice;

(d) Add up slices from $x = 0$ to the x-value at the right intersection point.

When $x+y=0$ and $y=x-x^2$ intersect $y=-x=x-x^2$.

$$\text{So } x^2-2x=0, \qquad \text{So } x(x-2)=0. \qquad \text{So } x=0 \text{ or } x=2.$$

$$\begin{aligned}
x\text{-coordinate of center of mass of area} &= \frac{\displaystyle\int_{x=0}^{x=2} (x\text{-position of slice})\delta L\,dx}{\displaystyle\int_{x=0}^{x=2} \delta L\,dx} \qquad \text{(where } \delta \text{ is the density)} \\
&= \frac{\displaystyle\int_{x=0}^{x=2} x\delta(y_{\text{top}-y_{\text{bottom}}})\,dx}{\displaystyle\int_{x=0}^{x=2} \delta(y_{\text{top}-y_{\text{bottom}}})\,dx} \\
&= \frac{\displaystyle\int_{x=0}^{x=2} x\delta((x-x^2)-(-x))\,dx}{\displaystyle\int_{x=0}^{x=2} \delta((x-x^2)-(-x))\,dx} \\
&= \frac{\displaystyle\int_{x=0}^{x=2} x\delta(2x-x^2)\,dx}{\displaystyle\int_{x=0}^{x=2} \delta(2x-x^2)\,dx} \\
&= \frac{\displaystyle\int_{x=0}^{x=2} \delta(2x^2-x^3)\,dx}{\displaystyle\int_{x=0}^{x=2} \delta(2x-x^2)\,dx} \\
&= \frac{\delta\Big(\frac{2x^3}{3}-\frac{x^4}{4})\Big]_{x=0}^{x=2}}{\delta\Big(\frac{2x^2}{2}-\frac{x^3}{3}\Big)\Big]_{x=0}^{x=2}\,dx} \\
&= \frac{\delta\Big(\frac{2\cdot 2^3}{3}-\frac{2^4}{4})-(0-0)}{\delta\Big(\frac{2\cdot 2^2}{2}-\frac{2^3}{3}\Big)-(0-0)\,dx} \\
&= \frac{2^4\big(\frac{1}{3}-\frac{1}{4}\big)}{2^3\big(\frac{1}{2}-\frac{1}{3}\big)} = \frac{2\big(\frac{1}{12}\big)}{\big(\frac{1}{6}\big)} = \frac{2}{2} = 1.
\end{aligned}$$

$$y\text{-coordinate of center of mass of area} = \frac{\displaystyle\int_{x=0}^{x=2} (y\text{-position of slice})\delta L\,dx}{\displaystyle\int_{x=0}^{x=2} \delta L\,dx} \qquad \text{(where } \delta \text{ is the density)}$$

$$= \frac{\displaystyle\int_{x=0}^{x=2} \Big(\big(\frac{y_{\mathrm{top}} - y_{\mathrm{bottom}}}{2}\big) + y_{\mathrm{bottom}}\Big)\delta(y_{\mathrm{top}} - y_{\mathrm{bottom}})\,dx}{\displaystyle\int_{x=0}^{x=2} \delta(y_{\mathrm{top}} - y_{\mathrm{bottom}})\,dx}$$

$$= \frac{\displaystyle\int_{x=0}^{x=2} \delta\Big(\frac{(x-x^2)-(-x)}{2} + (-x)\Big)((x-x^2)-(-x))\,dx}{\displaystyle\int_{x=0}^{x=2} \delta((x-x^2)-(-x))\,dx}$$

$$= \frac{\displaystyle\int_{x=0}^{x=2} \delta\Big(-\frac{x^2}{2}\Big)(2x-x^2)\,dx}{\displaystyle\int_{x=0}^{x=2} \delta(2x-x^2)\,dx} = \frac{\displaystyle\int_{x=0}^{x=2} \delta\Big(-\frac{2x^3}{2} + \frac{x^4}{2}\Big)\,dx}{\displaystyle\int_{x=0}^{x=2} \delta(2x-x^2)\,dx}$$

$$= \frac{\delta\Big(-\frac{x^4}{4} + \frac{x^5}{10}\Big)\Big]_{x=0}^{x=2}}{\delta(x^2 - \frac{x^3}{3})\Big]_{x=0}^{x=2}} = \frac{-\frac{2^4}{2} + \frac{2^5}{10}}{2^4 - \frac{2^3}{3}} = \frac{2^4\Big(-\frac{1}{4}+\frac{1}{5}\Big)}{2^3\big(2-\frac{1}{3}\big)} = \frac{2\big(-\frac{1}{20}\big)}{\frac{5}{3}}$$

$$= \frac{-2\cdot 3}{5\cdot 20} = \frac{-3}{5\cdot 10} = \frac{-3}{50}.$$

So the center of mass is at $\big(1, -\frac{3}{50}\big)$. □

5 Limits

5.1 Limits by algebra

Example 5.1. Evaluate $\displaystyle\lim_{x\to 7} \frac{x^2-49}{x-7}$.

Proof.

$$\lim_{x\to 7} \frac{x^2-49}{x-7} = \lim_{x\to 7} \frac{(x-7)(x+7)}{x-7} = \lim_{x\to 7}(x+7) = 7+7 = 14.$$

□

Example 5.2. Evaluate $\displaystyle\lim_{x\to 5} \frac{x^5-3125}{x-5}$.

Proof.

$$\begin{aligned}\lim_{x\to 5} \frac{x^5-3125}{x-5} &= \lim_{x\to 5} \frac{x^5-5^5}{x-5} = \lim_{x\to 5} \frac{(x-5)(x^4+5x^3+5^2x^2+5^3x+5^4)}{x-5} \\ &= \lim_{x\to 5} x^4+5x^3+5^2x^2+5^3x+5^4 = 5^4+5^4+5^4+5^4 = 5^5 = 3125.\end{aligned}$$

□

Example 5.3. Evaluate $\lim_{x\to a} \frac{x^{5/2}-a^{5/2}}{x-a}$.

Proof.

$$\begin{aligned}\lim_{x\to a}\frac{x^{5/2}-a^{5/2}}{x-a}\lim_{x\to a}\frac{(x^{5/2}-a^{5/2})}{(x-a)}\frac{(x^{5/2}+a^{5/2})}{(x^{5/2}+a^{5/2})} &= \lim_{x\to a}\frac{x^5-a^5}{x-a}\cdot\frac{1}{x^{5/2}-a^{5/2}} \\ &= \lim_{x\to a}\frac{(x-a)(x^4+ax^3+a^2x^2+a^3x+a^4)}{x-a}\cdot\frac{1}{x^{5/2}+a^{5/2}} = \lim_{x\to a}\frac{x^4+ax^3+a^2x^2+a^3x+a^4}{x^{5/2}+a^{5/2}} \\ &= \frac{a^4+a^4+a^4+a^4+a^4}{a^{5/2}+a^{5/2}} = \frac{5a^4}{2a^{5/2}} = \frac{5}{2}a^{3/2}.\end{aligned}$$

□

Particularly useful limits

Example 5.4. Evaluate $\lim_{x\to 0}\frac{\sin x}{x}$.

Proof.

$$\begin{aligned}\lim_{x\to 0}\frac{\sin x}{x} &= \lim_{x\to 0}\frac{x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\frac{x^9}{9!}-\cdots}{x} \\ &= \lim_{x\to 0}1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\frac{x^8}{9!}-\cdots = 1-0+0-0+0-\cdots = 1.\end{aligned}$$

□

Example 5.5. Evaluate $\lim_{x\to 0}\frac{\cos(x)-1}{x}$.

Proof.

$$\begin{aligned}\lim_{x\to 0}\frac{\cos(x)-1}{x} &= \lim_{x\to 0}\frac{\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\frac{x^8}{8!}-\cdots\right)-1}{x} \\ &= \lim_{x\to 0}\frac{-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\frac{x^8}{8!}-\cdots}{x} \\ &= \lim_{x\to 0}-\frac{x}{2!}+\frac{x^3}{4!}-\frac{x^5}{6!}+\frac{x^7}{8!}-\cdots = -0+0-0+0-\cdots = 0.\end{aligned}$$

□

Example 5.6. Evaluate $\lim_{x\to 0}\frac{e^x-1}{x}$.

Proof.

$$\begin{aligned}\lim_{x\to 0}\frac{e^x-1}{x} &= \lim_{x\to 0}\frac{\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots\right)-1}{x} \\ &= \lim_{x\to 0}\frac{x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots}{x} \\ &= \lim_{x\to 0}1+\frac{x}{2!}+\frac{x^2}{3!}+\frac{x^3}{4!}+\cdots = 1+0+0+0+0+\cdots = 1.\end{aligned}$$

□

Example 5.7. Evaluate $\lim_{x\to 0} \frac{\log(1+x)}{x}$.

Proof. Let $y = \log(1+x)$. Then

$$e^y = 1+x, \quad x = e^y - 1, \qquad \text{and} \quad y \to 0 \text{ as } x \to 0.$$

So

$$\lim_{x\to 0} \frac{\log(1+x)}{x} = \lim_{y\to 0} \frac{y}{e^y - 1} = \lim_{y\to 0} \frac{1}{\frac{e^y-1}{y}} = \frac{1}{1} = 1.$$

□

Example 5.8. Evaluate $\lim_{x\to 0}(1+x)^{1/x}$.

Proof.

$$\lim_{x\to 0}(1+x)^{1/x} = \lim_{x\to 0}\left(e^{\log(1+x)}\right)^{1/x} = \lim_{x\to 0} e^{\frac{1}{x}\log(1+x)} = \lim_{x\to 0} e^{\frac{\log(1+x)}{x}} = e^1 = e.$$

□

Note: $n \to \infty$ means as n gets larger and larger.

Example 5.9. Evaluate $\lim_{n\to\infty}(1+\frac{1}{n})^n$.

Proof. Let $x = \frac{1}{n}$. Then $x \to 0$ as $n \to \infty$. So

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{x\to 0}(1+x)^{1/x} = e.$$

□

Example 5.10. Evaluate $\lim_{x\to\pi} \frac{\sin(x)}{x-\pi}$.

Proof. Let $y = x - \pi$. Then $y \to 0$ as $x \to \pi$. So

$$\begin{aligned}\lim_{x\to\pi} \frac{\sin(x)}{x-\pi} &= \lim_{y\to 0} \frac{\sin(y+\pi)}{y} = \lim_{y\to 0} \frac{\sin(y)\cos(\pi) + \cos(y)\sin(\pi)}{y} \\ &= \lim_{y\to 0} \frac{\sin(y)(-1) + \cos(y)\cdot 0}{y} = \lim_{y\to 0} \frac{-\sin(y)}{y} = -1.\end{aligned}$$

□

Example 5.11. Evaluate $\lim_{x\to\infty} \frac{x^2 - 7x + 11}{3x^2 + 10}$.

Proof.

$$\lim_{x\to\infty} \frac{x^2 - 7x + 11}{3x^2 + 10} = \lim_{x\to\infty} \frac{1 - \frac{7}{x} + 11x^2}{3 + \frac{10}{x^2}} = \frac{1 - 0 + 0}{3 + 0} = \frac{1}{3}.$$

□

Example 5.12. Evaluate $\lim_{x\to 0} \frac{\sin(3x)}{\sin(5x)}$.

Proof.

$$\begin{aligned}\lim_{x\to 0}\frac{\sin(3x)}{\sin(5x)} &= \lim_{x\to 0}\frac{\sin(3x)}{3x}\cdot 3x\cdot\frac{5x}{\sin(5x)}\cdot\frac{1}{5x} = \lim_{x\to 0}\frac{\sin(3x)}{3x}\frac{1}{\frac{\sin(5x)}{5x}}\frac{3x}{5x}\\ &= \lim_{x\to 0}\frac{\sin(3x)}{3x}\frac{1}{\frac{\sin(5x)}{5x}}\frac{3}{5} = 1\cdot\frac{1}{1}\cdot\frac{3}{5} = \frac{3}{5}.\end{aligned}$$

□

Example 5.13. Evaluate $\lim_{x\to 1}\frac{1-x}{(\cos^{-1}(x))^2}$.

Proof. Let $y=\cos^{-1}x$. Then $y\to 0$ as $x\to 1$ and $x=\cos y$. So

$$\begin{aligned}\lim_{x\to 1}\frac{1-x}{(\cos^{-1}(x))^2} &= \lim_{y\to 0}\frac{1-\cos(y)}{y^2} = \lim_{y\to 0}\frac{(1-\cos(y))}{y^2}\cdot\frac{(1+\cos(y))}{(1+\cos(y))}\\ &= \lim_{y\to 0}\frac{(1-\cos(y)^2)}{y^2}\cdot\frac{1}{1+\cos(y)} = \lim_{y\to 0}\frac{\sin(y)}{y}\cdot\frac{\sin(y)}{y}\cdot\frac{1}{1+\cos(y)} = 1\cdot 1\cdot\frac{1}{2} = \frac{1}{2}.\end{aligned}$$

□

Example 5.14. Evaluate $\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$ when $f(x)=\sin(2x)$.

Proof.

$$\begin{aligned}\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} &= \lim_{\Delta x\to 0}\frac{\sin(2(x+\Delta x))-\sin(2x)}{\Delta x} = \lim_{\Delta x\to 0}\frac{\sin(2x+2\Delta x)-\sin(2x)}{\Delta x}\\ &= \lim_{\Delta x\to 0}\frac{\sin(2x)\cos(2\Delta x)+\cos(2x)\sin(2\Delta x)-\sin(2x)}{\Delta x}\\ &= \lim_{\Delta x\to 0}\sin(2x)\cdot\frac{(\cos(2\Delta x)-1)}{\Delta x}+\cos(2x)\frac{\sin(2\Delta x)}{\Delta x}\\ &= \lim_{\Delta x\to 0}\sin(2x)\cdot\frac{(\cos(2\Delta x)-1)}{2\Delta x}\cdot 2+\cos(2x)\frac{\sin(2\Delta x)}{2\Delta x}\cdot 2\\ &= \sin(2x)\cdot 0\cdot 2+\cos(2x)\cdot 1\cdot 2 = 2\cos(2x).\end{aligned}$$

□

Example 5.15. Evaluate $\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$ when $f(x)=\cos(x^2)$.

Proof.

$$
\begin{aligned}
\lim_{\Delta x\to 0}\frac{\cos((x+\Delta x)^2)-\cos(x^2)}{\Delta x} &= \lim_{\Delta x\to 0}\frac{\cos(x^2+2x\Delta x+(\Delta x)^2)-\cos(x^2)}{\Delta x} \\
&= \lim_{\Delta x\to 0}\frac{\cos(x^2)\cos(2x\Delta x+(\Delta x)^2)-\sin(x^2)\sin(2x\Delta x+(\Delta x)^2)-\cos(x^2)}{\Delta x} \\
&= \lim_{\Delta x\to 0}\cos(x^2)\cdot\frac{(\cos(2x\Delta x+(\Delta x)^2)-1)}{\Delta x}-\sin(x^2)\cdot\frac{\sin(2x\Delta x+(\Delta x)^2)}{\Delta x} \\
&= \lim_{\Delta x\to 0}\cos(x^2)\frac{(\cos(2x\Delta x+(\Delta x)^2)-1)}{2x\Delta x+(\Delta x)^2}\cdot\frac{2x\Delta x+(\Delta x)^2}{\Delta x} \\
&\qquad -\sin(x^2)\frac{\sin(2x\Delta x+(\Delta x)^2)}{2x\Delta x+(\Delta x)^2}\cdot\frac{(2x\Delta x+(\Delta x)^2)}{\Delta x} \\
&= \lim_{\Delta x\to 0}\cos(x^2)\cdot\frac{(\cos(\mathrm{STUFF})-1)}{\mathrm{STUFF}}\cdot(2x+\Delta x)-\sin(x^2)\frac{\sin(\mathrm{STUFF})}{\mathrm{STUFF}}\cdot(2x+\Delta x) \\
&= \cos(x^2)\cdot 0\cdot 2x-\sin x^2\cdot 1\cdot 2x = -2x\sin(x^2),
\end{aligned}
$$

where $\mathrm{STUFF} = 2x\Delta x+(\Delta x)^2)-1$. □

Example 5.16. Evaluate $\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$ when $f(x)=x^x$.

