12.08.2015 ①
Adv. Disc. Math
Lect. 7 A. Ram

<u>Theorem</u> Let $\mathbb{F}$ be a field and $V = \mathbb{F}^n$ an $\mathbb{F}$-vector space.
Let
$$\mathcal{G}(\mathbb{F}^n) = \{\mathbb{F}\text{-subspaces of } \mathbb{F}^n\},$$
partially ordered by inclusion.
Then
$$\mathcal{G}(V) \text{ is a modular lattice.}$$

<u>Proposition</u> Let $M, N, P \in \mathcal{G}(\mathbb{F}^n)$

(a) (infimums exist)
$$\inf(M,N) = M \cap N = \{v \in \mathbb{F}^n \mid v \in M \text{ and } v \in N\}.$$

(b) (supremums exist)
$$\sup(M,N) = M + N = \{m + n \mid m \in M \text{ and } n \in N\}$$

(c) (modular law) If $P \subseteq M$ then
$$M + (N \cap P) = (M+N) \cap P.$$

(d) (modular property)
$$\frac{M+N}{M} \cong \frac{N}{M \cap N}.$$

The function $\dim : \mathcal{G}(\mathbb{F}^n) \to \mathbb{Z}_{\geq 0}$ given by
$$\dim(V) = (\#\text{ of elements in an } \mathbb{F}\text{-basis of } V)$$
makes $\mathcal{G}(\mathbb{F}^n)$ a ranked modular lattice.

The subspace lattice

12.08.2025 (2)
Adv. Disc. Math
Lect. 7 A. Ram.

$$G(\mathbb{F}^n) = \bigsqcup_{k=0}^{n} G(\mathbb{F}^n)_k,$$

where

$$G(\mathbb{F}^n)_k = \left\{ \begin{matrix} \mathbb{F}\text{-subspaces } V \text{ of } \mathbb{F}^n \\ \text{with } \dim(V) = k \end{matrix} \right\}$$

Theorem Let $\mathbb{F}_q$ be a finite field with q elements. For $r \in \mathbb{Z}_{>0}$ let

$$[r] = \frac{q^r - 1}{q - 1} \quad \text{and} \quad [r]! = [r][r-1]\cdots[2][1].$$

For $k \in \{0, 1, \ldots, n\}$ let

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]!\,[n-k]!}$$

Then

$$\mathrm{Card}(G(\mathbb{F}_q^n)_k) = \begin{bmatrix} n \\ k \end{bmatrix}$$

and

$$\sum_{k=0}^{n} x^k q^{\frac{k(k-1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix} = (1+x)(1+xq)\cdots(1+xq^{n-1}).$$

The subset lattice $\mathfrak{S}(n)$

$$\mathfrak{S}(n) = \{\text{subsets of } \{1,\ldots,n\}\}$$

partially ordered by inclusion.

The function $\mathrm{Card}\colon \mathfrak{S}(n) \to \mathbb{Z}_{\geq 0}$ given by

$$\mathrm{Card}(S) = (\#\text{ of elements of } S)$$

makes $\mathfrak{S}(n)$ a ranked modular lattice.

$$\mathfrak{S}(n) = \bigsqcup_{k=0}^{n} \mathfrak{S}(n)_k,$$

where

$$\mathfrak{S}(n)_k = \left\{\begin{array}{l}\text{subsets } S \text{ of } \{1,\ldots,n\} \\ \text{with } \mathrm{Card}(S) = k\end{array}\right\}$$

Theorem For $r \in \mathbb{Z}_{>0}$ let

$$r! = r\cdot(r-1)\cdots 2\cdot 1.$$

For $k \in \{0,1,\ldots,n\}$ let

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

Then

$$\mathrm{Card}(\mathfrak{S}(n)_k) = \binom{n}{k}$$

and

$$\sum_{k=0}^{n} x^k \binom{n}{k} = \underbrace{(1+x)(1+x)\cdots(1+x)}_{n \text{ factors}} = (1+x)^n.$$

<u>Automorphisms of posets</u> Let P and Q be posets.

A <u>morphism of posets</u> is a function
$f: P \to Q$ such that

if $x, y \in P$ and $x \le y$ then $f(x) \le f(y)$.

An <u>isomorphism of posets</u> is a morphism
$f: P \to Q$ such that the inverse function
$f^{-1}: Q \to P$ exists and f^{-1} is a morphism
of posets.

An <u>automorphism of P</u> is an isomorphism
of posets $f: P \to P$.

