<u>Projective space and cosets</u>

Let $\mathbb{F}$ be a field.

Define an equivalence relation on $\mathbb{F}^n - \{0\}$ by

$$[a_1, \ldots, a_n] = [\lambda a_1, \ldots, \lambda a_n] \quad \text{if } \lambda \in \mathbb{F}^\times.$$

The <u>projective space</u> $\mathbb{P}^{n-1}(\mathbb{F})$ is

$$\begin{aligned}\mathbb{P}^{n-1}(\mathbb{F}) &= \{\text{equivalence classes}\} \\ &= \{\text{1-dimensional subspaces of } \mathbb{F}^n\} \\ &= G(\mathbb{F}^n)_1.\end{aligned}$$

Let

$$B = \left\{ g \in GL_n(\mathbb{F}) \;\middle|\; \begin{array}{l} g \text{ is upper} \\ \text{triangular} \end{array} \right\}$$

$$P_k = \left\{ \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \in GL_n(\mathbb{F}) \;\middle|\; \begin{array}{l} A \in M_{k\times k}(\mathbb{F}), \\ B \in M_{k\times(n-k)}(\mathbb{F}) \\ C \in M_{(n-k)\times(n-k)}(\mathbb{F}) \end{array} \right\}$$

Let $\{\varepsilon_1, \ldots, \varepsilon_n\}$ be the basis of $\mathbb{F}^n$ given by

$$\varepsilon_i = (0, \ldots 0, \underset{i\text{th}}{1}, 0, \ldots 0)^t$$

Let

$$E = (\{0\} \subsetneq E_1 \subsetneq E_2 \subsetneq \cdots \subsetneq E_n = \mathbb{F}^n)$$

where $E_k = \mathbb{F}\text{-span}\{\varepsilon_1, \ldots, \varepsilon_k\}$.

$\mathcal{G}(\mathbb{F}^n) = \{\mathbb{F}\text{-subspaces of } \mathbb{F}^n\}$

partially ordered by inclusion.

Then $\mathcal{G}(\mathbb{F}^n) = \bigsqcup_{k=0}^{n} \mathcal{G}(\mathbb{F}^n)_k$, where

$$\mathcal{G}(\mathbb{F}^n)_k = \left\{\begin{matrix}\text{subspaces of } \mathbb{F}^n \text{ of}\\ \text{dimension } k\end{matrix}\right\}$$

Let $G = GL_n(\mathbb{F})$.

<u>Proposition</u> (a) $GL_n(\mathbb{F})$ acts on $\mathcal{G}(\mathbb{F})$ by poset automorphisms.

$$\begin{aligned} GL_n(\mathbb{F}) \times \mathcal{G}(\mathbb{F}) &\to \mathcal{G}(\mathbb{F}) \\ (g, W) &\mapsto gW \end{aligned}$$

where $gW = \left\{ g\begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} \;\middle|\; \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} \in W \right\}$.

(b) $\mathrm{Stab}_{GL_n(\mathbb{F})}(E) = B$

(c) $\mathrm{Stab}_{GL_n(\mathbb{F})}(E_k) = P_k$

(d) Let $\mathcal{F}(\mathbb{F}^n) = \{\text{maximal chains in } \mathcal{G}(\mathbb{F}^n)\}$

Then $\mathcal{F}(\mathbb{F}^n) = G \cdot E$ and

$$G/B \xrightarrow{\sim} \mathcal{F}(\mathbb{F}^n) \quad (\text{as } G\text{-sets})$$

(e) $\mathcal{G}(\mathbb{F}^n)_k = GE_k$ and

$$G/P_k \xrightarrow{\sim} \mathcal{G}(\mathbb{F}^n)_k \quad (\text{as } G\text{-sets})$$

(f) $\mathcal{G}(\mathbb{F}^n)_1 = \mathbb{P}^{n-1} = GE_1$ and $G/P_1 \xrightarrow{\sim} \mathbb{P}^{n-1}$ as G-set.