Proof.

$$
\begin{aligned}
\lim_{\Delta x\to 0}\frac{(x+\Delta x)^{x+\Delta x}-x^x}{\Delta x} &= \lim_{\Delta x\to 0}\frac{(e^{\log(x+\Delta x)})^{x+\Delta x}-(e^{\log(x)})^x}{\Delta x} \\
&= \lim_{\Delta x\to 0}\frac{e^{(x+\Delta x)\log(x+\Delta x)}-e^{x\log(x)}}{\Delta x} \\
&= \lim_{\Delta x\to 0}e^{x\log(x)}\cdot\frac{\left(e^{(x+\Delta x)\log(x+\Delta x)-x\log(x)}-1\right)}{\Delta x} \\
&= \lim_{\Delta x\to 0}e^{x\log(x)}\frac{(e^{(x+\Delta x)\log(x+\Delta x)-x\log(x)}-1)}{\big((x+\Delta x)\log(x+\Delta x)-x\log(x)\big)}\cdot\frac{\big((x+\Delta x)\log(x+\Delta x)-x\log(x)\big)}{\Delta x} \\
&= \lim_{\Delta x\to 0}e^{x\log(x)}\left(\frac{e^{\mathrm{STUFF}}-1}{\mathrm{STUFF}}\right)\left(\frac{x\log(x+\Delta x)-x\log(x)}{\Delta x}+\log(x+\Delta x)\right) \\
&= \lim_{\Delta x\to 0}e^{x\log(x)}\left(\frac{e^{\mathrm{STUFF}}-1}{\mathrm{STUFF}}\right)\left(\frac{x\log\left(x(1+\frac{\Delta x}{x})\right)-x\log(x)}{\Delta x}+\log(x+\Delta x)\right) \\
&= \lim_{\Delta x\to 0}e^{x\log(x)}\left(\frac{e^{\mathrm{STUFF}}-1}{\mathrm{STUFF}}\right)\left(\frac{x\big(\log(x_+\log(1+\frac{\Delta x}{x})\big)-x\log(x)}{\Delta x}+\log(x+\Delta x)\right) \\
&= \lim_{\Delta x\to 0}e^{x\log(x)}\left(\frac{e^{\mathrm{STUFF}}-1}{\mathrm{STUFF}}\right)\left(\frac{x\log(1+\frac{\Delta x}{x})}{\Delta x}+\log(x+\Delta x)\right) \\
&= \lim_{\Delta x\to 0}e^{x\log(x)}\left(\frac{e^{\mathrm{STUFF}}-1}{\mathrm{STUFF}}\right)\left(\frac{\log(1+\frac{\Delta x}{x})}{\frac{\Delta x}{x}}+\log(x+\Delta x)\right) \\
&= e^{x\log(x)}\cdot 1(1+\log(x)) = x^x+x^x\log(x),
\end{aligned}
$$

where $\mathrm{STUFF} = (x+\Delta x)\log(x+\Delta x)-x\log(x)$. □

5.2 Additional limit examples

Example 5.17. Evaluate $\lim_{x\to 1}(6x^2 - 4x + 3)$.

Proof.

$$\lim_{x\to 1}(6x^2 - 4x + 3) = 6 \cdot 1^2 - 4 \cdot 1 + 3 = 6 - 4 + 3 = 5.$$

□

Example 5.18. Evaluate $\lim_{x\to 0} \frac{5x}{x}$.

Proof. $\lim_{x\to 0} \frac{5x}{x} \stackrel{?}{=} \frac{0}{0}$.

$$\lim_{x\to 0} \frac{5x}{x} = \lim_{x\to 0} 5 = 5.$$

□

Example 5.19. Evaluate $\lim_{x\to 0} \frac{17x}{2x}$.

Proof. $\lim_{x\to 0} \frac{17x}{2x} \stackrel{?}{=} \frac{0}{0}$.

$$\lim_{x\to 0} \frac{17x}{2x} = \lim_{x\to 0} \frac{17}{2} = \frac{17}{2}.$$

□

Example 5.20. Evaluate $\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x}$.

Proof.

$$\begin{aligned}
\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x} &= \lim_{x\to 0} \frac{(\sqrt{x+1}-1)}{x} \frac{(\sqrt{1+x}+1)}{(\sqrt{1+x}+1)} \\
&= \lim_{x\to 0} \frac{1+x-1}{x(\sqrt{1+x}+1)} \\
&= \lim_{x\to 0} \frac{x}{x(\sqrt{1+x}+1)} \\
&= \lim_{x\to 0} \frac{1}{\sqrt{1+x}+1} \\
&= \frac{1}{\sqrt{1+0}+1} = \frac{1}{2}.
\end{aligned}$$

□

Example 5.21. Evaluate $\lim_{x\to 7} \frac{x^2-49}{x-7}$.

Proof.

$$\lim_{x\to 7} \frac{x^2-49}{x-7} = \lim_{x\to 7} \frac{(x-7)(x+7)}{x-7} = \lim_{x\to 7}(x+7) = 7 + 7 = 14.$$

□

Example 5.22. Evaluate $\lim_{x\to 5} \frac{x^5 - 3125}{x - 5}$.

Proof.

$$\begin{aligned}\lim_{x\to 5} \frac{x^5 - 3125}{x - 5} &= \lim_{x\to 5} \frac{x^5 - 5^5}{x - 5}\\ &= \lim_{x\to 5} \frac{(x-5)(x^4 + 5x^3 + 5^2x^2 + 5^3x + 5^4)}{x - 5}\\ &= \lim_{x\to 5}(x^4 + 5x^3 + 5^2x^2 + 5^3x + 5^4)\\ &= 5^4 + 5^4 + 5^4 + 5^4 + 5^4 = 5^5 = 3125.\end{aligned}$$

□

Example 5.23. Evaluate $\lim_{x\to a} \frac{x^{\frac{5}{2}} - a^{\frac{5}{2}}}{x - a}$.

Proof.

$$\begin{aligned}\lim_{x\to a} \frac{x^{\frac{5}{2}} - a^{\frac{5}{2}}}{x - a} &= \lim_{x\to a} \frac{\left(x^{\frac{5}{2}} - a^{\frac{5}{2}}\right)}{(x-a)} \frac{\left(x^{\frac{5}{2}} + a^{\frac{5}{2}}\right)}{\left(x^{\frac{5}{2}} + a^{\frac{5}{2}}\right)}\\ &= \lim_{x\to a} \frac{(x^5 - a^5)}{(x-a)} \cdot \frac{1}{\left(x^{\frac{5}{2}} + a^{\frac{5}{2}}\right)}\\ &= \lim_{x\to a} \frac{(x-a)(x^4 + ax^3 + a^2x^2 + a^3x + a^4)}{(x-a)} \cdot \frac{1}{\left(x^{\frac{5}{2}} + a^{\frac{5}{2}}\right)}\\ &= \lim_{x\to a} \frac{(x^4 + ax^3 + a^2x^2 + a^3x + a^4)}{\left(x^{\frac{5}{2}} + a^{\frac{5}{2}}\right)}\\ &= \frac{(a^4 + a^4 + a^4 + a^4 + a^4)}{\left(a^{\frac{5}{2}} + a^{\frac{5}{2}}\right)} = \frac{5a^4}{2a^{\frac{5}{2}}} = \frac{5}{2}a^{\frac{3}{2}}.\end{aligned}$$

□

Example 5.24. Evaluate $\lim_{x\to 0} \frac{e^x - 1}{x}$.

Proof.

$$\begin{aligned}\lim_{x\to 0} \frac{e^x - 1}{x} &= \lim_{x\to 0} \frac{\left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots\right) - 1}{x}\\ &= \lim_{x\to 0} \frac{x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots}{x}\\ &= \lim_{x\to 0} 1 + \frac{1}{2!}x + \frac{1}{3!}x^2 + \frac{1}{4!}x^3 + \cdots\\ &= 1 + 0 + 0 + 0 + \cdots = 1.\end{aligned}$$

□

Example 5.25. Evaluate $\lim_{x\to 0} \frac{\cos(x) - 1}{x}$.

Proof.

$$\begin{aligned}
\lim_{x\to 0}\frac{\cos(x)-1}{x} &= \lim_{x\to 0}\frac{\left(1-\frac{1}{2!}x^2+\frac{1}{4!}x^4-\frac{1}{6!}x^6+\cdots\right)-1}{x} \\
&= \lim_{x\to 0}\frac{-\frac{1}{2!}x^2+\frac{1}{4!}x^4-\frac{1}{6!}x^6+\frac{1}{8!}x^8-\cdots}{x} \\
&= \lim_{x\to 0}\left(-\frac{1}{2!}x+\frac{1}{4!}x^3-\frac{1}{6!}x^5+\frac{1}{8!}x^7-\cdots\right) \\
&= -0+0-0+0-0+\cdots = 0.
\end{aligned}$$

□

Example 5.26. Evaluate $\lim_{x\to 0}\frac{\sin(x)}{x}$.

Proof.

$$\begin{aligned}
\lim_{x\to 0}\frac{\sin(x)}{x} &= \lim_{x\to 0}\frac{x-\frac{1}{3!}x^3+\frac{1}{5!}x^5-\frac{1}{7!}x^7+\frac{1}{9!}x^9-\cdots}{x} \\
&= \lim_{x\to 0}\left(1-\frac{1}{3!}x^2+\frac{1}{5!}x^4-\frac{1}{7!}x^6+\frac{1}{9!}x^8-\cdots\right) \\
&= 1-0+0-0+\cdots = 1.
\end{aligned}$$

□

Example 5.27. Evaluate $\lim_{x\to 0}\frac{\log(1+x)}{x}$.

Proof. Let $y=\log(1+x)$. Then $e^y=1+x$ and $x=e^y-1$. Also $y\to 0$ as $x\to 0$. So

$$\lim_{x\to 0}\frac{\log(1+x)}{x} = \lim_{y\to 0}\frac{y}{e^y-1} = \lim_{y\to 0}\frac{1}{\frac{e^y-1}{y}} = \frac{1}{1} = 1.$$

□

Example 5.28. Evaluate $\lim_{x\to 0}(1+x)^{\frac{1}{x}}$.

Proof.

$$\lim_{x\to 0}(1+x)^{\frac{1}{x}} = \lim_{x\to 0}\left(e^{\log(1+x)}\right)^{\frac{1}{x}} = \lim_{x\to 0}e^{\frac{1}{x}\log(1+x)} = \lim_{x\to 0}e^{\frac{\log(1+x)}{x}} = e^1.$$

□

Example 5.29. Evaluate $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$.

Proof. Let $x=\frac{1}{n}$. Then $x\to 0$ and $n\to\infty$. So

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{x\to 0}(1+x)^{\frac{1}{x}} = e^1.$$

□

Example 5.30. Evaluate $\lim_{x\to\pi} \frac{\sin(x)}{x-\pi}$.

Proof. Let $y = x - \pi$. Then $y \to 0$ as $x \to \pi$. So

$$\begin{aligned}
\lim_{x\to\pi} \frac{\sin(x)}{x-\pi} &= \lim_{y\to 0} \frac{\sin(y+\pi)}{y} \\
&= \lim_{y\to 0} \frac{\sin(y)\cos(\pi) + \cos(y)\sin(\pi)}{y} \\
&= \lim_{y\to 0} \frac{\sin(y)(-1) + \cos(y)\cdot 0}{y} \\
&= \lim_{y\to 0} \Big(-\frac{\sin(y)}{y} \Big) = -1.
\end{aligned}$$

□

Example 5.31. Evaluate $\lim_{x\to 2} \frac{x^3+2x^2-1}{5-3x}$.

Proof. Since addition, multiplication, scalar multiplication and division away from 0 are continuous in $\mathbb{R}$ then

$$\lim_{x\to 2} \frac{x^3+2x^2-1}{5-3x} = \frac{(\lim_{x\to 2} x)^3 + 2(\lim_{x\to 2} x)^2 - 1}{5 - 3\lim_{x\to 2} x} = \frac{2^3 + 22^2 - 1}{5 - 32} = \frac{8+8-1}{-27} = -\frac{215}{27}.$$

□

Example 5.32. Evaluate $\lim_{x\to\infty} \frac{3x^2-2x+3}{x^2+4x+4}$.

Proof.

$$\begin{aligned}
\lim_{x\to\infty} \frac{3x^2-2x+3}{x^2+4x+4} &= \lim_{x\to\infty} \frac{(3x^2-2x+3)\frac{1}{x^2}}{(x^2+4x+4)\frac{1}{x^2}} = \lim_{x\to\infty} \frac{(3-2\frac{1}{x}+3\frac{1}{x^2})}{(1+4\frac{1}{x}+4\frac{1}{x^2})} \\
&= \frac{(3 - 2\lim_{x\to\infty}\left(\frac{1}{x}\right) + 3\lim_{x\to\infty}\left(\frac{1}{x}\right)^2)}{(1 + 4\lim_{x\to\infty}\left(\frac{1}{x}\right) + 4\lim_{x\to\infty}\left(\frac{1}{x}\right)^2)} \qquad \begin{pmatrix} \text{since addition, multiplication,} \\ \text{scalar multiplication,} \\ \text{and division away from 0} \\ \text{are continuous in } \mathbb{R} \end{pmatrix} \\
&= \frac{(3 - 2\cdot 0 + 3\cdot 0^2)}{(1 + 4\cdot 0 + 4\cdot 0^2)} = \frac{3}{1} = 3.
\end{aligned}$$

□

Example 5.33. Evaluate $\lim_{x\to\infty} \big(\sqrt{x^2+1} - x\big)$.

Proof.

$$\begin{aligned}
\lim_{x\to\infty} \big(\sqrt{x^2+1} - x\big) &= \lim_{x\to\infty} \frac{(\sqrt{x^2+1} - x)(\sqrt{x^2+1} + x)}{(\sqrt{x^2+1} + x)} \\
&= \lim_{x\to\infty} \frac{x^2+1-x^2}{(\sqrt{x^2+1} + x)} \le \lim_{x\to\infty} \frac{1}{(\sqrt{x^2} + x)} = \lim_{x\to\infty} \frac{1}{2x} = 0.
\end{aligned}$$

If x is large then $x \in \mathbb{R}_{>0}$ and if $x \in \mathbb{R}_{>0}$ then $\sqrt{x^2+1} - x \in \mathbb{R}_{>0}$. So $\lim_{x\to\infty} \left(\sqrt{x^2+1} - x\right) \geq 0$.

$$\text{So} \quad 0 \leq \lim_{x\to\infty} \left(\sqrt{x^2+1} - x\right) \leq 0 \qquad \text{giving that} \qquad \lim_{x\to\infty} \left(\sqrt{x^2+1} - x\right) = 0.$$

□

Example 5.34. Evaluate $\lim_{x\to 0} \frac{\sin(x)}{x}$.

Proof.