<u>HW</u>: Give an example of a morphism of finite
posets $f: P \to Q$ that is bijective
but is not an isomorphism of posets.

$$\overset{a}{\cdot} \quad \overset{b}{\cdot} \quad \longmapsto \quad \begin{array}{l} f(b) \\ | \\ f(a) \end{array}$$

<u>Theorem</u>

(a) $\mathrm{Aut}(G(\mathbb{F}^n)) = GL_n(\mathbb{F}_q)$

(b) $\mathrm{Aut}(S(n)) = S_n$.

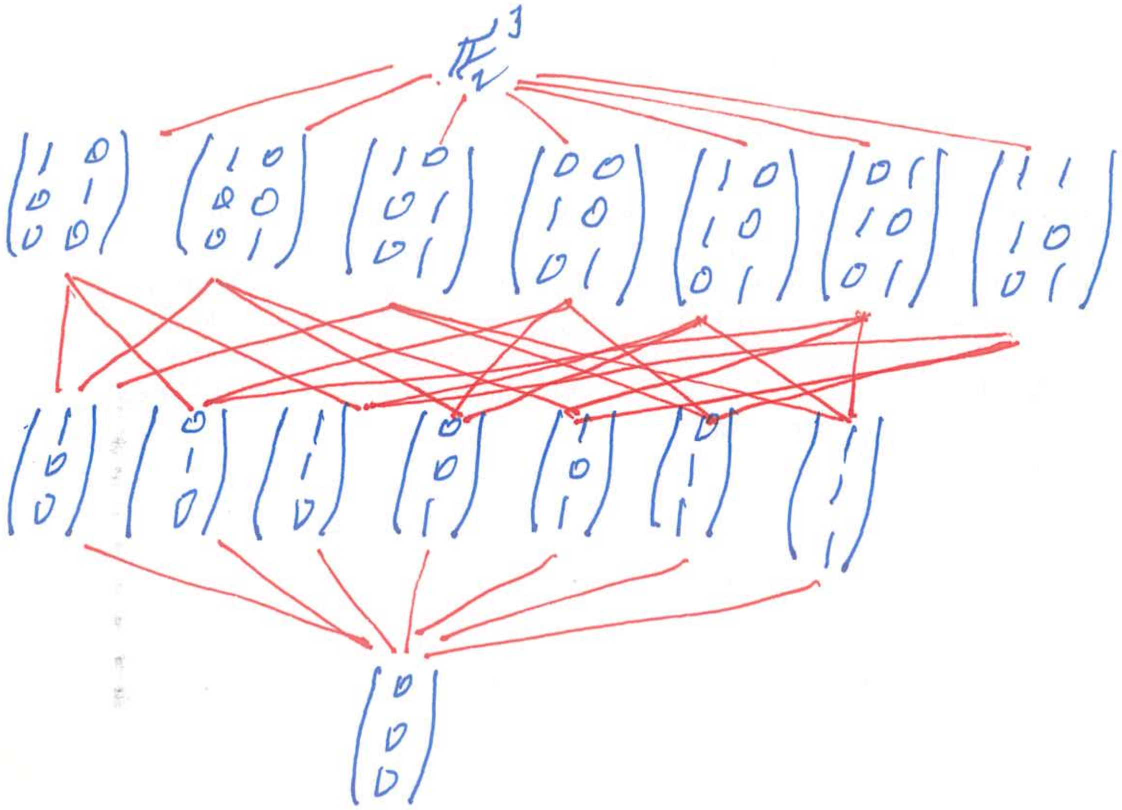
(So S_n "=" $GL_n(\mathbb{F}_1)$.)

The Fano plane

Let $q=2$ and $n=3$ so that

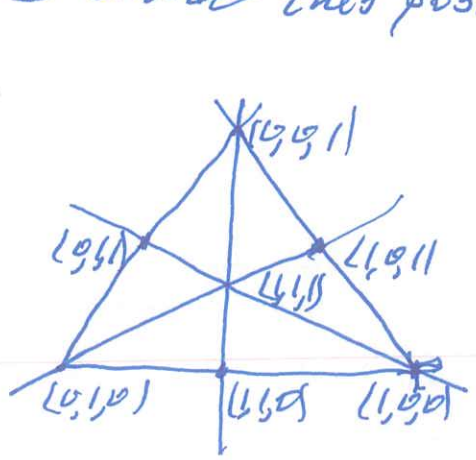
$$\mathbb{F}_2=\{0,1\} \quad \text{and} \quad \mathbb{F}_2^3=\left\{\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \;\middle|\; a_i\in\mathbb{F}_2\right\}$$

has cardinality 8. