4 Special functions and manipulatorics

4.1 exponential, trig functions, hyperbolic functions, elliptic functions

Define

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots,$$
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \frac{x^{13}}{13!} - \cdots,$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \frac{x^{12}}{12!} - \cdots,$$

and

$$\tan x = \frac{\sin x}{\cos x}, \quad \cot x = \frac{1}{\tan x}, \quad \sec x = \frac{1}{\cos x}, \quad \csc x = \frac{1}{\sin x}.$$

Define

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots,$$
$$\sinh x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \frac{x^{13}}{13!} - \cdots,$$
$$\cosh x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \frac{x^{12}}{12!} - \cdots,$$

and

$$\tanh x = \frac{\sinh x}{\cosh x}, \quad \coth x = \frac{1}{\tanh x}, \quad \text{sech } x = \frac{1}{\cosh x}, \quad \text{csch } x = \frac{1}{\sinh x}.$$

HW: Prove the following:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots, \qquad \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots,$$

$$\text{If } n \in \mathbb{Z}_{>0} \text{ then } \frac{dx^n}{dx} = nx^{n-1}, \qquad \text{If } n \in \mathbb{Z}_{\geq 0} \text{ then } \frac{dx^n}{dx} = nx^{n-1}.$$

$$\text{If } n \in \mathbb{Z} \text{ then } \frac{dx^n}{dx} = nx^{n-1}, \qquad \text{If } r \in \mathbb{Q} \text{ then } \frac{dx^r}{dx} = rx^{r-1}.$$

$$\text{If } r \in \mathbb{R} \text{ then } \frac{dx^r}{dx} = rx^{r-1}, \qquad \text{If } r \in \mathbb{C} \text{ then } \frac{dx^r}{dx} = rx^{r-1},$$

HW: Let $i^2 = 1$. Prove the following:

$$e^{ix} = \cos x + i \sin x, \qquad e^x = \cosh x + \sinh x,$$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2i}, \qquad \cosh x = \frac{e^x + e^{-x}}{2},$$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2}, \qquad \sinh x = \frac{e^x - e^{-x}}{2},$$

HW: Let $i^2 = 1$. Prove the following:

$$\cos(-x) = \cos x,\ \sin(-x) = -\sin x, \qquad \cosh(-x) = \cosh x,\ \sinh(-x) = -\sinh x,$$

$$\cos^2 x + \sin^2 x = 1, \qquad \cosh^2 x - \sinh^2 x = 1,$$

$$\cos(x+y) = \cos x \cos y - \sin x \sin y, \qquad \cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y,$$

$$\sin(x+y) = \sin x \cos y + \cos x \sin y, \qquad \sinh(x+y) = 2 \sinh x \cosh y,$$

$$\mathrm{arcsinh}(x) = \ln(x + \sqrt{x^2+1}), \qquad \mathrm{arccosh}(x) = \ln(x + \sqrt{x^2-1}),$$

$$\mathrm{arctanh}(x) = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right),$$

HW: Prove the following:

$$\ln 1 = 0, \qquad \ln(ab) = \ln a + \ln b,$$

$$\ln(\tfrac{1}{a}) = -\ln a, \quad \ln(a^b) = b \ln a,$$

HW: Prove the following:

$$\frac{de^x}{dx} = e^x, \qquad \frac{d \ln x}{dx} = \frac{1}{x},$$

$$\frac{d \sin x}{dx} = \cos x, \qquad \frac{d \sin^{-1} x}{dx} = \frac{1}{\sqrt{1-x^2}},$$

$$\frac{d \cos x}{dx} = -\sin x, \qquad \frac{d \cos^{-1} x}{dx} = \frac{-1}{\sqrt{1-x^2}},$$

$$\frac{d \tan x}{dx} = \sec^2 x, \qquad \frac{d \tan^{-1} x}{dx} = \frac{1}{\sqrt{x^2+1}},$$

$$\frac{d \cot x}{dx} = \csc^2 x, \qquad \frac{d \cot^{-1} x}{dx} = \frac{-1}{\sqrt{x^2+1}},$$

$$\frac{d \sec x}{dx} = \tan x \sec x, \qquad \frac{d \sec^{-1} x}{dx} = \frac{1}{x\sqrt{x^2-1}},$$

$$\frac{d \csc x}{dx} = -\csc x \cot x, \qquad \frac{d \csc^{-1} x}{dx} = \sqrt{x^2-1},$$

Example: Explain why $e^{ix} = \cos x + i \sin x$, if $i = \sqrt{-1}$.

$$\begin{aligned}
e^{ix} &= 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \frac{(ix)^7}{7!} + \cdots, \\
&= 1 + ix + \frac{i^2x^2}{2!} + \frac{i^3x^3}{3!} + \frac{i^4x^4}{4!} + \frac{i^5x^5}{5!} + \frac{i^6x^6}{6!} + \frac{i^7x^7}{7!} + \cdots, \\
&= 1 + ix + \frac{i^2x^2}{2!} + \frac{i \cdot i^2x^3}{3!} + \frac{((i^2)^2x^4}{4!} + \frac{i \cdot (i^2)^2x^5}{5!} + \frac{(i^2)^3x^6}{6!} + \frac{i \cdot (i^2)^3x^7}{7!} + \cdots, \\
&= 1 + ix + \frac{(-1)x^2}{2!} + \frac{i \cdot (-1)x^3}{3!} + \frac{(-1)^2x^4}{4!} + \frac{i \cdot (-1)^2x^5}{5!} + \frac{(-1)^3x^6}{6!} + \frac{i \cdot (-1)^3x^7}{7!} + \cdots, \\
&= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \frac{x^6}{6!} - i\frac{x^7}{7!} + \cdots, \\
&= (1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots) + i(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots) \\
&= \cos x + i \sin x.
\end{aligned}$$

Example: Explain why $\cos(-x) = \cos x$ and $\sin(-x) = -\sin x$.

$$\begin{aligned}
\cos(-x) &= 1 - \frac{(-x)^2}{2!} + \frac{(-x)^4}{4!} - \frac{(-x)^6}{6!} + \frac{(-x)^8}{8!} - \frac{(-x)^{10}}{10!} + \frac{(-x)^{12}}{12!} - \cdots, \\
&= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \frac{x^{12}}{12!} - \cdots, \\
&= \cos x,
\end{aligned}$$

and

$$\begin{aligned}
\sin(-x) &= (-x) - \frac{(-x)^3}{3!} + \frac{(-x)^5}{5!} - \frac{(-x)^7}{7!} + \frac{(-x)^9}{9!} - \frac{(-x)^{11}}{11!} + \frac{(-x)^{13}}{13!} - \cdots, \\
&= -x + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \frac{x^9}{9!} + \frac{x^{11}}{11!} - \frac{x^{13}}{13!} + \cdots, \\
&= -\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \frac{x^{13}}{13!} - \cdots\right), \\
&= -\sin x.
\end{aligned}$$

Example: Explain why $\cos^2 x + \sin^2 x = 1$.

$$\begin{aligned}
1 = e^0 &= e^{ix+(-ix)} \\
&= e^{ix}e^{-ix} = e^{ix}e^{i(-x)} \\
&= (\cos x + i \sin x)(\cos(-x) + i \sin(-x)) \\
&= (\cos x + i \sin x)(\cos x + i(-\sin x)) \\
&= \cos^2 x - i \sin x \cos x + i \sin x \cos x - i^2 \sin^2 x \\
&= \cos^2 x - (-1) \sin^2 x \\
&= \cos^2 x + \sin^2 x.
\end{aligned}$$

Example: Explain why

$$\begin{aligned} \cos(x+y) &= \cos x\cos y - \sin x\sin y, \qquad \text{and} \\ \sin(x+y) &= \sin x\cos y + \cos x\sin y. \end{aligned}$$

$$\begin{aligned} \cos(x+y) + i\sin(x+y) &= e^{i(x+y)} \\ &= e^{ix+iy} = e^{ix}e^{iy} \\ &= (\cos x + i\sin x)(\cos y + i\sin y) \\ &= \cos x\cos y + i\cos x\sin y + i\sin x\cos y + i^2\sin x\sin y \\ &= \big(\cos x\cos y + (-1)\sin x\sin y\big) + i\big(\cos x\sin y + \sin x\cos y\big). \end{aligned}$$

Comparing terms on each side gives

$$\begin{aligned} \cos(x+y) &= \cos x\cos y - \sin x\sin y, \qquad \text{and} \\ \sin(x+y) &= \sin x\cos y + \cos x\sin y. \end{aligned}$$

Example: Explain why $e^x = \cosh x + \sinh x$.

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots \\ &= \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots\right) + \left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots\right) \\ &= \cosh x + \sinh x. \end{aligned}$$

Example: Explain why $\cosh(-x) = \cosh x$ and $\sinh(-x) = -\sinh x$.

$$\begin{aligned} \cosh(-x) &= 1 + \frac{(-x)^2}{2!} + \frac{(-x)^4}{4!} + \frac{(-x)^6}{6!} + \frac{(-x)^8}{8!} + \frac{(-x)^{10}}{10!} + \frac{(-x)^{12}}{12!} + \cdots \\ &= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \frac{x^{10}}{10!} + \frac{x^{12}}{12!} + \cdots \\ &= \cosh x, \end{aligned}$$

and

$$\begin{aligned} \sinh(-x) &= (-x) + \frac{(-x)^3}{3!} + \frac{(-x)^5}{5!} + \frac{(-x)^7}{7!} + \frac{(-x)^9}{9!} + \frac{(-x)^{11}}{11!} + \frac{(-x)^{13}}{13!} + \cdots \\ &= -x - \frac{x^3}{3!} - \frac{x^5}{5!} - \frac{x^7}{7!} - \frac{x^9}{9!} - \frac{x^{11}}{11!} - \frac{x^{13}}{13!} - \cdots \\ &= -\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \frac{x^{11}}{11!} + \frac{x^{13}}{13!} + \cdots\right) \\ &= -\sinh x. \end{aligned}$$

Example: Explain why $\cosh x = \dfrac{e^x + e^{-x}}{2}$ and $\sinh x = \dfrac{e^x - e^{-x}}{2}$.

$$\begin{aligned}
\tfrac{1}{2}(e^x + e^{-x}) &= \tfrac{1}{2}\big(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots \\
&\qquad + 1 + (-x) + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \frac{(-x)^4}{4!} + \frac{(-x)^5}{5!} + \frac{(-x)^6}{6!} + \frac{(-x)^7}{7!} + \cdots\big) \\
&= \tfrac{1}{2}\big(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots \\
&\qquad + 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \frac{x^6}{6!} - \frac{x^7}{7!} + \cdots\big) \\
&= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \cdots \\
&= \cosh x.
\end{aligned}$$

$$\begin{aligned}
\tfrac{1}{2}(e^x - e^{-x}) &= \tfrac{1}{2}\big((1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots) \\
&\qquad - (1 + (-x) + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \frac{(-x)^4}{4!} + \frac{(-x)^5}{5!} + \frac{(-x)^6}{6!} + \frac{(-x)^7}{7!} + \cdots)\big) \\
&= \tfrac{1}{2}\big((1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots) \\
&\qquad - (1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \frac{x^6}{6!} - \frac{x^7}{7!} + \cdots)\big) \\
&= \tfrac{1}{2}\big(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots \\
&\qquad - 1 + x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \frac{x^5}{5!} - \frac{x^6}{6!} + \frac{x^7}{7!} - \cdots\big) \\
&= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \cdots \\
&= \sinh x.
\end{aligned}$$

Example: Explain why $\cosh^2 x - \sinh^2 x = 1$.

$$\begin{aligned}
1 = e^0 &= e^{x+(-x)} \\
&= e^x e^{-x} \\
&= (\cosh x + \sinh x)(\cosh(-x) + \sinh(-x)) \\
&= (\cosh x + \sinh x)(\cosh x - \sinh x)) \\
&= \cosh^2 x - \sinh x \cosh x + \sinh x \cosh x - \sinh^2 x \\
&= \cosh^2 x - \sinh^2 x.
\end{aligned}$$