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin(x)}{x} &= \lim_{x\to 0} \frac{x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots}{x} \\
&= \lim_{x\to 0} (1 - \frac{1}{3!}x^2 + \frac{1}{5!}x^4 - \frac{1}{7!}x^6 + \cdots) \\
&= 1 - \frac{1}{3!}(\lim_{x\to 0} x)^2 + \frac{1}{5!}(\lim_{x\to 0} x)^4 - \frac{1}{7!}(\lim_{x\to 0} x)^6 + \cdots \qquad \begin{pmatrix} \text{since addition, multiplication,} \\ \text{and scalar multiplication,} \\ \text{are continuous in } \mathbb{R} \end{pmatrix} \\
&= 1.
\end{aligned}$$

□

Example 5.35. Evaluate $\lim_{x\to\infty} \left(x^{-\frac{1}{2}} \log(x)\right)$.

Proof. Let $x = e^y$ so that $y = \log(x)$ and x gets larger and larger exactly when y gets larger and larger. Then

$$\begin{aligned}
\lim_{x\to\infty} \left(x^{-\frac{1}{2}} \log(x)\right) &= \lim_{y\to\infty} \left((e^y)^{-\frac{1}{2}} \log(e^y)\right) = \lim_{y\to\infty} \left((e^{-\frac{1}{2}y})y\right) = \lim_{y\to\infty} \frac{y}{e^{\frac{1}{2}y}} \\
&= \lim_{y\to\infty} \frac{y}{1 + \frac{1}{2}y + \frac{1}{2}\cdot(\frac{1}{2})^2 y^2 + \frac{1}{3!}(\frac{1}{2})^3 y^3 + \cdots} \\
&= \lim_{y\to\infty} \frac{1}{\frac{1}{y} + \frac{1}{2} + \frac{1}{2}\cdot(\frac{1}{2})^2 y + \frac{1}{3!}(\frac{1}{2})^3 y^2 + \cdots} \\
&\leq \lim_{y\to\infty} \frac{1}{\frac{1}{2}\cdot(\frac{1}{2})^2 y} = 0.
\end{aligned}$$

If $x \in \mathbb{R}_{>1}$ then $x^{-\frac{1}{2}} \log(x) \in \mathbb{R}_{>0}$, Thus, if $\lim_{x\to\infty} \left(x^{-\frac{1}{2}} \log(x)\right)$ exists in $\mathbb{R}$ then $0 \leq \lim_{x\to\infty} \left(x^{-\frac{1}{2}} \log(x)\right)$. So

$$0 \leq \lim_{x\to\infty} \left(x^{-\frac{1}{2}} \log(x)\right) \leq 0 \qquad \text{giving} \qquad \lim_{x\to\infty} \left(x^{-\frac{1}{2}} \log(x)\right) = 0.$$

□

Example 5.36. Evaluate $\lim_{n\to\infty} \frac{3^n+2}{4^n+2^n}$.

Proof. If $n \in \mathbb{Z}_{>0}$ then $\frac{3^n+2}{4^n+2^n} \in \mathbb{R}_{>0}$. Thus, if $\lim_{n\to\infty} \frac{3^n+2}{4^n+2^n}$ exists then $\lim_{n\to\infty} \frac{3^n+2}{4^n+2^n} \geq 0$.

$$\begin{aligned}
\lim_{n\to\infty} \frac{3^n+2}{4^n+2^n} &\leq \lim_{n\to\infty} \frac{3^n+2}{4^n+4^n} = \lim_{n\to\infty} \frac{3^n+2}{2\cdot 4^n} = \lim_{n\to\infty} \left(\left(\tfrac{3}{4}\right)^n + 2\left(\tfrac{1}{4}\right)^n\right) \\
&= \lim_{n\to\infty} \left(\tfrac{3}{4}\right)^n + 2\lim_{n\to\infty} \left(\tfrac{1}{4}\right)^n \qquad \begin{pmatrix} \text{since addition, multiplication,} \\ \text{and scalar multiplication,} \\ \text{are continuous in } \mathbb{R} \end{pmatrix} \\
&= 0 + 0 = 0.
\end{aligned}$$

So

$$0 \le \lim_{n\to\infty} \frac{3^n+2}{4^n+2^n} \le 0 \qquad \text{giving that} \qquad \lim_{n\to\infty} \frac{3^n+2}{4^n+2^n} = 0.$$

□

Example 5.37. Evaluate $\lim\limits_{n\to\infty} \dfrac{1+\sin^2\left(\frac{n\pi}{3}\right)}{\sqrt{n}}$.

Proof. If $n \in \mathbb{Z}_{>0}$ then $\frac{1+\sin^2\left(\frac{n\pi}{3}\right)}{\sqrt{n}} \in \mathbb{R}_{>0}$.
Thus, if $\lim\limits_{n\to\infty} \dfrac{1+\sin^2\left(\frac{n\pi}{3}\right)}{\sqrt{n}}$ exists in $\mathbb{R}$ then $\lim\limits_{n\to\infty} \dfrac{1+\sin^2\left(\frac{n\pi}{3}\right)}{\sqrt{n}} \ge 0$.

If $n \in \mathbb{Z}_{>0}$ then $1+\sin^2\left(\frac{n\pi}{3}\right) \le 2$.
So

$$\lim_{n\to\infty} \frac{1+\sin^2\left(\frac{n\pi}{3}\right)}{\sqrt{n}} \le \lim_{n\to\infty} \frac{2}{\sqrt{n}} = 0,$$

since $\frac{2}{\sqrt{n}}$ gets closer and closer to 0 as n gets larger and larger. So

$$0 \le \lim_{n\to\infty} \frac{1+\sin^2\left(\frac{n\pi}{3}\right)}{\sqrt{n}} \le 0 \qquad \text{giving that} \qquad \lim_{n\to\infty} \frac{1+\sin^2\left(\frac{n\pi}{3}\right)}{\sqrt{n}} = 0.$$

□

Example 5.38. Evaluate $\lim\limits_{n\to\infty} \log(3n^2+2) - \log(n^2)$.

Proof. Let $f\colon \mathbb{R}_{>0} \to \mathbb{R}$ be the function given by $f(x) = \log(3n^2+2) - \log(n^2)$.
If $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \log(3x^2+2) - \log(x^2)$ exists then

$$\lim_{n\to\infty} \log(3n^2+2) - \log(n^2) = \lim_{x\to\infty} f(x) = \lim_{x\to\infty} \log(3x^2+2) - \log(x^2).$$

Since

$$\begin{aligned}
\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \log(3x^2+2) - \log(x^2) = \lim_{x\to\infty} \log\left(\frac{3x^2+2}{x^2}\right) = \lim_{x\to\infty} \log\left(3+2\frac{1}{x^2}\right) \\
&= \lim_{x\to\infty} \log\left(3+2(\lim_{x\to\infty}\frac{1}{x^2})\right) \qquad \begin{pmatrix} \text{since log, addition,} \\ \text{and scalar multiplication,} \\ \text{are continuous in } \mathbb{R}_{>0} \end{pmatrix} \\
&= \log(3+0) = \log 3.
\end{aligned}$$

So

$$\lim_{n\to\infty} \log(3n^2+2) - \log(n^2) = \log 3.$$

□

Example 5.39. Let $p \in \mathbb{R}_{>0}$. Evaluate $\lim\limits_{n\to\infty} \dfrac{\log n}{n^p}$.

Proof. Let $f\colon \mathbb{R}_{>0} \to \mathbb{R}$ be the function given by $f(x) = \frac{\log(x)}{x^p}$. If $\lim_{x\to\infty} \dfrac{\log x}{x^p}$ exists in $\mathbb{R}$ then $\lim_{n\to\infty} \dfrac{\log x}{x^p}$ exists in $\mathbb{R}$. Llet $x = e^y$ so that y gets larger and larger as x gets larger and larger. Then

$$\begin{aligned}
\lim_{x\to\infty} \frac{\log x}{x^p} &= \lim_{y\to\infty} \frac{\log e^y}{(e^y)^p} = \lim_{y\to\infty} \frac{\log e^y}{e^{py}} = \lim_{y\to\infty} \frac{y}{e^{py}} \\
&= \lim_{y\to\infty} \frac{y}{1 + py + \frac{1}{2!}p^2y^2 + \frac{1}{3!}p^3y^3 + \cdots} \\
&= \lim_{y\to\infty} \frac{1}{\frac{1}{y} + p + \frac{1}{2!}p^2y + \frac{1}{3!}p^3y^2 + \cdots} \\
&\le \lim_{y\to\infty} \frac{1}{\frac{1}{2!}p^2y} = 0.
\end{aligned}$$

If $x \in \mathbb{R}_{>1}$ then $\frac{\log x}{x^p} \in \mathbb{R}_{>0}$. So if $\lim_{x\to\infty} \dfrac{\log x}{x^p}$ exists in $\mathbb{R}$ then $\lim_{x\to\infty} \dfrac{\log x}{x^p} \ge 0$. Thus

$$0 \le \lim_{x\to\infty} \frac{\log x}{x^p} \le 0 \qquad \text{giving that} \qquad \lim_{x\to\infty} \frac{\log x}{x^p} = 0.$$

So

$$\lim_{n\to\infty} \frac{\log n}{n^p} = 0.$$

□

Example 5.40. Evaluate $\lim_{n\to\infty} \Big(\Big(\dfrac{n-2}{n}\Big)^n + \dfrac{4n^2}{3^n} \Big)$.

Proof. Step 1.

$$\lim_{n\to\infty} \Big(\frac{n-2}{n}\Big)^n = \lim_{n\to\infty} \Big(1 - \frac{2}{n}\Big)^n = \lim_{n\to\infty} \Big(e^{\log(1-\frac{2}{n})}\Big)^n = \lim_{n\to\infty} e^{n\log(1-\frac{2}{n})} = e^{\lim_{n\to\infty} n\log(1-\frac{2}{n})}.$$

Step 2. Then

$$\begin{aligned}
\lim_{n\to\infty} n\log(1-\frac{2}{n}) &= \lim_{n\to\infty} n\Big(\frac{2}{n} + \frac{1}{2}\Big(\frac{2}{n}\Big)^2 + \frac{1}{3}\Big(\frac{2}{n}\Big)^3 + \cdots\Big) \\
&= \lim_{n\to\infty} (2 + \frac{1}{2}\frac{2^2}{n} + \frac{1}{3}\frac{2^3}{n^2} + \cdots) \\
&= 2 + 0 + 0 + \cdots = 2.
\end{aligned}$$

So

$$\lim_{n\to\infty} \Big(\frac{n-2}{n}\Big)^n = e^{\lim_{n\to\infty} n\log(1-\frac{2}{n})} = e^2.$$

Step 3.

$$\begin{aligned}
\lim_{n\to\infty} \frac{4n^2}{3^n} &= \lim_{n\to\infty} \frac{4n^2}{(e^{\log 3})^n} = \lim_{n\to\infty} \frac{4n^2}{(e^{n\log 3})} \\
&= \lim_{n\to\infty} \frac{4n^2}{1 + n\log 3 + \frac{1}{2!}n^2(\log 3)^2 + \frac{1}{3!}n^3(\log 3)^3 + \cdots)} \\
&= \lim_{n\to\infty} 4\frac{1}{\frac{1}{n^2} + \frac{1}{n}\log 3 + \frac{1}{2!}(\log 3)^2 + \frac{1}{3!}n(\log 3)^3 + \cdots)} \\
&\le \lim_{n\to\infty} 4\frac{1}{\frac{1}{3!}n(\log 3)^3} = 0.
\end{aligned}$$

If $n \in \mathbb{Z}_{>0}$ then $\frac{4n^2}{3^n} \in \mathbb{R}_{>0}$. Thus, if $\lim_{n\to\infty} \dfrac{4n^2}{3^n}$ exists in $\mathbb{R}$ then $\lim_{n\to\infty} \dfrac{4n^2}{3^n} \geq 0$. Thus,

$$0 \leq \lim_{n\to\infty} \frac{4n^2}{3^n} \leq 0 \qquad \text{giving that} \qquad \lim_{n\to\infty} \frac{4n^2}{3^n} = 0.$$

Final step. Since $\lim_{n\to\infty} \dfrac{4n^2}{3^n}$ and $\lim_{n\to\infty} \left(\dfrac{n-2}{n}\right)^n$ exist in $\mathbb{R}$ and addition is continuous in $\mathbb{R}$ then

$$\lim_{n\to\infty} \left(\left(\frac{n-2}{n}\right)^n + \frac{4n^2}{3^n}\right) = \lim_{n\to\infty} \left(\frac{n-2}{n}\right)^n + \lim_{n\to\infty} \frac{4n^2}{3^n} = e^2 + 0 = e^2.$$

□

Example 5.41. Let $c \in \mathbb{R}_{>0}$. Show that $\lim_{n\to\infty} \arctan(cn) = \dfrac{\pi}{2}$.

Proof.

$$\lim_{n\to\infty} \arctan(cn) = \lim_{cn\to\infty} \arctan(cn) = \lim_{x\to\infty} \arctan(x) = \lim_{y\to\frac{\pi}{2}^+} \arctan(\tan(y)) = \lim_{y\to\frac{\pi}{2}^+} y = \frac{\pi}{2}.$$

□

Example 5.42. Evaluate $\lim_{x\to 0} \dfrac{5x}{x}$.

Proof.

$$\lim_{x\to 0} \frac{5x}{x} = \lim_{x\to 0} 5 = 5.$$

□

Example 5.43. Evaluate $\lim_{x\to 0} \dfrac{642x}{x}$.

Proof.

$$\lim_{x\to 0} \frac{642x}{x} = \lim_{x\to 0} 642 = 642.$$

□

Example 5.44. Evaluate $\lim_{x\to 0} \dfrac{x^2}{x}$.

Proof.

$$\lim_{x\to 0} \frac{x^2}{x} = \lim_{x\to 0} x = 0.$$

□

Example 5.45. Evaluate $\lim_{x\to 0} \dfrac{x}{x^2}$.

Proof.

$$\lim_{x\to 0} \frac{x}{x^2} = \lim_{x\to 0} \frac{1}{x} \quad \text{does not exist in } \mathbb{R}$$

since

$$\lim_{x\to 0^+} \frac{x}{x^2} = \lim_{x\to 0^+} \frac{1}{x} \quad \text{is very large and positive, and}$$
$$\lim_{x\to 0^-} \frac{x}{x^2} = \lim_{x\to 0^-} \frac{1}{x} \quad \text{is very large and negative.}$$

□

Example 5.46. Evaluate $\lim\limits_{x\to 1} \dfrac{\log(x)}{x-1}$.

Proof. Let $y = x-1$. Then $y \to 0$ as $x \to 1$. So

$$\lim_{x\to 1} \frac{\log(x)}{x-1} = \lim_{y\to 0} \frac{\log(1+y)}{y} = 1,$$

by Example ???. ☐

Example 5.47. Evaluate $\lim\limits_{x\to\infty} \dfrac{\log(x)}{e^x}$.

Proof. Let $x = e^y$. Then $y = \log(x)$ and $y \to \infty$ as $x \to \infty$. So

$$\begin{aligned}\lim_{x\to\infty} \frac{\log(x)}{e^x} &= \lim_{y\to\infty} \frac{\log(e^y)}{e^{e^y}} = \lim_{y\to\infty} \frac{y}{e^{e^y}} = \lim_{y\to\infty} \frac{1}{\frac{1}{y}e^{e^y}} \\ &= \lim_{y\to\infty} \frac{1}{\frac{1}{y}\big(1+e^y+\frac{1}{2!}e^{2y}+\cdots} = 0,\end{aligned}$$

since the denominator is very very large when y is very large. ☐

Example 5.48. Evaluate $\lim\limits_{x\to\infty} \dfrac{\log(x)^2}{x^2}$.