<u>Remark</u> If $n=2$ then $P_1 = B = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in GL_2(\mathbb{F}) \right\}$

and
$$\mathcal{F}(\mathbb{F}^2) = \mathbb{P}^1 = GL_2(\mathbb{F})/B.$$

<u>Counting</u> Let $\mathbb{F}_q$ be a field with q elements.

$$
\begin{aligned}
\mathrm{Card}(GL_n(\mathbb{F}_q)) &= (q^n-1)(q^n-q)(q^n-q^2)\cdots(q^n-q^{n-1}) \\
&= (q^n-1)\, q(q^{n-1}-1)\, q^2(q^{n-2}-1) \cdots q^{n-1}(q-1) \\
&= q^{1+2+\cdots+(n-1)} \frac{(q^n-1)}{(q-1)} \frac{(q^{n-1}-1)}{(q-1)} \cdots \frac{(q-1)}{(q-1)} (q-1)^n \\
&= q^{\frac{n(n-1)}{2}} (q-1)^n [n]!
\end{aligned}
$$

$$\mathrm{Card}(B) = (q-1)^n q^{n-1} q^{n-2} \cdots q^1 = q^{\frac{1}{2}n(n-1)} (q-1)^n$$

$$
\begin{aligned}
\mathrm{Card}(P_k) &= q^{k(n-k)} q^{\frac{k(k-1)}{2}} (q-1)^k [k]!\, q^{\frac{(n-k)(n-k-1)}{2}} (q-1)^{n-k} [n-k]! \\
&= q^{\frac{k}{2}(n-k)} q^{k\frac{(n-k)}{2}} q^{\frac{k}{2}(k-1)} q^{(n-k-1)\frac{(n-k)}{2}} (q-1)^n [k]! [n-k]! \\
&= q^{\frac{k}{2}(n-1)} q^{\frac{(n-k)}{2}(n-1)} (q-1)^n [k]! [n-k]! \\
&= q^{\frac{n(n-1)}{2}} (q-1)^n [k]! [n-k]!
\end{aligned}
$$

$$\mathrm{Card}(\mathcal{F}(\mathbb{F}_q^n)) = \mathrm{Card}(G/B) = \frac{\mathrm{Card}(G)}{\mathrm{Card}(B)} = [n]!$$

$$\mathrm{Card}(G(\mathbb{F}_q^n, k)) = \mathrm{Card}(G/P_k) = \frac{\mathrm{Card}(G)}{\mathrm{Card}(P_k)} = \begin{bmatrix} n \\ k \end{bmatrix}.$$

In particular,

$$\mathrm{Card}(\mathbb{P}^1(\mathbb{F}_q)) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{[2][1]}{[1][1]} = \frac{q^2-1}{q-1} = q+1.$$

and

$$\mathcal{F}(\mathbb{F}_1^n) = \{\text{maximal chains in } S(n)\}$$
$$= \{\text{permutations of } \{1,2,\ldots,n\}\}$$

and $\mathrm{Card}(\mathcal{F}(\mathbb{F}_1^n)) = n!$

and $G(\mathbb{F}_1^n)_k = \left\{ \begin{array}{l} \text{subsets of } \{1,\ldots,n\} \\ \text{of cardinality } k \end{array} \right\}$

and $\mathrm{Card}(G(\mathbb{F}_1^n)_k) = \binom{n}{k}$

and $\mathbb{P}^1(\mathbb{F}_1) = G(\mathbb{F}^2)_1 = \left\{ \begin{array}{l} \text{subsets of } \{1,\ldots,2\} \\ \text{of cardinality } 1 \end{array} \right\}$
$= \{\{1\} \ldots \{2\}\}$

and $\mathrm{Card}(\mathbb{P}^1(\mathbb{F}_1)) = 2 = 1+1.$

<u>Remark</u> $\mathbb{Q}_{>0} = \left\{ \frac{a}{b} \;\middle|\; \begin{array}{l} b \in \mathbb{Z}_{>0} \\ a \in \mathbb{Z}_{>0} \end{array} \right\}$ with

$$\frac{a}{b} = \frac{\lambda a}{\lambda b} \quad \text{if } \lambda \in \mathbb{Z}_{>0}$$

So $\mathbb{Q}_{>0} = \mathbb{P}^1(\mathbb{Z}_{>0}).$