Example: Explain why

$$\begin{aligned}
\cosh(x+y) &= \cosh x \cosh y + \sinh x \sinh y, \qquad \text{and} \\
\sinh(x+y) &= 2 \sinh x \cosh y.
\end{aligned}$$

$$\begin{aligned}
\cosh x \cosh y + \sinh x \sinh y &= \left(\frac{e^x + e^{-x}}{2}\right)\left(\frac{e^y + e^{-y}}{2}\right) + \left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^y - e^{-y}}{2}\right) \\
&= \frac{e^x e^y + e^{-x} e^y + e^x e^{-y} + e^{-x} e^{-y}}{4} \\
&\quad + \frac{e^x e^y - e^{-x} e^y - e^x e^{-y} + e^{-x} e^{-y}}{4} \\
&= \frac{2e^x e^y + 2e^{-x} e^{-y}}{4} \\
&= \frac{e^{(x+y)} + e^{-(x+y)}}{2} \\
&= \cosh(x+y).
\end{aligned}$$

and

$$\begin{aligned}
\sinh x \cosh y + \cosh x \sinh y &= \left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^y + e^{-y}}{2}\right) + \left(\frac{e^x + e^{-x}}{2}\right)\left(\frac{e^y - e^{-y}}{2}\right) \\
&= \frac{e^x e^y - e^{-x} e^y + e^x e^{-y} - e^{-x} e^{-y}}{4} \\
&\quad + \frac{e^x e^y + e^{-x} e^y - e^x e^{-y} - e^{-x} e^{-y}}{4} \\
&= \frac{2e^{x+y} - 2e^{-(x+y)}}{4} \\
&= \sinh(x+y).
\end{aligned}$$

4.2 Inverse functions

$\sqrt{x}$ is the function that undoes x^2. This means that

$$\sqrt{x^2} = x \qquad \text{and} \qquad (\sqrt{x})^2 = x.$$

$\ln x$ is the function that undoes e^x. This means that

$$\ln(e^x) = x \qquad \text{and} \qquad e^{\ln x} = x.$$

$\sin^{-1} x$ is the function that undoes $\sin x$. This means that

$$\sin^{-1}(\sin x) = x \qquad \text{and} \qquad \sin(\sin^{-1} x) = x.$$

$\cos^{-1} x$ is the function that undoes $\cos x$. This means that

$$\cos^{-1}(\cos x) = x \qquad \text{and} \qquad \cos(\cos^{-1} x) = x.$$

$\tan^{-1} x$ is the function that undoes $\tan x$. This means that

$$\tan^{-1}(\tan x) = x \qquad \text{and} \qquad \tan(\tan^{-1} x) = x.$$

$\cot^{-1} x$ is the function that undoes $\cot x$. This means that

$$\cot^{-1}(\cot x) = x \qquad \text{and} \qquad \cot(\cot^{-1} x) = x.$$

$\sec^{-1} x$ is the function that undoes $\sec x$. This means that

$$\sec^{-1}(\sec x) = x \qquad \text{and} \qquad \sec(\sec^{-1} x) = x.$$

$\csc^{-1} x$ is the function that undoes $\csc x$. This means that

$$\csc^{-1}(\csc x) = x \qquad \text{and} \qquad \csc(\csc^{-1} x) = x.$$

$\log_a x$ is the function that undoes a^x. This means that

$$\log_a(a^{\sqrt{7}\pi i \sin 32}) = \sqrt{7}\pi i \sin 32 \qquad \text{and} \qquad a^{\log_a(\sqrt{7}\pi i \sin 32)} = \sqrt{7}\pi i \sin 32.$$

WARNING: $\sin^{-1} x$ is VERY DIFFERENT from $(\sin x)^{-1}$. For example,

$$\sin^{-1} 0 = \sin^{-1}(\sin 0) = 0, \qquad \text{BUT} \qquad (\sin 0)^{-1} = \frac{1}{\sin 0} = \frac{1}{0} = \text{UNDEFINED}.$$