Proof. Let $x = e^y$. Then $y = \log(x)$ and $y \to \infty$ as $x \to \infty$. So

$$\begin{aligned}\lim_{x\to\infty} \frac{\log(x)^2}{x^2} &= \lim_{y\to\infty} \frac{\log(e^y)}{(e^y)^2} = \lim_{y\to\infty} \frac{y}{e^{2y}} = \lim_{y\to\infty} \frac{1}{\frac{1}{y}e^{2y}} \\ &= \lim_{y\to\infty} \frac{1}{\frac{1}{y}\big(1+e^{2y}+\frac{1}{2!}e^{4y}+\cdots\big)} = 0,\end{aligned}$$

since the denominator is very very large when y is very large. ☐

Example 5.49. Evaluate $\lim\limits_{x\to 0} x\log(x)$.

Proof. Let $x = e^{-y}$. Then $-y = \log(x)$ and $y \to \infty$ as $x \to 0$. So

$$\begin{aligned}\lim_{x\to 0} x\log(x) &= \lim_{y\to\infty} e^{-y}\log(e^{-y}) = \lim_{y\to\infty} -ye^{-y} = \lim_{y\to\infty} \frac{-y}{e^y} = \lim_{y\to\infty} \frac{-1}{\frac{1}{y}e^y} \\ &= \lim_{y\to\infty} \frac{-1}{\frac{1}{y}(1+y+\frac{1}{2!}y^2+\cdots)} = 0,\end{aligned}$$

since the denominator is very very large when y is very large. ☐

Example 5.50. Evaluate $\lim\limits_{x\to\pi} (x-\pi)\cot(x)$.

Proof. Let $y = x-\pi$. Then $y \to 0$ as $x \to \pi$. So

$$\begin{aligned}\lim_{x\to\pi} (x-\pi)\cot(x) &= \lim_{x\to\pi} \frac{(x-\pi)\cos(x)}{\sin(x)} = \lim_{y\to .0} \frac{y\cos(y+\pi)}{\sin(y+\pi)} \\ &= \lim_{y\to .0} \frac{y(-\cos(y)}{-\sin(y)} = \lim_{y\to .0} \frac{y}{\sin(y)}\cos(y) = 1\cdot 1 = 1.\end{aligned}$$

☐

Example 5.51. Evaluate $\lim_{x\to 0}(1-2x)^{\frac{1}{x}}$.

Proof.

$$\begin{aligned}\lim_{x\to 0}(1-2x)^{\frac{1}{x}} &= \lim_{x\to 0}\left(e^{\log(1-2x)}\right)^{\frac{1}{x}} = \lim_{x\to 0} e^{\frac{1}{x}\log(1-2x)} \\ &= \lim_{x\to 0} e^{\frac{-2\log(1-2x)}{-2x}} = e^{-2\cdot 1} = e^{-2}.\end{aligned}$$

□

Example 5.52. Evaluate $\lim_{x\to 0} x^x$.

Proof.

$$\lim_{x\to 0} x^x = \lim_{x\to 0}\left(e^{\log(x)}\right)^x = \lim_{x\to 0} e^{x\log(x)} = e^0,$$

by Example 5.49. □

Example 5.53. Evaluate $\lim_{x\to 0}(x^{-1}-\csc(x))$.

Proof.

$$\begin{aligned}\lim_{x\to 0}(x^{-1}\csc(x)) &= \lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{\sin(x)}\right) = \lim_{x\to 0}\frac{\sin(x)-x}{x\sin(x)} \\ &= \lim_{x\to 0}\frac{(x-\frac{1}{3!}x^3+\frac{1}{5!}x^5-\cdots)-x}{x(x-\frac{1}{3!}x^3+\frac{1}{5!}x^5-\cdots)} \\ &= \lim_{x\to 0}\frac{\frac{1}{x^2}(-\frac{1}{3!}x^3+\frac{1}{5!}x^5-\cdots)}{\frac{1}{x^2}(x^2-\frac{1}{3!}x^4+\frac{1}{5!}x^6-\cdots)} \\ &= \lim_{x\to 0}\frac{(-\frac{1}{3!}x+\frac{1}{5!}x^3-\cdots)}{1-\frac{1}{3!}x^2+\frac{1}{5!}x^4-\cdots} \\ &= \lim_{x\to 0} x\cdot\frac{-\frac{1}{3!}+\frac{1}{5!}x^2-\cdots}{1-\frac{1}{3!}x^2+\frac{1}{5!}x^4-\cdots} \\ &= 0\cdot\frac{-1}{3!} = 0.\end{aligned}$$

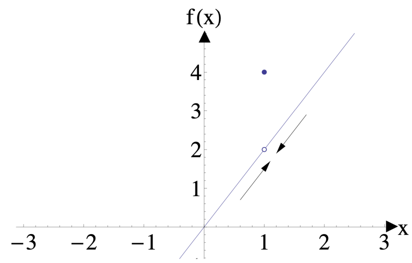
□

5.3 Limits by graphing

For now, the pictures are taken, by screenshot, from the Melbourne University Lecture slides for MAST 10006 Semester 1 2024.

Example 5.54. Let $f\colon \mathbb{R}\to\mathbb{R}$ be the function given by $f(x) = \begin{cases} 2x, & \text{if } x\neq 1, \\ 4, & \text{if } x = 1.\end{cases}$
Compute $\lim_{x\to 1} f(x)$ and determine if $f(x)$ is continuous at $x=1$.

Proof. The graph of $\{(x, f(x)) \in \mathbb{R}^2\}$ is



As x gets closer and closer to 1 then $f(x)$ gets closer and closer to 2. So

$$\lim_{x\to 1} f(x) = 2.$$

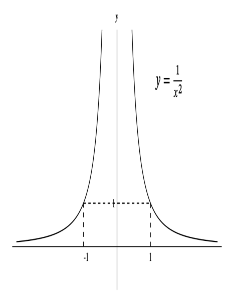
Since $f(1) = 4$ then

$$\lim_{x\to 1} f(x) \neq f(1).$$

So $f(x)$ is *not* continuous at $x = 1$. □

Example 5.55. Let $f\colon \mathbb{R}_{\neq 0} \to \mathbb{R}$ be the function given by $f(x) = \dfrac{1}{x^2}$.
Compute $\lim\limits_{x\to 0} \dfrac{1}{x^2}$ and determine if $f(x)$ is continuous at $x = 0$.

Proof. The graph of $\{(x, f(x)) \in \mathbb{R}^2\}$ is



the graph of $\frac{1}{x^2}$

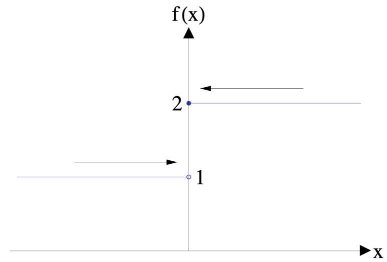
As x gets closer and closer to 0 then $f(x) = \frac{1}{x^2}$ gets larger and larger. So

$$\lim_{x\to 0} \frac{1}{x^2} \quad \text{does not exist in } \mathbb{R}.$$

Since $f(0)$ is not defined it doesn't make sense to ask if $\lim_{x\to 0} f(x)$ is equal to $f(0)$. So, because $f(0)$ is not defined it does not make sense to ask if $f(x)$ is continuous at $x = 0$. □

Example 5.56. Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function given by $f(x) = \begin{cases} 1, & \text{if } x < 0, \\ 2,, & \text{if } x \geq 0. \end{cases}$
Compute $\lim\limits_{x\to 0} f(x)$ and determine if $f(x)$ is continuous at $x = 0$.

Proof. The graph of $\{(x, f(x)) \in \mathbb{R}^2\}$ is



As x gets closer and closer to 1 from the positive side then $f(x)$ gets closer and closer to 2.
As x gets closer and closer to 1 from the negitive side then $f(x)$ gets closer and closer to 1. So

$$\lim_{x\to 1^+} f(x) = 2 \qquad \text{and} \qquad \lim_{x\to 1^-} f(x) = 1.$$

As x gets closer and closer to 1 then $f(x)$ does not get closer and closer to a single real number. So

$$\lim_{x\to 1} f(x) \quad \text{does not exist in } \mathbb{R}.$$

Since $f(1) = 2$ and $\lim\limits_{x\to 1} f(x)$ is not equal to $f(1)$ then $f(x)$ is not contnuous at $x = 1$. □

Example 5.57. Let $f: \mathbb{R} \to \mathbb{R}$ be the function given by $f(x) = \begin{cases} \frac{x^2-4}{x-2}, & \text{if } x \neq 2, \\ 4, & \text{if } x = 2. \end{cases}$
Compute $\lim\limits_{x\to 2} f(x)$ and determine if $f(x)$ is continuous at $x = 2$.

Proof.

$$f(x) = \begin{cases} \frac{x^2-4}{x-2}, & \text{if } x \neq 2, \\ 4, & \text{if } x = 2, \end{cases} = \begin{cases} x+2, & \text{if } x \neq 2, \\ 4, & \text{if } x = 2. \end{cases}$$

The graph of $\{(x, f(x)) \in \mathbb{R}^2\}$ is

PICTURE

As x gets closer and closer to 2 (from either the positive or negative side) then $f(x)$ gets closer and closer to 4.
So

$$\lim_{x\to 2} f(x) = 4.$$

Since $f(2) = 4$ then

$$\lim_{x\to 2} f(x) = f(2) \quad \text{and } f(x) \text{ is continuous at } x = 2.$$

□

Example 5.58. Let $c \in \mathbb{R}$. Let $f: \mathbb{R} \to \mathbb{R}$ be the function given by $f(x) = \begin{cases} x+2, & \text{if } x \leq 1, \\ (x-3)^2 + c,, & \text{if } x > 1. \end{cases}$
For which values of c is $f(x)$ continuous for $x \in \mathbb{R}$?

Proof. The graph of $\{(x, f(x)) \in \mathbb{R}^2\}$ is

PICTURE

Then

$$\lim_{x\to 1^-} f(x) = 1+2 \qquad \text{and} \qquad \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x-3)^2 + c = (-2)^2 + c = 4 + c.$$

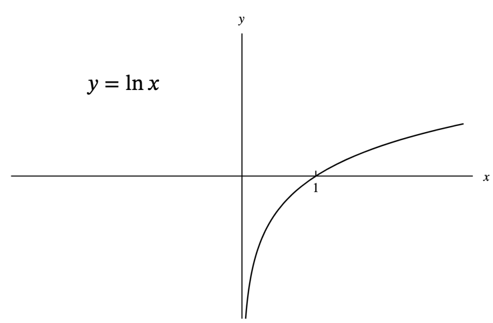
So $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x)$ exactly when $4 + c = 3$.
So, if $c = -1$ then

$$\lim_{x\to 1} f(x) = 3 \qquad \text{and} \qquad f(1) = 3.$$

So, if $c = -1$ then $f(x)$ is continuous at $x = 1$. If $c \neq 1$ then $\lim_{x\to 1} f(x)$ does not exist in $\mathbb{R}$. □

Example 5.59. Let $h(x) = \sin(2\log x)$.
For which values of $x \in \mathbb{R}$ is $h(x)$ defined?
For which values of $x \in \mathbb{R}$ is $h(x)$ continuous?
Always carefully justify your answers.

Proof. Let $f\colon \mathbb{R}_{>0} \to \mathbb{R}$ be the function given by $f(x) = 2\log(x)$. The graph of $\{(x, f(x)) \in \mathbb{R}^2\}$ is



the graph of $2\log(x)$

The function $f(x)$ is continuous for $x \in \mathbb{R}_{>0}$.

Let $g\colon \mathbb{R} \to \mathbb{R}$ be the function given by $g(y) = \sin(y)$. The graph of $\{(y, g(y)) \in \mathbb{R}^2\} g(y)$ is

$$PICTURE$$

The function $g(y)$ is continuous for $y \in \mathbb{R}$.

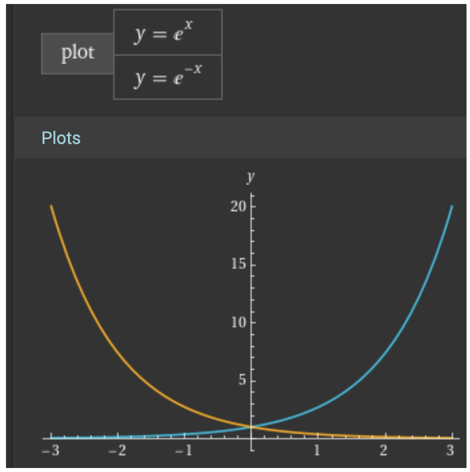
Let $h\colon \mathbb{R}_{>0} \to \mathbb{R}$ be the function given by $h(x) = g(f(x) = \sin(2\log(x))$. The graph of $\{(x, h(x)) \in \mathbb{R}^2\}$ is

$$PICTURE$$

The function $h(x)$ is defined for $x \in \mathbb{R}_{>0}$ and continuous for $x \in \mathbb{R}_{>0}$. □

Example 5.60. Evaluate $\lim_{x\to\infty} e^{-x}$.

Proof. Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function given by $f(x) = e^{-x}$. The graph of $\{(x, f(x)) \in \mathbb{R}^2\}$ is



solutions of $y = e^x$ and $y = e^{-x}$ from Wolfram alpha

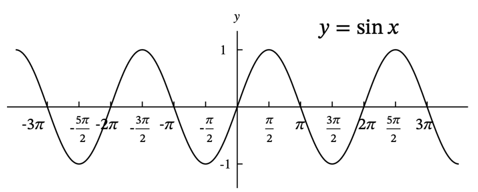
As x gets larger and larger then $f(x)$ gets closer and closer to 0. So

$$\lim_{x\to\infty} e^{-x} = 0.$$

□

Example 5.61. Evaluate $\lim_{x\to\infty} \sin(x)$.

Proof. Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function given by $f(x) = \sin(x)$. The graph of $\{(x, f(x)) \in \mathbb{R}^2\}$ is



real solutions of $y = \sin(x)$

As x gets larger and larger then $f(x)$ oscillates between 1 and -1 and does not get closer and closer to any single real number. So

$$\lim_{x\to\infty} \sin(x) \quad \text{does not exist in } \mathbb{R}.$$

□

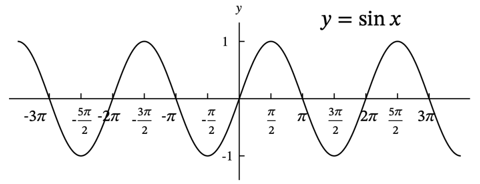
Example 5.62. Evaluate $\lim_{x\to\infty} \sin(e^{-x})$.

Proof. Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function given by $f(x) = e^{-x}$. The graph of $\{(x, f(x)) \in \mathbb{R}^2\}$ is

$$PICTURE$$

As x gets larger and larger e^{-x} gets closer and closer to 0.

Let $g\colon \mathbb{R} \to \mathbb{R}$ be the function given by $g(y) = \sin(y)$. The graph of $\{(y, g(y)) \in \mathbb{R}^2\}$ is



real solutions of $y = \sin(x)$

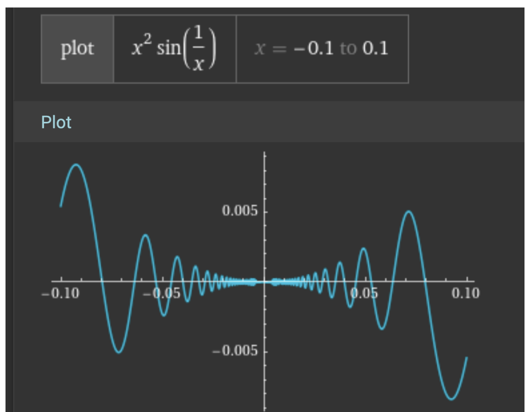
As y gets closer and closer to 0 then $\sin(y)$ gets closer and closer to 0. The function $g(y) = \sin(y)$ is continuous. As e^{-x} gets closer and closer to 0 then $g(e^{-x})$ gets closer and closer to 0. Thus

$$\lim_{x\to\infty} \sin(e^{-x}) = \sin\big(\lim_{x\to\infty} e^{-x}\big) = \sin(0) = 0.$$

□

Example 5.63. Evaluate $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right)$.

