Projective space and cosets

Let $\mathbb{F}$ be a field.

Define an equivalence relation on $\mathbb{F}^n - \{(0,\ldots,0)\}$ by

$$[a_1,\ldots,a_n] = [\lambda a_1,\ldots,\lambda a_n], \text{ if } a_1,\ldots,a_n \in \mathbb{F} \text{ and } \lambda \in \mathbb{F}^\times.$$

The projective space $\mathbb{P}^{n-1}$ is

$$\mathbb{P}^{n-1} = \{\text{equivalence classes}\}.$$

Let $\{e_1,\ldots,e_n\}$ be an $\mathbb{F}$-basis of $\mathbb{F}^n$ and let

$$E = (0 = E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_n = \mathbb{F}^n)$$

where $E_k = \mathbb{F}\text{-span}\{e_1,\ldots,e_k\}$, for $k \in \{0,1,\ldots,n\}$.

Let

$$B = \{ b \in GL_n(\mathbb{F}) \mid b \text{ is upper triangular}\}$$

and

$$P_k = \left\{ \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \in GL_n(\mathbb{F}) \;\middle|\; \begin{array}{l} A \in M_{k\times k}(\mathbb{F}), \\ B \in M_{k\times(n-k)}(\mathbb{F}), \\ C \in M_{(n-k)\times(n-k)}(\mathbb{F}) \end{array} \right\}$$

Then B and $P_1, P_2, \ldots, P_n$ are subgroups of $GL_n(\mathbb{F})$.

14.08.2025 ②
Adv. Disc Math.
Lect. 8 A. Ram.

Proof idea

The flag variety is the set of maximal chains in $\mathcal{B}(\mathbb{F}^n)$.
(paths ~~from~~ $\emptyset$ to $\mathbb{F}^n$ in $\mathcal{B}(\mathbb{F}^n)$).

An automorphism of $\mathcal{B}(\mathbb{F}^n)$ must take a maximal chain to a maximal chain.
A maximal chain in $\mathcal{B}(\mathbb{F}^n)$ is equivalent to an ordered basis of $\mathbb{F}^n$

If $\varepsilon_i = (0, \ldots, 0, \overset{i\text{th}}{1}, 0, \ldots 0)^t$ in $\mathbb{F}^n$
then $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ is an ordered basis of $\mathbb{F}^n$. The corresponding maximal chain is

$$E = (0 \subsetneq E_1 \subsetneq E_2 \subsetneq \cdots \subsetneq E_n = \mathbb{F}^n)$$

where $E_k = \mathbb{F}\text{-span}\{\varepsilon_1, \ldots, \varepsilon_k\}$.

An automorphism g is determined by the ordered basis
$(b_1, \ldots, b_n) = (g\varepsilon_1, g\varepsilon_2, \ldots g\varepsilon_n)$
which is the transition matrix between bases, i.e. equivalent to an element of $GL_n(\mathbb{F})$.