Example: Explain why $\ln 1 = 0$.

$$\ln 1 = \ln(e^0) = 0.$$

Example: Explain why $\ln(ab) = \ln a + \ln b$.

$$\ln(ab) = \ln(e^{\ln a} \cdot e^{\ln b}) = \ln(e^{\ln a + \ln b}) = \ln a + \ln b.$$

Example: Explain why $\ln\left(\frac{1}{a}\right) = -\ln a$.

$$\ln\left(\frac{1}{a}\right) = \ln\left(\frac{1}{e^{\ln a}}\right) = \ln\left(e^{-\ln a}\right) = -\ln a.$$

Example: Explain why $\ln\left(a^b\right) = b \ln a$.

$$\ln(a^b) = \ln\left(\left(e^{\ln a}\right)^b\right) = \ln\left(e^{b \ln a}\right) = b \ln a.$$

Thus

$$\begin{array}{ccc} e^0 = 1 & \text{turns into} & \ln 1 = 0, \\ e^x e^y = e^{x+y} & \text{turns into} & \ln(ab) = \ln a + \ln b, \\ e^{-x} = \dfrac{1}{e^x} & \text{turns into} & \ln\left(\dfrac{1}{a}\right) = -\ln a, \quad \text{and} \\ (e^x)^y = e^{yx} & \text{turns into} & \ln(a^b) = b \ln a. \end{array}$$

Example: Explain why $\frac{d \ln x}{dx} = \frac{1}{x}$.

Since $e^{\ln x} = x$, $\quad \frac{de^{\ln x}}{dx} = \frac{dx}{dx}.$

So $\quad e^{\ln x}\dfrac{d\ln x}{dx}=1.$ $\qquad$ So $\quad x\dfrac{d\ln x}{dx}=1.$ $\qquad$ So $\quad \dfrac{d\ln x}{dx}=\dfrac{1}{x}.$

Example: Find $\dfrac{d\sin^{-1}x}{dx}$.

Since $\quad \sin(\sin^{-1}x)=x,\qquad \dfrac{d\sin(\sin^{-1}x)}{dx}=\dfrac{dx}{dx}.$

So $\quad \cos(\sin^{-1}x)\dfrac{d\sin^{-1}x}{dx}=1.$ $\qquad$ So $\quad \dfrac{d\sin^{-1}x}{dx}=\dfrac{1}{\cos(\sin^{-1}x)}.$

So we would like to “simplify” $\cos(\sin^{-1}x)$.

Since $1-\cos^2(\sin^{-1}x)=\sin^2(\sin^{-1}x),\qquad 1-\big(\cos(\sin^{-1}x)\big)^2=\big(\sin(\sin^{-1}x)\big)^2.$

So $\quad 1-\big(\cos(\sin^{-1}x)\big)^2=x^2.$ $\qquad$ So $\quad 1-x^2=\big(\cos(\sin^{-1}x)\big)^2.$

So $\quad \cos(\sin^{-1}x)=\sqrt{1-x^2}.$ $\qquad$ So $\quad \dfrac{d\sin^{-1}x}{dx}=\dfrac{1}{\cos(\sin^{-1}x)}=\dfrac{1}{\sqrt{1-x^2}}.$

Example: Find $\dfrac{d\cos^{-1}x}{dx}$.

Since $\quad \cos(\cos^{-1}x)=x,\qquad \dfrac{d\cos(\cos^{-1}x)}{dx}=\dfrac{dx}{dx}.$

So $\quad -\sin(\cos^{-1}x)\dfrac{d\cos^{-1}x}{dx}=1.$ $\qquad$ So $\quad \dfrac{d\cos^{-1}x}{dx}=\dfrac{-1}{\sin(\cos^{-1}x)}.$ So we would like to “simplify” $\sin(\cos^{-1}x)$.