Proof. Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function given by $f(x) = x^2 \sin\left(\frac{1}{x}\right)$. The graph of $\{(x, f(x)) \in \mathbb{R}^2\}$ is



solutions of $y = x^2 \sin(\frac{1}{x})$ from Wolfram alpha

As x gets closer and loser to 0 then $f(x)$ gets closer and closer to 0. So

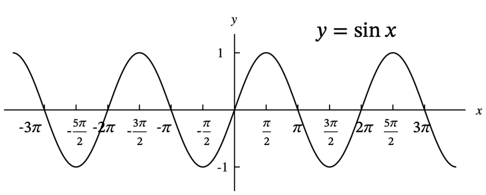
$$\lim_{x\to\infty} x^2 \sin\left(\tfrac{1}{x}\right) = 0.$$

□

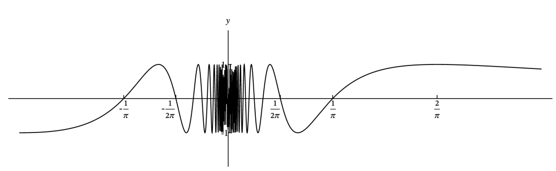
If $a \in \mathbb{R}_{\geq 1}$ then $\frac{1}{a} \in \mathbb{R}_{(0,1]}$. The graph of $\{(x, \sin(\frac{1}{x})) \in \mathbb{R}^2\}$ is the same as the graph of $\{(x, \sin(x)) \in \mathbb{R}^2\}$ with

(a) the region $\mathbb{R}_{\geq 1}$ flipped with the region $\mathbb{R}_{(0,1]}$ on the x-axis and

(b) the region $\mathbb{R}_{\leq -1}$ flipped with the region $\mathbb{R}_{[-1,0)}$ on the x-axis.



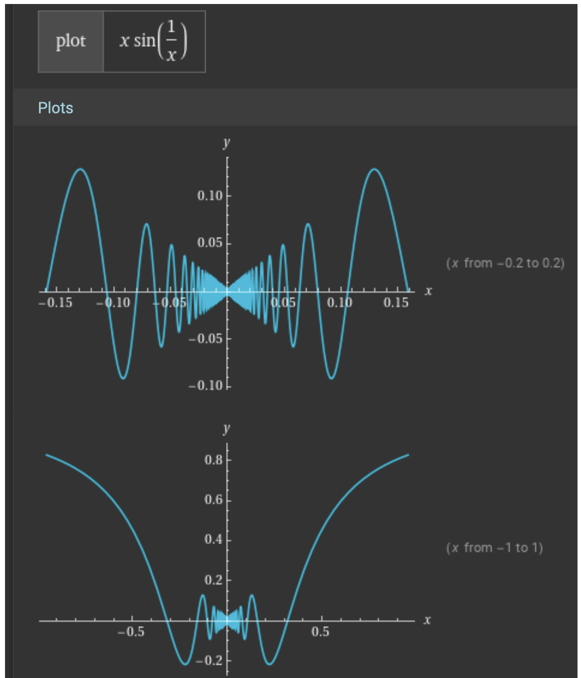
the graph of $\{(x, \sin(x) \in \mathbb{R}^2\}$



the graph of $\{(x, \sin(\frac{1}{x}) \in \mathbb{R}^2\}$

Example 5.64. Evaluate $\lim_{x\to 0} x\sin\left(\frac{1}{x}\right)$.

Proof. Let $f\colon \mathbb{R}\to\mathbb{R}$ be the function given by $f(x) = x\sin\left(\frac{1}{x}\right)$. The graph of $\{(x, f(x)) \in \mathbb{R}^2\}$ is



solutions of $y = x\sin(\frac{1}{x})$ from Wolfram alpha

As x gets closer and loser to 0 then $f(x)$ gets closer and closer to 0. So

$$\lim_{x\to\infty} x\sin\left(\frac{1}{x}\right) = 0.$$

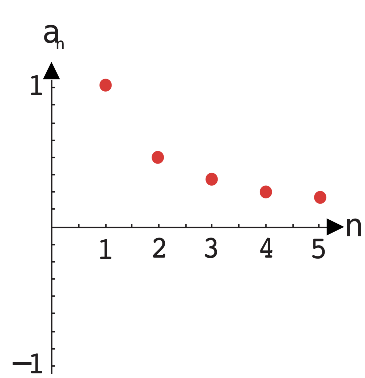
□

Example 5.65. Evaluate $\lim_{n\to\infty} \frac{1}{n}$.

Proof. Let

$$f\colon \begin{matrix} \mathbb{Z}_{>0} & \to & \mathbb{R} \\ n & \mapsto & a_n \end{matrix} \qquad \text{be the function given by} \qquad a_n = \frac{1}{n}.$$

Then $(a_1, a_2, \ldots) = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots)$ and the graph of $\{(n, f(n)) \in \mathbb{Z}_{>0}\times\mathbb{R}\}$ is



As n gets larger and larger $a_n = \frac{1}{n}$ gets closer and closer to 0. So

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

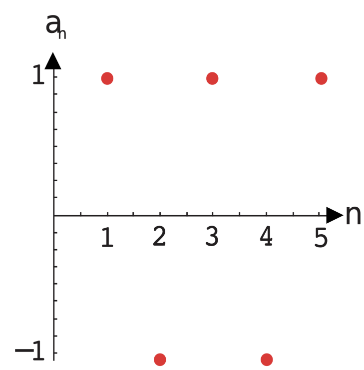
$\square$

Example 5.66. Evaluate $\lim_{n\to\infty} (-1)^{n-1}$.

Proof. Let

$$f\colon \begin{array}{ccc} \mathbb{Z}_{>0} & \to & \mathbb{R} \\ n & \mapsto & a_n \end{array} \quad \text{be the function given by} \quad a_n = (-1)^{n-1}.$$

Then $(a_1, a_2, \ldots) = (1, -1, 1, -1, 1, -1, \ldots)$ and the graph of $\{(n, f(n)) \in \mathbb{Z}_{>0} \times \mathbb{R}\}$ is



As n gets larger and larger $a_n = (-1)^{n-1}$ oscillates between 1 and -1 and does not get closer and closer to any single real number. So

$$\lim_{n\to\infty} (-1)^{n-1} \quad \text{does not exist in } \mathbb{R}.$$

$\square$

Example 5.67. Evaluate $\lim_{n\to\infty} n$.

Proof. Let

$$f\colon \begin{array}{ccc} \mathbb{Z}_{>0} & \to & \mathbb{R} \\ n & \mapsto & a_n \end{array} \quad \text{be the function given by} \quad a_n = n.$$

Then $(a_1, a_2, \ldots) = (1, 2, 3, 4, 5, 6 \ldots)$ and the graph of $\{(n, f(n)) \in \mathbb{Z}_{>0} \times \mathbb{R}\}$ is

PICTURE

As n gets larger and larger $a_n = n$ gets larger and larger. So

$$\lim_{n\to\infty} n \quad \text{does not exist in } \mathbb{R}.$$

$\square$

5.4 Continuity and behavior of x^n and e^x

Example 5.68. (x^n is continuous) Show that if $n \in \mathbb{Z}_{\ge 0}$ and $a \in \mathbb{C}$ then $\lim\limits_{x\to a} x^n = a^n$.

Proof. Assume $n \in \mathbb{Z}_{\ge 0}$ and $a \in \mathbb{C}$. Then

$$\begin{aligned}
\lim_{x\to a} x^n &= \lim_{x\to a} (\underbrace{x \cdot x \cdots x}_{n \text{ times}}) \\
&= \underbrace{\big(\lim_{x\to a} x\big) \cdot \big(\lim_{x\to a} x\big) \cdots \big(\lim_{x\to a} x\big)}_{n \text{ times}} \qquad \text{(by continuity of multiplication)} \\
&= \underbrace{a \cdot a \cdots a}_{n \text{ times}} = a^n.
\end{aligned}$$

□

Example 5.69. (e^x is continuous) Show that if $a \in \mathbb{C}$ then $\lim_{x\to a} e^x = e^a$.

Proof.
Case 1: $a = 0$. To show: $\lim\limits_{x\to 0} e^x = e^0$.
Using Theorem **??**(a), To show $\lim\limits_{x\to 0} |e^x - 1| = 0$.

$$\begin{aligned}
\lim_{x\to 0} |e^x - 1| &= \lim_{x\to 0} \left| \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \right) - 1 \right| \\
&= \lim_{x\to 0} \left| x \left(1 + x + \frac{x}{2!} + \frac{x^2}{3!} + \cdots \right) \right| \\
&\le \lim_{x\to 0} |x| \left(1 + |x| + \frac{|x|}{2!} + \frac{|x|^2}{3!} + \cdots \right) && \text{(triangle inequality for } |\ |) \\
&\le \lim_{x\to 0} |x| \left(1 + |x| + |x| + |x|^2 + \cdots \right) && \text{(by term by term comparison)} \\
&= \lim_{x\to 0} |x| \frac{1}{1 - |x|} = 0 \cdot 1 && \text{(by geometric series)} \\
&= 0.
\end{aligned}$$

Case 2: $a \neq 0$. To show $\lim\limits_{x\to a} e^x = e^a$. Let $x = y + a$. Then $y \to 0$ as $x \to a$ and

$$\begin{aligned}
\lim_{x\to a} e^x &= \lim_{y\to 0} e^{y+a} \\
&= \lim_{y\to 0} e^a e^y = e^a \lim_{y\to 0} e^y && \text{(by continuity of scalar multiplication)} \\
&= e^a \cdot e^0 && \text{(by Case 1)} \\
&= e^{a+0} = e^a.
\end{aligned}$$

□

Example 5.70. (Behaviour of x^n as n gets large) Let $x \in \mathbb{C}$. Show that

$$\lim_{n\to\infty} x^n = \begin{cases} 0, & \text{if } |x| < 1, \\ \text{diverges in } \mathbb{C}, & \text{if } |x| > 1, \\ 1, & \text{if } x = 1, \\ \text{diverges in } \mathbb{C}, & \text{if } |x| = 1 \text{ and } x \neq 1. \end{cases}$$

Proof. Let $x \in \mathbb{C}$.
Case $|x| < 1$. To show: $\lim_{n\to\infty} x^n = 0$.
Let $N \in \mathbb{Z}_{>0}$ such that $|x| < 1 - \frac{1}{N+1}$. Then

$$\begin{aligned}
\lim_{n\to\infty} |x^n - 0| &= \lim_{n\to\infty} |x|^n \le \lim_{n\to\infty} \left(1 - \frac{1}{N+1}\right)^n \\
&= \lim_{n\to\infty} \left(\frac{N+1-1}{N+1}\right)^n = \lim_{n\to\infty} \left(\frac{N}{N+1}\right)^n = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{N}\right)^n} \\
&= \lim_{n\to\infty} \frac{1}{1 + n\frac{1}{N} + \cdots + \left(\frac{1}{N}\right)^n} \\
&\le \lim_{n\to\infty} \frac{1}{1+\frac{n}{N}} = \lim_{n\to\infty} \frac{N}{n+N} = N \cdot \lim_{n\to\infty} \frac{1}{n+N} = N \cdot 0 = 0.
\end{aligned}$$

Case $|x| > 1$. To show: $\lim_{n\to\infty} x^n$ diverges in $\mathbb{C}$.
Let $N \in \mathbb{Z}_{>0}$ such that $|x| > 1 - \frac{1}{N}$. Then

$$|x|^n > \left(1 + \frac{1}{N}\right)^n = 1 + n\left(\frac{1}{N}\right) + \cdots \left(\frac{1}{N}\right)^n > \left(\frac{1}{N}\right) n.$$

Since $\left(\frac{1}{N}\right) n$ is unbounded as n gets larger and larger then $|x|^n$ is unbounded as $n \to \infty$.
So $\lim_{n\to\infty} x^n$ diverges in $\mathbb{C}$.
Case $x = 1$. In this case $(x, x^2, x^3, x^4, \ldots) = (1, 1^2, 1^3, 1^4, \ldots) = (1, 1, 1, 1, \ldots)$.
So $\lim_{n\to\infty} x^n = \lim_{n\to\infty} 1^n = \lim_{n\to\infty} 1 = 1$.
Case $|x| = 1$ *and* $x \neq 1$. Then $x = e^{i\theta}$ with $\theta \in \mathbb{R}_{(0,2\pi)}$. FINISH THE PROOF to show that this case diverges in $\mathbb{C}$. IGUESS THE point is that if $a \in \mathbb{R}$ then

$$|e^{ia} - e^{i(a+\theta)}| = |e^{ia}| \cdot |1 - e^{i\theta}| = |1 - e^{i\theta}| \neq 0.$$

YUP clean THIS UP. □

Example 5.71. (exponentials dominate polynomials) Assume $n \in \mathbb{Z}_{>0}$. Show that, in $\mathbb{R}$,

$$\lim_{x\to\infty} \frac{x^n}{e^x} = 0.$$

Proof. Let $n \in \mathbb{Z}_{>0}$. Then, in $\mathbb{R}$,

$$0 \le \lim_{x\to\infty} \frac{x^n}{e^x} \le \lim_{x\to\infty} \frac{x^n}{\frac{1}{(n+1)!}x^{n+1}} = \lim_{x\to\infty} \frac{(n+1)!}{x} = (n+1)! \lim_{x\to\infty} \frac{1}{x} = 0.$$

□

Example 5.72. (polynomials dominate logs) Assume $\alpha \in \mathbb{R}_{>0}$. Show that

$$\lim_{x\to\infty} \frac{\log(x)}{x^\alpha} = 0.$$

Proof. (b) Let $\alpha \in \mathbb{R}_{>0}$. Then

$$0 \le \lim_{x\to\infty} \frac{\log(x)}{x^\alpha} = \lim_{y\to\infty} \frac{\log(e^y)}{(e^y)^\alpha} = \lim_{y\to\infty} \frac{y}{e^{\alpha y}} = \lim_{y\to\infty} \frac{1}{\frac{1}{y} + \alpha + \frac{1}{2}\alpha^2 y + \cdots} \le \lim_{y\to\infty} \frac{1}{\frac{1}{2}\alpha^2 y} = 0.$$

□

Example 5.73. Let $p \in \mathbb{R}_{>0}$. Show that

$$\lim_{n\to\infty} \frac{1}{n^p} = 0.$$

Proof. (c) Let $p \in \mathbb{R}_{>0}$. Let $k \in \mathbb{Z}_{>0}$ and let $N_k = \lceil e^{\frac{k}{p}\log(10)} \rceil$ so that $n_k \in \mathbb{Z}_{>0}$ and $e^{p\log(N_k)} = (N_k)^p > 10^k = e^{k\log(10)}$.

$$\text{If } n > N_k \text{ then} \qquad 0 \le \frac{1}{n^p} < \frac{1}{N_k^p} \le \frac{1}{10^k}. \qquad \text{So } \lim_{n\to\infty} \frac{1}{n^p} = 0.$$

□

Example 5.74. Let $p \in \mathbb{R}_{>0}$. Then $\lim_{n\to\infty} p^{1/n} = 0$.

