1.16 Continuity

Let $n, m \in \mathbb{Z}_{>0}$ and let $p \in \mathbb{R}^m$. A function $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is *continuous at p* if

$$\lim_{x\to p} f(x) = f(p).$$

1.16.1 x^n and e^x are continuous

Proposition 1.8.

(a) Let $n \in \mathbb{Z}_{>0}$. The function $f\colon \mathbb{C} \to \mathbb{C}$ given by $f(x) = x^n$ is continuous.

(b) The function $f\colon \mathbb{C} \to \mathbb{C}$ given by $f(x) = e^x$ is continuous.

1.16.2 Behavior of x^n as $n \in \mathbb{Z}_{>0}$ gets large

HW: Let $x \in \mathbb{C}$. Show that

$$\lim_{n\to\infty} x^n = \begin{cases} 0, & \text{if } |x| < 1, \\ \text{diverges in } \mathbb{C}, & \text{if } |x| > 1, \\ 1, & \text{if } x = 1, \\ \text{diverges in } \mathbb{C}, & \text{if } |x| = 1 \text{ and } x \neq 1. \end{cases}$$

1.16.3 Behavior of $1 + x + x^2 + \cdots + x^n$ as $n \in \mathbb{Z}_{>0}$ gets large

HW: Let $x \in \mathbb{C}$. Show that

$$\lim_{n\to\infty} (1 + x + x^2 + \cdots + x^n) = \lim_{n\to\infty} \frac{1 - x^{n+1}}{1 - x} = \begin{cases} \dfrac{1}{1-x}, & \text{if } |x| < 1, \\ \text{diverges in } \mathbb{C}, & \text{if } |x| \geq 1. \end{cases}$$

For example, if $x = \frac{1}{2}$ then

$$\lim_{n\to\infty} \left(1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots + \left(\frac{1}{2}\right)^n\right) = \lim_{n\to\infty} \frac{1 - \left(\frac{1}{2}\right)^{n+1}}{1 - \frac{1}{2}} = \frac{1}{1 - \frac{1}{2}} = 2.$$

1.16.4 Favorite limits

Proposition 1.9. *(a) If $n \in \mathbb{Z}_{>0}$ then, in $\mathbb{R}$, $\lim_{x\to\infty} x^n e^{-x} = 0$.*

(b) If $\alpha \in \mathbb{R}_{>0}$ then $\lim_{x\to\infty} x^{-\alpha} \log x = 0$.

(c) Let $p \in \mathbb{R}_{>0}$. Then $\lim_{n\to\infty} \frac{1}{n^p} = 0$.

(d) Let $p \in \mathbb{R}_{>0}$. Then $\lim_{n\to\infty} p^{1/n} = 0$.

(e) $\lim_{n\to\infty} n^{1/n} = 1$.