Since $1-\sin^2(\cos^{-1}x)=\cos^2(\cos^{-1}x),\qquad 1-\big(\sin(\cos^{-1}x)\big)^2=\big(\cos(\cos^{-1}x)\big)^2.$

So $\quad 1-\big(\sin(\cos^{-1}x)\big)^2=x^2.$ $\qquad$ So $\quad 1-x^2=\big(\sin(\cos^{-1}x)\big)^2.$

So $\quad \sin(\cos^{-1}x)=\sqrt{1-x^2}.$ $\qquad$ So $\quad \dfrac{d\cos^{-1}x}{dx}=\dfrac{-1}{\sin(\cos^{-1}x)}=\dfrac{-1}{\sqrt{1-x^2}}.$

Example: Find $\dfrac{d\tan^{-1}x}{dx}$.

Since $\quad \tan(\tan^{-1}x)=x,\qquad \dfrac{d\tan(\tan^{-1}x)}{dx}=\dfrac{dx}{dx}.$

So $\quad \sec^2(\tan^{-1}x)\dfrac{d\tan^{-1}x}{dx}=1.$ $\qquad$ So $\quad \dfrac{d\tan^{-1}x}{dx}=\dfrac{1}{\sec^2(\tan^{-1}x)}.$

So we would like to “simplify” $\sec^2(\tan^{-1}x)$.

Since $\quad \sin^2x+\cos^2x=1,$

$$\frac{\sin^2x}{\cos^2x}+\frac{\cos^2x}{\cos^2x}=\frac{1}{\cos^2x}.$$

So $\quad \tan^2x+1=\sec^2x.$

So $\quad \sec^2(\tan^{-1}x)=\tan^2(\tan^{-1}x)+1=\big(\tan(\tan^{-1}x)\big)^2+1=x^2+1.$

So $\quad \dfrac{d\tan^{-1}x}{dx}=\dfrac{1}{x^2+1}.$

Example: Find $\dfrac{d\cot^{-1}x}{dx}$.

Since $\cot(\cot^{-1}x) = x$, $\dfrac{d\cot(\cot^{-1}x)}{dx} = \dfrac{dx}{dx}$.

So $-\csc^2(\cot^{-1}x)\dfrac{d\cot^{-1}x}{dx} = 1$. So $\dfrac{d\cot^{-1}x}{dx} = \dfrac{-1}{\csc^2(\cot^{-1}x)}$.

So we would like to "simplify" $\csc^2(\cot^{-1}x)$. Since $\sin^2 x + \cos^2 x = 1$,

$$\frac{\sin^2 x}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x} = \frac{1}{\sin^2 x}.$$

So $1 + \cot^2 x = \csc^2 x$. So $\csc^2(\cot^{-1}x) = 1 + \cot^2(\cot^{-1}x) = 1 + \big(\cot(\cot^{-1}x)\big)^2 = 1 + x^2$.

So $\dfrac{d\cot^{-1}x}{dx} = \dfrac{-1}{1+x^2}$.

Example: Find $\dfrac{d\sec^{-1}x}{dx}$.

Since $\sec(\sec^{-1}x) = x$, $\dfrac{d\sec(\sec^{-1}x)}{dx} = \dfrac{dx}{dx}$.

So $\tan(\sec^{-1}x)\sec(\sec^{-1}x)\dfrac{d\sec^{-1}x}{dx} = 1$. So $\tan(\sec^{-1}x)\cdot x\cdot\dfrac{d\sec^{-1}x}{dx} = 1$.

So $\dfrac{d\sec^{-1}x}{dx} = \dfrac{1}{x\tan(\sec^{-1}x)}$.

So we would like to "simplify" $\tan(\sec^{-1}x)$.

Since $\sin^2 x + \cos^2 x = 1$,

$$\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}.$$

So $\tan^2 x + 1 = \sec^2 x$.

So $\tan^2(\sec^{-1}x) + 1 = \sec^2(\sec^{-1}x)$. So $\big(\tan(\sec^{-1}x)\big)^2 + 1 = \big(\sec(\sec^{-1}x)\big)^2$.

So $\big(\tan(\sec^{-1}x)\big)^2 + 1 = x^2$. So $\tan(\sec^{-1}x) = \sqrt{x^2-1}$.

So $\dfrac{d\sec^{-1}x}{dx} = \dfrac{1}{x\sqrt{x^2-1}}$.