Proof. Let $p \in \mathbb{R}_{>0}$.

$$\lim_{n\to\infty} p^{1/n} = \lim_{n\to\infty} (e^{\log p})^{1/n} = \lim_{n\to\infty} e^{\frac{1}{n}\log p} = e^{\log p \cdot \lim_{n\to\infty}(1/n)} = e^{\log p \cdot 0} = e^0 = 1.$$

□

Example 5.75. Show that $\lim_{n\to\infty} n^{1/n} = 1$.

Proof. Using that polynomials dominate logs (Example 5.72),

$$\lim_{n\to\infty} n^{1/n} = \lim_{n\to\infty} (e^{\log(n)})^{1/n} = \lim_{n\to\infty} (e^{\frac{1}{n}\log(n)}) = e^{\lim_{n\to\infty}(\frac{\log(n))}{n})} = e^0 = 1.$$

□

Example 5.76. Show that $\lim_{n\to\infty} \left(1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots + \left(\frac{1}{2}\right)^n\right) = 2$.

Proof.

$$\lim_{n\to\infty} \left(1 + \tfrac{1}{2} + \left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^3 + \cdots + \left(\tfrac{1}{2}\right)^n\right) = \lim_{n\to\infty} \frac{1 - \left(\frac{1}{2}\right)^{n+1}}{1 - \frac{1}{2}} = \frac{1}{1 - \frac{1}{2}} = 2.$$

□

Example 5.77. (Behaviour of $1 + x + x^2 + \cdots + x^n$ as n gets large) Let $x \in \mathbb{C}$. Show that if $|x| < 1$ then

$$\lim_{n\to\infty} (1 + x + x^2 + \cdots + x^n) = \frac{1}{1-x}.$$

Proof. By continiuity of addition and division away from 0,

$$\lim_{n\to\infty} (1 + x + x^2 + \cdots + x^n) = \lim_{n\to\infty} \frac{1 - x^{n+1}}{1-x} = \frac{1 - \left(\lim_{n\to\infty} x^{n+1}\right)}{1-x} = \frac{1-0}{1-x} = \frac{1}{1-x}.$$

□

Example 5.78. (Behaviour of $1 + x + x^2 + \cdots + x^n$ as n gets large) Let $x \in \mathbb{C}$. Show that if $|x| > 1$ then

$$\lim_{n\to\infty} (1 + x + x^2 + \cdots + x^n) = \frac{1}{1-x} \qquad \text{does not exist in } \mathbb{C}.$$

Proof. By continuity of addition and scalar multiplication,

$$\lim_{n\to\infty}(1+x+x^2+\cdots+x^n) = \lim_{n\to\infty}\frac{1-x^{n+1}}{1-x} = \frac{1-\big(\lim_{n\to\infty}x^{n+1}\big)}{1-x}$$

and the right hand side does not converge in $\mathbb{C}$. ☐

Example 5.79. Show that $\lim_{x\to 0}\frac{\sin(x)}{x} = 1$.

Proof.

$$\lim_{x\to 0}\frac{\sin(x)}{x} = \lim_{x\to 0}\big(1-\frac{1}{3!}x^2+\cdots\big) = 1+0+0+\cdots = 1.$$

This proof has the right reason, but it is not quite complete because 'the sum of an infinite number of 0s' is really 'the sum of a infinite number of really tiny numbers' and we to be sure that sum really is 0. How should the proof be tightened up? (with a comparison to a geometric series, see the proof of Example 5.69.) ☐

Example 5.80. Show that $\lim_{x\to 0}\frac{\log(1+x)}{x} = 1$.

Proof.

$$\lim_{x\to 0}\frac{\log(1+x)}{x} = \lim_{x\to 0}\big(1-\frac{1}{2}x+\frac{1}{3}x^2-\frac{1}{4}x^3+\cdots\big) = 1+0+0+\cdots = 1.$$

This proof has the right reason, but it is not quite complete because 'the sum of an infinite number of 0s' is really 'the sum of a infinite number of really tiny numbers' and we to be sure that sum really is 0. How should the proof be tightened up? (with a comparison to a geometric series, see the proof of Example 5.69.) ☐

5.5 Continuity of addition, multiplication, composition and order

Example 5.81. (scalar multiplication is continuous) Let $n\in\mathbb{Z}_{>0}$. Let $f\colon\mathbb{R}^n\to\mathbb{R}$ and $g\colon\mathbb{R}^n\to\mathbb{R}$ be functions and let $a\in\mathbb{R}^n$.

$$\text{Assume that}\qquad \lim_{x\to a}f(x)\qquad\text{exists.}$$

Then, if $c\in\mathbb{R}$ then $\lim_{x\to a}cf(x) = c\lim_{x\to a}f(x)$,

Proof.

Assume $c\in\mathbb{R}$ and let $l = \lim_{x\to a}f(x)$.
To show: $\lim_{x\to a}cf(x) = cl$.
To show: If $e\in\mathbb{Z}_{>0}$ then there exists $d\in\mathbb{Z}_{>0}$ such that

$$\text{if } x\in\mathbb{R}^n \text{ is within } 10^{-d} \text{ of } a \text{ then } cf(x) \text{ is within } 10^{-e} \text{ of } cl.$$

Assume $e\in\mathbb{Z}_{>0}$.
Let $r\in\mathbb{Z}_{>0}$ be such that $c<10^r$.
Since $l = \lim_{x\to a}f(x)$ then we know that there exists $d\in\mathbb{Z}_{>0}$ such that

$$\text{if } x\in\mathbb{R}^n \text{ is within } 10^{-d} \text{ of } a \text{ then } f(x) \text{ is within } 10^{-(e+r)} \text{ of } l.$$

To show: If $x \in \mathbb{R}^n$ is within 10^{-d} of a then $cf(x)$ is within 10^{-e} of cl.
Assume $x \in \mathbb{R}^n$ is within 10^{-d} of a.
To show: $cf(x)$ is within 10^{-e} of cl.

$$d(cf(x), cl) = |cf(x) - cl| = |c| \cdot |f(x) - l| < |c| \cdot 10^{-(e+r)} < 10^r 10^{-(e+r)} = 10^{-e}.$$

So $cf(x)$ is within 10^{-e} of cl.
So $\lim_{x \to a} cf(x) = cl$.

□

Example 5.82. (Addition is continuous) Let $n \in \mathbb{Z}_{>0}$. Let $f\colon \mathbb{R}^n \to \mathbb{R}$ and $g\colon \mathbb{R}^n \to \mathbb{R}$ be functions and let $a \in \mathbb{R}^n$.

$$\text{Assume that} \quad \lim_{x \to a} f(x) \quad \text{and} \quad \lim_{x \to a} g(x) \quad \text{exist.}$$

Then $\lim_{x \to a}(f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$,

Proof.

Let $l_1 = \lim_{x \to a} f(x)$ and $l_2 = \lim_{x \to a} g(x)$.
To show: $\lim_{x \to a}(f(x) + g(x)) = l_1 + l_2$.
To show: If $e \in \mathbb{Z}_{>0}$ then there exists $d \in \mathbb{Z}_{>0}$ such that

$$\text{if } x \text{ is within } 10^{-d} \text{ of } a \text{ then } f(x) + g(x) \text{ is within } 10^{-e} \text{ of } l_1 + l_2.$$

Assume $e \in \mathbb{Z}_{>0}$.
Since $\lim_{x \to a} f(x) = l_1$ then we know that there exists $d_1 \in \mathbb{Z}_{>0}$ such that

$$\text{if } x \text{ is within } 10^{-d_1} \text{ of } a \text{ then } f(x) \text{ is within } 10^{-(e+1)} \text{ of } l_1.$$

Since $\lim_{x \to a} g(x) = l_2$ then we know that there exists $d_2 \in \mathbb{Z}_{>0}$ such that

$$\text{if } x \text{ is within } 10^{-d_2} \text{ of } a \text{ then } g(x) \text{ is within } 10^{-(e+1)} \text{ of } l_2.$$

Let $d = \max(d_1, d_2)$.
To show: if x is within 10^{-d} of a then $f(x) + g(x)$ is within 10^{-e} of l_1+l_2.
Assume x is within 10^{-d} of a.
To show: $f(x) + g(x)$ is within 10^{-e} of l_1+l_2.

$$\begin{aligned} |(f(x) + g(x)) - (l_1 + l_2)| &= |(f(x) - l_1) + (g(x) - l_2)| \\ &\le |f(x) - l_1| + |g(x) - l_2| \\ &\le 10^{-e+1} + 10^{-(e+1)} = \frac{2}{10} 10^{-e} < 10^{-e}. \end{aligned}$$

So $(f(x) + g(x)$ is within 10^{-e} of l_1+l_2.
So $\lim_{x \to a}(f(x) + g(x)) = l_1 + l_2 = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$.

□

Example 5.83. (multiplication is continuous) Let $n \in \mathbb{Z}_{>0}$. Let $f\colon \mathbb{R}^n \to \mathbb{R}$ and $g\colon \mathbb{R}^n \to \mathbb{R}$ be functions and let $a \in \mathbb{R}^n$.

$$\text{Assume that}\qquad \lim_{x\to a} f(x) \quad\text{and}\quad \lim_{x\to a} g(x) \qquad \text{exist.}$$

Then $\lim_{x\to a}(f(x)g(x)) = \big(\lim_{x\to a} f(x)\big)\big(\lim_{x\to a} g(x)\big)$.

Proof.

Let $l_1 = \lim_{x\to a} f(x)$ and $l_2 = \lim_{x\to a} g(x)$.
To show: $\lim_{x\to a}(f(x)g(x)) = l_1 l_2$.
To show: If $e \in \mathbb{Z}_{>0}$ then there exists $d \in \mathbb{Z}_{>0}$ such that

$$\text{if } x \in \mathbb{R}^n \text{ is within } 10^{-d} \text{ of } a \text{ then } f(x)g(x) \text{ is within } 10^{-e} \text{ of } l_1 l_2.$$

Assume $e \in \mathbb{Z}_{>0}$.
Let $r, s \in \mathbb{Z}_{>0}$ such that $|\ell_1| < 10^r$ and $|\ell_2| < 10^s$.
Since $\lim_{x\to a} f(x) = l_1$ then we know that there exists $d_1 \in \mathbb{Z}_{>0}$ such that

$$\text{if } x \in \mathbb{R}^n \text{ is within } 10^{-d_1} \text{ of } a \text{ and } f(x) \text{ is within } 10^{-(e+s+1)} \text{ of } l_1.$$

Since $\lim_{x\to a} f(x) = l_2$ then we know that there exists $d_2 \in \mathbb{Z}_{>0}$ such that

$$\text{if } x \in \mathbb{R}^n \text{ is within } 10^{-d_2} \text{ of } a \text{ and } f(x) \text{ is within } 10^{-(e+r+1)} \text{ of } l_2.$$

Let $d = \max(d_1, d_2)$.
Assume $x \in \mathbb{R}^n$ is within 10^{-d} of a.
To show: $f(x)g(x)$ is within 10^{-e} of $l_1 l_2$.

$$\begin{aligned}
|f(x)g(x) - l_1 l_2| &= |(f(x)-l_1)g(x) + l_1(g(x)-l_2)| \\
&\le |(f(x)-l_1)g(x)| + |l_1(g(x)-l_2)|, \quad \text{by the triangle inequality,} \\
&= |(f(x)-l_1)(g(x)-l_2) + (f(x)-l_1)l_2| + |l_1|\,|g(x)-l_2| \\
&\le |(f(x)-l_1)(g(x)-l_2)| + |(f(x)-l_1)l_2| + |l_1|\,|g(x)-l_2| \\
&\le |f(x)-l_1|\,|g(x)-l_2| + |f(x)-l_1|\,|l_2| + |l_1|\,|g(x)-l_2| \\
&\le |f(x)-l_1|\,|g(x)-l_2| + |f(x)-l_1|\,10^s + 10^r\,|g(x)-l_2| \\
&\le 10^{-(e+r+1)}\cdot 10^{-(e+s+1)} + 10^{-(e+s+1)}10^s + 10^r 10^{-(e+r+1)} \\
&= 10^{-e}(10^{-(e+r+s+2)} + 10^{-1} + 10^{-1}) < 10^{-e}\cdot 1 = 10^{-e}.
\end{aligned}$$

So $f(x)g(x)$ is within 10^{-e} of $l_1 l_2$.
So there exists $d \in \mathbb{Z}_{>0}$ such that

$$\text{if } x \in \mathbb{R}^n \text{ is within } 10^{-d} \text{ of } a \text{ then } f(x)g(x) \text{ is within } 10^{-e} \text{ of } l_1 l_2.$$

So $\lim_{x\to a}(f(x)g(x)) = l_1 l_2$.

□

Example 5.84. (Limits and composition of functions) Let $m, n, p \in \mathbb{Z}_{>0}$. Let Let $f\colon \mathbb{R}^n \to \mathbb{R}^p$ and $g\colon \mathbb{R}^m \to \mathbb{R}^n$ be functions and let $a \in \mathbb{R}^m$ and $\ell \in \mathbb{R}^n$.

$$\text{Assume that}\qquad \lim_{x\to a} g(x) = \ell \quad \text{and } \lim_{y\to\ell} f(y) \text{ exists.}$$

Then

$$\lim_{y\to\ell} f(y) = \lim_{x\to a} f(g(x)).$$

Proof.

Let $L = \lim_{y\to\ell} f(y)$.
To show: $\lim_{x\to a} f(g(x)) = L$.
To show: If $e \in \mathbb{Z}_{>0}$ then there exists $d \in \mathbb{Z}_{>0}$ such that

$$\text{if } x \in \mathbb{R}^m \text{ is within } 10^{-d} \text{ of } a \text{ then } f(g(x)) \text{ is within } 10^{-e} \text{ of } L.$$

Assume $e \in \mathbb{Z}_{>0}$.
To show: There exists $d \in \mathbb{Z}_{>0}$ such that

$$\text{if } x \in \mathbb{R}^m \text{ is within } 10^{-d} \text{ of } a \text{ then } f(g(x)) \text{ is within } 10^{-e} \text{ of } L.$$

Since $\lim_{y\to\ell} f(y) = L$ we know that there exists $d_1 \in \mathbb{Z}_{>0}$ such that

$$\text{if } y \in \mathbb{R}^n \text{ is within } 10^{-d_1} \text{ of } \ell \text{ then } f(y) \text{ is within } 10^{-e} \text{ of } L.$$

Since $\lim_{x\to a} g(x) = \ell$ we know that there exists $d \in \mathbb{Z}_{>0}$ such that

$$\text{if } x \in \mathbb{R}^m \text{ is within } 10^{-d} \text{ of } a \text{ then } g(x) \text{ is within } 10^{-d_1} \text{ of } \ell.$$

To show: If $x \in \mathbb{R}^n$ is within 10^{-d} of a then $f(g(x))$ is within 10^{-e} of L.
Assume $x \in \mathbb{R}^n$ is within 10^{-d} of a.
To show: $f(g(x))$ is within 10^{-e} of L.
Since x is within 10^{-d} of a then $g(x)$ is within 10^{-d_1} of ℓ,
and so $f(g(x))$ is within 10^{-e} of L.
So, if $x \in \mathbb{R}^n$ is within 10^{-d} of a then $f(g(x))$ is within 10^{-e} of L.
So there exists $d \in \mathbb{Z}_{>0}$ such that if $x \in \mathbb{R}^n$ is within 10^{-d} of a then $f(g(x))$ is within 10^{-e} of L.
So $\lim_{x\to a} f(g(x)) = L$.

□

Example 5.85. (Limits and order) Let $n \in \mathbb{Z}_{>0}$ and let $f\colon \mathbb{R}^n \to \mathbb{R}$ and $g\colon \mathbb{R}^n \to \mathbb{R}$ be functions. Let $a \in \mathbb{R}^n$. Assume that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist and

$$\text{if } x \in X \text{ then } f(x) \le g(x).$$

Then $\lim_{x\to a} f(x) \le \lim_{x\to a} g(x)$.

Proof.

Let $\ell_1 = \lim_{x\to a} f(x)$ and $\ell_2 = \lim_{x\to a} g(x)$.
To show: If f and g satisfy the condition

$$\text{if } x \in X \text{ then } f(x) \le g(x),$$

then $\ell_1 \le \ell_2$.
Proof by contrapositive.
Assume $\ell_1 > \ell_2$ (the opposite of $\ell_1 \le \ell_2$ is $\ell_1 > \ell_2$).
To show: There exists $x \in \mathbb{R}^n$ such that $f(x) > g(x)$
(the opposite of 'if $x \in \mathbb{R}^n$ then $f(x) \le g(x)$' is 'there exists $x \in \mathbb{R}^n$ such that $f(x) > g(x)$.').
Let $r \in \mathbb{Z}_{>0}$ be such that $10^{-r} < \ell_1 - \ell_2$.
Since $\lim_{x\to a} f(x) = \ell_1$ then we know that there exists $d_1 \in \mathbb{Z}_{>0}$ such that

$$\text{if } x \in \mathbb{R}^n \text{ is within } 10^{-d_1} \text{ of } a \text{ then } f(x) \text{ is within } 10^{-(r+1)} \text{ of } \ell_1.$$

Since $\lim_{x\to a} g(x) = \ell_2$ then we know that there exists $d_2 \in \mathbb{Z}_{>0}$ such that

$$\text{if } x \in \mathbb{R}^n \text{ is within } 10^{-d_2} \text{ of } a \text{ then } f(x) \text{ is within } 10^{-(r+1)} \text{ of } \ell_2.$$

Let $d = \max(d_1, d_2)$ and let $x \in \mathbb{R}^n$ be within 10^{-d} of a (so that $x \ne a$ but x is quite close to a).
To show: $f(x) > g(x)$.

$$f(x) > \ell_1 - 10^{-(r+1)} = \ell_1 - \ell_2 + \ell_2 - 10^{-(r+1)} > 10^{-r} + \ell_2 - 10^{-(r+1)} > \ell_2 + 10^{-(r+1)} > g(x).$$

This proves that if f and g satisfy the condition 'if $x \in X$ then $f(x) \le g(x)$' then $\ell_1 \le \ell_2$.

□

Example 5.86. (Limits and order for sequences) Let $(a_1, a_2, \ldots)$ and $(b_1, b_2, \ldots)$ be sequences in $\mathbb{R}$. Assume that $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ exist and

$$\text{if } n \in \mathbb{Z}_{>0} \text{ then } a_n \le b_n.$$

Then $\lim_{n\to\infty} a_n \le \lim_{n\to\infty} b_n$.

Proof.

Let $\ell_1 = \lim_{n\to\infty} a_n$ and $\ell_2 = \lim_{n\to\infty} b_n$.
To show: If $(a_1, a_2, \ldots)$ and $(b_1, b_2, \ldots)$ satisfy the condition

$$\text{if } n \in \mathbb{Z}_{>0} \text{ then } a_n \le b_n,$$

then $\ell_1 \le \ell_2$.
Proof by contrapositive.
Assume $\ell_1 > \ell_2$ (the opposite of $\ell_1 \le \ell_2$ is $\ell_1 > \ell_2$).
To show: There exists $N \in \mathbb{Z}_{>0}$ such that $a_N > b_N$
(the opposite of 'if $n \in \mathbb{Z}_{>0}$ then $a_n \le b_n$' is 'there exists $N \in \mathbb{Z}_{>0}$ such that $a_N > b_N$').
Let $r \in \mathbb{Z}_{>0}$ be such that $10^{-r} < \ell_1 - \ell_2$.

Since $\lim_{n\to\infty} a_n = \ell_1$ then we know that there there exists $N_1 \in \mathbb{Z}_{>0}$ such that

$$\text{if } n \in \mathbb{Z}_{>0} \text{ is at least } N_1 \text{ then } a_n \text{ is within } 10^{-(r+1)} \text{ of } \ell_1.$$

Since $\lim_{n\to\infty} b_n = \ell_2$ then we know that there there exists $N_2 \in \mathbb{Z}_{>0}$ such that

$$\text{if } n \in \mathbb{Z}_{>0} \text{ is at least } N_2 \text{ then } b_n \text{ is within } 10^{-(r+1)} \text{ of } \ell_2.$$

Let $N = \max(N_1, N_2)$.
To show: $a_N > b_N$.

$$\begin{aligned} a_N &> \ell_1 - 10^{-(r+1)} = \ell_1 - \ell_2 + \ell_2 - 10^{-(r+1)} \\ &> 10^{-r} + \ell_2 - 10^{-(r+1)} > \ell_2 + 10^{-(r+1)} > b_N. \end{aligned}$$

This proves that if $(a_1, a_2, \ldots)$ and $(b_1, b_2, \ldots)$ satisfy the condition 'if $n \in \mathbb{Z}_{>0}$ then $a_n \le b_n$' then $\ell_1 \le \ell_2$.

□

5.6 The interest sequence

Example 5.87. If you borrow \$500 on your credit card at 14% interest, find the amounts due at the end of two years if the interest is compounded

(a) annually,
(b) quarterly,
(c) monthly,
(d) daily,
(e) hourly,
(f) every second,
(g) every nanosecond,
(h) continuously.

Proof.
(a) You owe

$$500 + 500(.14) = 500(1 + .14) \text{ after one year} \qquad \text{and} \qquad 500(1 + .14)(1 + .14) \text{ after two years.}$$

(b) You owe

$$500 + 500\Big(\frac{.14}{12}\Big) = 500\Big(1 + \frac{.14}{12}\Big) \quad \text{after one month.}$$

You owe

$$500\Big(1 + \frac{.14}{12}\Big)\Big(1 + \frac{.14}{12}\Big) \quad \text{after two months.}$$

You owe

$$500\Big(1 + \frac{.14}{12}\Big)^{24} \quad \text{after two years.}$$

(f) You owe

$$500 + 500\Big(\frac{.14}{365 \cdot 24 \cdot 3600}\Big) \quad \text{after one second.}$$

and

$$500\Big(1 + \frac{.14}{365 \cdot 24 \cdot 3600}\Big)^{2\cdot 365\cdot 24\cdot 3600} \qquad \text{after two years.}$$

In fact,

$$\begin{aligned}
\lim_{n\to\infty} 500\Big(1 + \frac{.14}{n}\Big)^{2n} &= 500 \lim_{n\to\infty} \Big(e^{\log\left(1+\frac{.14}{n}\right)}\Big)^{2n} \\
&= 500 \lim_{n\to\infty} e^{2n\log\left(1+\frac{.14}{n}\right)} \\
&= 500 \lim_{n\to\infty} e^{2\cdot .14\frac{\log(1+\frac{.14}{n})}{\frac{.14}{n}}} \\
&= 500 \lim_{n\to\infty} e^{.28\frac{\log(1+\frac{.14}{n})}{\frac{.14}{n}}} = 500e^{.28},
\end{aligned}$$

since

$$\lim_{x\to 0} \frac{\log(1+x)}{x} = 1.$$

So you owe $500e^{.28}$ after two years if the interest is compounded continuously.

Note: $500(1 + .14)^2 = 649.80$, $\quad 500\Big(1 + \frac{.14}{12}\Big)^{24} \approx 660.49$, and $\quad 500e^{.28} \approx 661.58$. ☐

5.7 Series

Example 5.88. (Constant series) Show that

$$\sum_{n=1}^{\infty} 1 \qquad \text{does not converge in } \mathbb{R} \text{ or } \mathbb{C}.$$

Proof.

$$\sum_{n=1}^{\infty} 1 = \lim_{r\to\infty}\Big(\sum_{n=1}^{r} 1\Big) = \lim_{r\to\infty} r \quad \text{does not converge in } \mathbb{R} \text{ or } \mathbb{C},$$

since r is growing and is unbounded. □

Example 5.89. (Geometric series) Let $x \in \mathbb{C}$. Show that if $|x| < 1$ then

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

Proof. By continuity of addition and division away from 0,

$$\sum_{n=0}^{\infty} x^n = \lim_{r\to\infty}(1 + x + x^2 + \cdots + x^r) = \lim_{r\to\infty}\frac{1-x^{r+1}}{1-x} = \frac{1-\big(\lim_{r\to\infty} x^{r+1}\big)}{1-x} = \frac{1}{1-x}.$$

□

Example 5.90. (The zeta function) Show that

$$\sum_{n=1}^{\infty} \frac{1}{n} \quad \text{does not converge in } \mathbb{R}.$$

Proof. This proof is by comparison to a constant series. The sum $\zeta(1) = \sum_{n=1}^{\infty} \frac{1}{n}$ does not converge in $\mathbb{R}$ since

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}} + \cdots > 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots .$$

and the right hand side is growing and is unbounded. □

Example 5.91. (The zeta function) Show that

$$\text{if } p \in \mathbb{R}_{(0,1)} \quad \text{then} \qquad \sum_{n=1}^{\infty} \frac{1}{n^p} \quad \text{does not converge in } \mathbb{R}.$$

Proof. This proof is by comparison to $\zeta(1)$. The sum $\zeta(p) = \sum_{n=1}^{\infty} \frac{1}{n^p}$ does not converge in $\mathbb{R}$ since

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots > 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \zeta(1)$$

and $\zeta(1)$ is growing and is unbounded. □

Example 5.92. (The zeta function) Show that

$$\text{if } p \in \mathbb{R}_{>1} \quad \text{then} \qquad \sum_{n=1}^{\infty} \frac{1}{n^p} \quad \text{converges in } \mathbb{R}.$$

Proof. This proof is by comparison to a geometric series. The sum $\zeta(p) = \sum_{n=1}^{\infty} \frac{1}{n^p}$ converges in $\mathbb{R}$ since

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n^p} &= 1 + \underbrace{\frac{1}{2^p} + \frac{1}{3^p}} + \underbrace{\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}} + \cdots \\
&< 1 + \frac{2}{2^p} + \frac{4}{4^p} + \frac{8}{8^p} + \cdots \\
&= 1 + \frac{1}{2^{p-1}} + \frac{1}{4^{p-1}} + \frac{1}{8^{p-1}} + \cdots \\
&= 1 + \frac{1}{2^{p-1}} + \left(\frac{1}{2^{p-1}}\right)^2 + \left(\frac{1}{2^{p-1}}\right)^3 + \cdots \\
&= \frac{1}{1 - \frac{1}{2^{p-1}}} = \frac{2^{p-1}}{2^{p-1} - 1}.
\end{aligned}$$

□

Example 5.93. (The zeta function) Assume $p \in \mathbb{R}_{>0}$. Show that

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \quad \text{converges if and only if } p \in \mathbb{R}_{>1}.$$

Proof. Case 1: $p = 1$. In this case $\zeta(1) = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges since

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}} + \cdots > 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots .$$

Case 2: $p \in \mathbb{R}_{<1}$. Then $\zeta(p) = \sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges since

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots > 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots .$$

Case 3: $p \in \mathbb{R}_{>1}$. Then $\zeta(p) = \sum_{n=1}^{\infty} \frac{1}{n^p}$ converges since

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n^p} &= 1 + \underbrace{\frac{1}{2^p} + \frac{1}{3^p}} + \underbrace{\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}} + \cdots \\
&< 1 + \frac{2}{2^p} + \frac{4}{4^p} + \frac{8}{8^p} + \cdots \\
&= 1 + \frac{1}{2^{p-1}} + \frac{1}{4^{p-1}} + \frac{1}{8^{p-1}} + \cdots \\
&= 1 + \frac{1}{2^{p-1}} + \left(\frac{1}{2^{p-1}}\right)^2 + \left(\frac{1}{2^{p-1}}\right)^3 + \cdots \\
&= \frac{1}{1 - \frac{1}{2^{p-1}}} = \frac{2^{p-1}}{2^{p-1} - 1}.
\end{aligned}$$

□

Example 5.94. (Ratio test for convergence) Let $(a_1, a_2, a_3, \ldots)$ be a sequence in $\mathbb{R}$. Assume $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = a$ exists and $a < 1$. Show that

$$\sum_{n=1}^{\infty} |a_n| \qquad \text{exists in } \mathbb{R}.$$

Proof. Assume $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = a$ exists and $a < 1$.
Let $\varepsilon \in \mathbb{R}_{>0}$ be such that $a + \varepsilon < 1$.
Since $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = a$ there exists $N \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{\geq N}$ then $\frac{|a_{n+1}|}{|a_n|} < a + \varepsilon$.
Then

$$\begin{aligned}
\sum_{n=0}^{\infty} |a_n| &= |a_0| + |a_1| + \cdots + |a_N| + |a_{N+1}| + |a_{N+2}| + \cdots \\
&= |a_0| + \cdots + |a_N| + |a_{N+1}| + |a_{N+1}|\left(\frac{|a_{N+2}|}{|a_{N+1}|}\right) + |a_{N+1}|\left(\frac{|a_{N+2}|}{|a_{N+1}|}\right)\left(\frac{|a_{N+3}|}{|a_{N+2}|}\right) + \cdots \\
&< |a_0| + \cdots + |a_N| + |a_{N+1}| + |a_{N+1}|(a+\varepsilon) + a_{N+1}(a+\varepsilon)^2 + \cdots \\
&= |a_0| + \cdots + |a_N + |a_{N+1}|(1 + (a+\varepsilon) + (a+\varepsilon)^2 + \cdots) \\
&= |a_0| + \cdots + |a_N| + |a_{N+1}|\left(\frac{1}{1-(a+\varepsilon)}\right).
\end{aligned}$$

Then, since $a + \varepsilon < 1$, $\sum_{n=0}^{\infty} |a_n|$ converges. □

Example 5.95. (Ratio test for divergence) Let $(a_1, a_2, a_3, \ldots)$ be a sequence in $\mathbb{R}$. Assume $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = a$ exists and $a > 1$. Show that

$$\sum_{n=1}^{\infty} |a_n| \qquad \text{does not exist in } \mathbb{R}.$$

Proof. Assume $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = a$ exists and $a > 1$.
Let $\varepsilon \in \mathbb{R}_{>0}$ be such that $a + \varepsilon > 1$.
Since $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = a$ there exists $N \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{\geq N}$ then $\frac{|a_{n+1}|}{|a_n|} < a - \varepsilon$.
Then

$$\begin{aligned}
\sum_{n=0}^{\infty} |a_n| &= |a_0| + |a_1| + \cdots + |a_N| + |a_{N+1}| + |a_{N+2}| + \cdots \\
&= |a_0| + \cdots + |a_N| + |a_{N+1}| + |a_{N+1}| \left(\frac{|a_{N+2}|}{|a_{N+1}|}\right) + |a_{N+1}| \left(\frac{|a_{N+2}|}{|a_{N+1}|}\right)\left(\frac{|a_{N+3}|}{|a_{N+2}|}\right) + \cdots \\
&= |a_0| + \cdots + |a_N| + |a_{N+1}| + |a_{N+1}|(a-\varepsilon) + a_{N+1}(a-\varepsilon)^2 + \cdots \\
&> |a_0| + \cdots + |a_N| + |a_{N+1}|(1 + (a-\varepsilon) + (a-\varepsilon)^2 + \cdots) \\
&> |a_0| + \cdots + |a_N| + |a_{N+1}|(1 + 1 + 1 + \cdots).
\end{aligned}$$