Example: Find $\dfrac{d\csc^{-1}x}{dx}$.

Since $\csc(\csc^{-1}x) = x$, $\dfrac{d\csc(\csc^{-1}x)}{dx} = \dfrac{dx}{dx}$.

So $-\csc(\csc^{-1}x)\cot(\csc^{-1}x)\dfrac{d\csc^{-1}x}{dx} = 1$. So $-x\cot(\csc^{-1}x)\dfrac{d\csc^{-1}x}{dx} = 1$.

So $\dfrac{d\csc^{-1}x}{dx} = \dfrac{-1}{x\cot(\csc^{-1}x)}$.

So we would like to "simplify" $\cot(\csc^{-1}x)$.

Since $\sin^2 x + \cos^2 x = 1$,

$$\frac{\sin^2 x}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x} = \frac{1}{\sin^2 x}.$$

So $\quad 1+\cot^2 x = \csc^2 x$.

So $\quad 1+\cot^2(\csc^{-1} x) = \csc^2(\csc^{-1} x)$. $\qquad$ So $\quad 1+\big(\cot(\csc^{-1} x)\big)^2 = \big(\csc(\csc^{-1} x)\big)^2$.

So $\quad 1+\big(\cot(\csc^{-1} x)\big)^2 = x^2$. $\qquad$ So $\quad \cot(\csc^{-1} x) = \sqrt{x^2-1}$.

So $\quad \dfrac{d\csc^{-1} x}{dx} = \dfrac{-1}{x\sqrt{x^2-1}}$.

Example: Find $\dfrac{dy}{dx}$ when $y = \log_x 10$.

$$x^y = x^{\log_x 10} = 10.$$

Take the derivative:

$$\frac{d\ x^y}{dx} = \frac{d\ (e^{\ln x})^y}{dx} = \frac{d\ e^{y}\ln x}{dx} = e^{y\ln x}\left(y\cdot\frac{1}{x}+\frac{dy}{dx}\ln x\right)$$
$$= \frac{d10}{dx} = 0.$$

So $\qquad e^{y\ln x}\left(y\cdot\dfrac{1}{x}+\dfrac{dy}{dx}\ln x\right) = 0.$

Solve for $\dfrac{dy}{dx}$.

$$e^{y\ln x}\frac{dy}{dx}\ln x = \frac{-e^{y\ln x}y}{x}. \qquad \text{So} \qquad \frac{dy}{dx} = \frac{-e^{y\ln x}y}{xe^{y\ln x}\ln x} = \frac{-y}{x\ln x} = \frac{\log_x 10}{x\ln x}.$$

Example: Find the third derivative of 2^x with respect to x.

$y = 2^x$.

$$\frac{dy}{dx} = \frac{d2^x}{dx} = \frac{2(e^{\ln 2})^x}{dx} = \frac{de^{x\ln 2}}{dx} = e^{x\ln 2}(\ln 2) = (e^{\ln 2})^x\ln 2 = 2^x\ln 2.$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d2^x\ln 2}{dx} = \ln 2\cdot 2^x\ln 2 = (\ln 2)^2 2^x.$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d}{dx}\big(\ln 2)^2 2^x\big) = (\ln 2)^2 2^x\ln 2 = (\ln 2)^3 2^x.$$

Example: If $y = a\cos(\ln x)+b\sin(\ln x)$ show that $x^2\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}+y = 0$.

$$\begin{aligned}\frac{dy}{dx} &= a(-\sin(\ln x))\frac{1}{x}+b\cos(\ln x)\frac{1}{x}\\ &= -a\sin(\ln x)x^{-1}+b\cos(\ln x)x^{-1},\end{aligned}$$

$$\begin{aligned}\frac{d^2y}{dx^2} &= -a\cos(\ln x)\frac{1}{x}x^{-1}+-a\sin(\ln x)(-1)x^{-2}+\ -b\sin(\ln x)\frac{1}{x}x^{-1}+b\cos(\ln x)(-1)x^{-2}\\ &= \frac{-a\cos(\ln x)+a\sin(\ln x)-b\sin(\ln x)-b\cos(\ln x)}{x^2}\\ &= \frac{1}{x^2}\big((a-b)\sin(\ln x)-(a+b)\cos(\ln x)\big).\end{aligned}$$