So $\sum_{n=0}^{\infty} |a_n|$ does not exist in $\mathbb{R}$. □

Example 5.96. (Root test for convergence) Let $(a_1, a_2, a_3, \ldots)$ be a sequence in $\mathbb{R}$.
Assume If $\lim_{n\to\infty} |a_n|^{1/n} = a$ exists and $a < 1$. Show that

$$\sum_{n=1}^{\infty} |a_n| \qquad \text{exists in } \mathbb{R}.$$

Proof. Assume $\lim_{n\to\infty} |a_n|^{1/n} = a$ exists and $a < 1$.
Let $\varepsilon \in \mathbb{R}_{>0}$ be such that $a + \varepsilon < 1$.
Since $\lim_{n\to\infty} |a_n|^{1/n} = a$ then there exists $N \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{\geq N}$ then $|a_n|^{1/n} < a + \varepsilon$.
Then

$$\begin{aligned}
\sum_{n=0}^{\infty} |a_n| &= |a_0| + |a_1| + \cdots + |a_N| + |a_{N+1}| + |a_{N+2}| + \cdots \\
&= |a_0| + \cdots + a_N| + \left(|a_{N+1}|^{1/(N+1)}\right)^{N+1} + \left(|a_{N+2}|^{1/(N+2)}\right)^{N+2} + \cdots \\
&< |a_0| + \cdots + |a_N| + (a+\varepsilon)^{N+1} + (a+\varepsilon)^{N+2} + \cdots \\
&= |a_0| + \cdots + |a_N| + (a+\varepsilon)^{N+1}(1 + (a+\varepsilon) + (a+\varepsilon)^2 + \cdots) \\
&= |a_0| + \cdots + |a_N| + (a+\varepsilon)^{N+1}\left(\frac{1}{1-(a+\varepsilon)}\right).
\end{aligned}$$

Then, since $a + \varepsilon < 1$, $\sum_{n=0}^{\infty} |a_n|$ exists in $\mathbb{R}$. □

Example 5.97. (Root test for divergence) Let $(a_1, a_2, a_3, \ldots)$ be a sequence in $\mathbb{R}$.
Assume If $\lim_{n\to\infty} |a_n|^{1/n} = a$ exists and $a > 1$. Show that

$$\sum_{n=1}^{\infty} |a_n| \qquad \text{does not exist in } \mathbb{R}.$$

Proof. Assume $\lim_{n\to\infty} |a_n|^{1/n} = a$ exists and $a > 1$.
Assume $\lim_{n\to\infty} |a_n|^{1/n} = a$ exists and $a > 1$. Let $\varepsilon \in \mathbb{R}_{>0}$ be such that $a+\varepsilon > 1$.
Since $\lim_{n\to\infty} |a_n|^{1/n} = a$ there exists $N \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{\geq N}$ then $|a_n|^{1/n} < a - \varepsilon$.
Then

$$\begin{aligned}\sum_{n=0}^{\infty} |a_n| &= |a_0| + |a_1| + \cdots + |a_N| + |a_{N+1}| + |a_{N+2}| + \cdots \\ &= |a_0| + \cdots + |a_N| + \left(|a_{N+1}|^{1/(N+1)}\right)^{N+1} + \left(|a_{N+2}|^{1/(N+2)}\right)^{N+2} + \cdots \\ &> |a_0| + \cdots + |a_N| + (a-\varepsilon)^{N+1} + (a-\varepsilon)^{N+2} + \cdots \\ &= |a_0| + \cdots + |a_N| + (a-\varepsilon)^{N+1}(1 + (a-\varepsilon) + (a-\varepsilon)^2 + \cdots) \\ &> |a_0| + \cdots + |a_N| + (a-\varepsilon)^{N+1}(1+1+1+\cdots).\end{aligned}$$

So $\sum_{n=0}^{\infty} |a_n|$ does not exist in $\mathbb{R}$. □

Example 5.98. (Absolute convergence gives convergence) Let $(a_1, a_2, a_3, \ldots)$ be a sequence in $\mathbb{R}$ or $\mathbb{C}$.

$$\text{If} \quad \sum_{n=1}^{\infty} |a_n| \text{ converges} \qquad \text{then} \qquad \sum_{n=1}^{\infty} a_n \text{ converges.}$$

Proof.

Assume that $\sum_{n=0}^{\infty} |a_n|$ converges.
To show: $\sum_{n=0}^{\infty} a_n$ converges.
Let $A_n = |a_0| + |a_1| + \cdots + |a_n|$ and $s_n = a_0 + a_1 + \cdots + a_n$.
Since $\sum_{n=0}^{\infty} |a_n| = (A_0, A_1, \ldots)$ converges then the sequence $(A_0, A_1, \ldots)$ is Cauchy.
Let $m, n \in \mathbb{Z}_{\geq 0}$ with $m \leq n$.
Since

$$|s_n - s_m| = |a_{m+1} + \cdots + a_n| \leq |a_{m+1}| + \cdots + |a_n| = |A_n - A_m|,$$

then the sequence $(s_0, s_1, \ldots)$ is Cauchy.
Since Cauchy sequences converge in $\mathbb{R}$ and $\mathbb{C}$ (in any complete metric space),
then the sequence $(s_0, s_1, \ldots) = \sum_{n=1}^{\infty} a_n$ converges.

□

Example 5.99. (Radius of convergence) Let $(a_0, a_1, a_2, a_3, \ldots)$ be a sequence in $\mathbb{R}$ or $\mathbb{C}$. Let $r, s \in \mathbb{C}$ and

$$\text{assume} \quad \sum_{n=0}^{\infty} a_n s^n \text{ converges.} \quad \text{If } |r| < |s| \quad \text{then} \quad \sum_{n=0}^{\infty} a_n |r|^n \text{ converges.}$$

Proof.

Since $\sum_{n=0}^{\infty} a_n s^n$ converges, $\lim_{n\to\infty} |a_n s^n| = 0$.
Let $\varepsilon \in \mathbb{R}_{>0}$.
Then there exists $N \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{\geq N}$ then $|a_n s^n| < \varepsilon$.
Then

$$\begin{aligned}
\sum_{n=0}^{\infty} |a_n r^n| &= |a_0| + |a_1 r| + \cdots + |a_N r^N| + |a_{N+1} r^{N+1}| + \cdots \\
&= |a_0| + \cdots + |a_N r^N| + |a_{N+1} s^{N+1}| \left|\frac{r^{N+1}}{s^{N+1}}\right| + |a_{N+2} s^{N+2}| \left|\frac{r^{N+2}}{s^{N+2}}\right| + \cdots \\
&< |a_0| + \cdots + |a_N r^N| + \varepsilon \left|\frac{r^{N+1}}{s^{N+1}}\right| + \varepsilon \left|\frac{r^{N+2}}{s^{N+2}}\right| + \cdots \\
&= |a_0| + \cdots + |a_N r^N| + \varepsilon \left|\frac{r^{N+1}}{s^{N+1}}\right| \left(1 + \left|\frac{r}{s}\right| + \left|\frac{r^2}{s^2}\right| + \cdots\right) \\
&= |a_0| + |a_1 r| + \cdots + |a_N r^N| + \varepsilon \left|\frac{r^{N+1}}{s^{N+1}}\right| \left(\frac{1}{1 - \left|\frac{r}{s}\right|}\right).
\end{aligned}$$

Thus, since $|r| < |s|$ then $\sum_{n=0}^{\infty} |a_n r^n|$ converges.

So, by the previous Proposition, $\sum_{n=0}^{\infty} a_n |r|^n$ converges.

□

Example 5.100. (Alternating series) If $(a_1, a_2, a_3, \ldots)$ is a decreasing sequence in $\mathbb{R}_{\geq 0}$

$$\text{such that} \quad \lim_{n\to\infty} a_n = 0 \qquad \text{then} \qquad \sum_{n=1}^{\infty} (-1)^n a_n \text{ converges.}$$

Proof.
Assume $(a_0, a_1, \ldots)$ is a sequence in $\mathbb{R}_{\geq 0}$, $\lim_{n\to\infty} a_n = 0$ and if $n \in \mathbb{Z}_{\geq 0}$ then $a_n \geq a_{n+1}$.

To show: $\sum_{n=0}^{\infty} (-1)^{n-1} a_n = a_1 - a_2 + a_3 - a_4 + a_5 - \cdots$ converges.

Let

$$s_{2m} = (a_1 - a_2) + (a_3 - a_4) + \cdots + (a_{2m-1} - a_{2m}).$$

Then $s_{2m} \leq s_{2(m+1)}$.
Since $s_{2m} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (a_{2m-2} - a_{2m-1}) - a_{2m}$, then $s_{2m} \leq a_1$.
So the sequence $(s_2, s_2, s_6, \ldots)$ is increasing and bounded above.
So $\lim_{m\to\infty} s_{2m}$ exists.
Let $\ell = \lim_{m\to\infty} s_{2m}$.
Let $s_{2m+1} = s_{2m} + a_{2m+1}$.
Then

$$\lim_{m\to\infty} s_{2m+1} = \lim_{m\to\infty} s_{2m} + \lim_{m\to\infty} a_{2m+1} = \ell + 0 = \ell.$$

So $\lim_{m\to\infty} s_m = \ell$.

So $\sum_{n=0}^{\infty}(-1)^{n-1}a_n = \ell$.

□

5.8 Additional series examples

Example 5.101. Evaluate $\sum_{n=1}^{\infty} 2$.

Proof.

$$\sum_{n=1}^{\infty} 2 = \lim_{r\to\infty}\Big(\sum_{n=1}^{r} 2\Big) = \lim_{r\to\infty} 2r,$$

which gets larger and larger as r gets larger and larger.

So $\sum_{n=1}^{\infty} 2$ does not get closer and closer to a single real number. So

$$\sum_{n=1}^{\infty} 2 \quad \text{does not exist in } \mathbb{R}.$$

□

Example 5.102. Evaluate $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$.

Proof. Since

$$\sum_{n=1}^{\infty} \left(\tfrac{1}{2}\right)^n = -1 + \Big(\sum_{n=0}^{\infty} \left(\tfrac{1}{2}\right)^n\Big)$$

and

$$\begin{aligned}\sum_{n=0}^{\infty} \left(\tfrac{1}{2}\right)^n &= \lim_{r\to\infty}\Big(\sum_{n=0}^{r} \left(\tfrac{1}{2}\right)^n\Big) = \lim_{r\to\infty}\left(\frac{1-(\frac{1}{2})^{r+1}}{1-\frac{1}{2}}\right) \\ &= \lim_{r\to\infty} 2\big(1-(\tfrac{1}{2})^{r+1}\big) = 2 - 2\lim_{r\to\infty}\frac{1}{2^{r+1}} = 2 - 2\cdot 0 = 2,\end{aligned}$$

then

$$\sum_{n=1}^{\infty} \left(\tfrac{1}{2}\right)^n = -1 + \Big(\sum_{n=0}^{\infty} \left(\tfrac{1}{2}\right)^n\Big) = -1 + 2 = 1.$$

□

Example 5.103. Evaluate $\sum_{n=1}^{\infty} \frac{10^n}{n!}$.

Proof. By the definition of e^x,

$$\sum_{n=1}^{\infty} \frac{10^n}{n!} = -1 + \Big(\sum_{n=0}^{\infty} \frac{10^n}{n!}\Big) = -1 + e^{10}.$$

□

Example 5.104. Evaluate $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n}$.

Proof. Since

$$\sum_{n=1}^{\infty} \frac{n+1}{n} = \lim_{r\to\infty} \Big(\sum_{n=1}^{r} \frac{n+1}{n}\Big) \geq \lim_{r\to\infty} \Big(\sum_{n=1}^{r} \frac{n}{n}\Big) = \lim_{r\to\infty} r,$$

then $\displaystyle\sum_{n=1}^{r} \frac{n+1}{n}$ gets larger and larger as r gets larger and larger. So

$$\sum_{n=1}^{\infty} \frac{n+1}{n} \qquad \text{does not exist in } \mathbb{R}.$$

□

Example 5.105. Evaluate $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{n!n!}$.

Proof.

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{(2n)!}{n!n!} &= \sum_{n=1}^{\infty} \frac{1\cdot 2\cdots (n-1)\cdot n\cdot (n+1)(n+2)\cdots 2n}{1\cdot 2\cdots (n-1)n\cdot 1\cdot 2\cdots (n-1)n} \\ &= \sum_{n=1}^{\infty} \frac{(n+1)(n+2)\cdots 2n}{1\cdot 2\cdots (n-1)n} = \sum_{n=1}^{\infty} \frac{(n+1)}{1}\cdot\frac{(n+2)}{2}\cdots\frac{2n}{n} \\ &\geq \sum_{n=1}^{\infty} 1\cdot 1\cdots 1 = \sum_{n=1}^{\infty} 1,\end{aligned}$$

which gets larger and larger and does not get closer and closer to a single real number. So

$$\sum_{n=1}^{\infty} \frac{(2n)!}{n!n!} \quad \text{does not exist in } \mathbb{R}.$$

□

Example 5.106. Determine whether $\displaystyle\sum_{n=1}^{\infty} \frac{3+\frac{5}{n}}{2n^2+n+2}$ exists in $\mathbb{R}$. Always carefully justify your answers.

Proof.

$$\sum_{n=1}^{\infty} \frac{3+\frac{5}{n}}{2n^2+n+2} \leq \sum_{n=1}^{\infty} \frac{8}{2n^2+n^2+2n^2} = \sum_{n=1}^{\infty} \frac{8}{5n^2} = \frac{8}{5}\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{8}{5}\cdot\frac{\pi^2}{6}.$$

So the sequence $(a_1, a_2, \ldots)$ given by

$$a_r = \sum_{n=1}^{r} \frac{3+\frac{5}{n}}{2n^2+n+2} \qquad \text{is an increasing sequence in } \mathbb{R} \text{ bounded by } \quad \frac{8}{5}\cdot\frac{\pi^2}{6}.$$

So

$$\sum_{n=1}^{\infty} \frac{3+\frac{5}{n}}{2n^2+n+2} = \lim_{r\to\infty} a_r \qquad \text{converges in } \mathbb{R}.$$

□

Example 5.107. Determine whether $\sum_{n=1}^{\infty} \frac{n^2+4}{n^3+5}$ exists in $\mathbb{R}$. Always carefully justify your answers.

Proof. Let $(a_1, a_2, \ldots)$ be the sequence given by $a_r = \sum_{n=1}^{r} \frac{n^2+4}{n^3+5}$.

$$\sum_{n=1}^{r} \frac{n^2+4}{n^3+5} \geq \sum_{n=1}^{r} \frac{n^2}{n^3+5n^3} = \sum_{n=1}^{r} \frac{1}{6n} = \frac{1}{6}\sum_{n=1}^{r} \frac{1}{n}.$$

Let $(b_1, b_2, \ldots)$ be the sequence given by $b_r = \sum_{n=1}^{r} \frac{1}{n}$ is a p-series with $p = 1$. The sequence $(b_1, b_2, \ldots)$ gets larger and larger as r gets larger and larger.

Since $a_r \geq \frac{1}{6} b_r$ then the sequence $(a_1, a_2, \ldots)$ gets larger and larger as r gets larger and larger. So

$$\sum_{n=1}^{\infty} \frac{n^2+4}{n^3+5} \qquad \text{does not exist in } \mathbb{R}.$$

□

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