So

$$\begin{aligned}
LHS &= x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y \\
&= x^2\,\frac{1}{x^2}\big((a-b)\sin(\ln x) - (a+b)\cos(\ln x)\big) \\
&\qquad + x\big(-a\sin(\ln x)x^{-1} + b\cos(\ln x)x^{-1}\big) \\
&\qquad + a\cos(\ln x) + b\sin(\ln x) \\
&= (a-b)\sin(\ln x) - (a+b)\cos(\ln x) \\
&\quad - a\sin(\ln x) + b\cos(\ln x) \\
&\quad + b\sin(\ln x) + a\cos(\ln x) \\
&= 0.
\end{aligned}$$

Example: Find $\frac{dy}{dx}$ when $a\sin(xy) + b\cos\left(\frac{x}{y}\right) = 0$.

Take the derivative:

$$\begin{aligned}
0 &= a\cos(xy)\left(x\frac{dy}{dx} + 1\cdot y\right) + \; -b\sin\left(\frac{x}{y}\right)\left(x(-1)y^{-2}\frac{dy}{dx} + 1\cdot y^{-1}\right) \\
&= a\cos(xy)x\frac{dy}{dx} + a\cos(xy)y + b\sin\left(\frac{x}{y}\right)\frac{x}{y^2}\frac{dy}{dx} - b\sin\left(\frac{x}{y}\right)y^{-1}.
\end{aligned}$$

Solve for $\frac{dy}{dx}$.

$$a\cos(xy)x\frac{dy}{dx} + b\sin\left(\frac{x}{y}\right)\frac{x}{y^2}\frac{dy}{dx} = a\cos(xy)y - b\sin\left(\frac{x}{y}\right)y^{-1}.$$

So

$$\frac{dy}{dx} = \frac{a\cos(xy)y - b\sin\left(\frac{x}{y}\right)y^{-1}}{a\cos(xy)x + b\sin\left(\frac{x}{y}\right)\frac{x}{y^2}}$$

$$= \frac{a\cos(xy)y^3 - b\sin\left(\frac{x}{y}\right)y}{a\cos(xy)xy^2 + b\sin\left(\frac{x}{y}\right)x}$$

Example: Find $\frac{dy}{dx}$ when $y = \tan^{-1}\left(\frac{a}{x}\right)\cdot\cot^{-1}\left(\frac{x}{a}\right)$.

$$\frac{dy}{dx} = \tan^{-1}\left(\frac{a}{x}\right)\left(\frac{-1}{1+\left(\frac{x}{a}\right)^2}\right)\frac{1}{a} + \frac{1}{1+\left(\frac{x}{a}\right)^2}(-1)ax^{-2}\cot^{-1}\left(\frac{x}{a}\right)$$

$$= \frac{-\tan^{-1}\left(\frac{a}{x}\right)}{a+\frac{x^2}{a}} + \frac{-\cot^{-1}\left(\frac{x}{a}\right)a}{x^2+a^2}$$

$$= \frac{-\tan^{-1}\left(\frac{a}{x}\right)a}{a^2+x^2} + \frac{-\cot^{-1}\left(\frac{x}{a}\right)a}{x^2+a^2}$$

$$= \left(\frac{-a}{a^2+x^2}\right)\left(\tan^{-1}\left(\frac{a}{x}\right) + \cot^{-1}\left(\frac{x}{a}\right)\right).$$

If $\frac{a}{x} = \tan z$ then $\frac{x}{a} = \cot z$ and $z = \tan^{-1}\left(\frac{a}{x}\right) = \cot^{-1}\left(\frac{x}{a}\right)$.

So

$$\frac{dy}{dx} = \left(\frac{-a}{a^2+x^2}\right)\left(\tan^{-1}\left(\frac{a}{x}\right) + \tan^{-1}\left(\frac{a}{x}\right)\right) = \frac{-2a\tan^{-1}\left(\frac{a}{x}\right)}{a^2+x^2}.$$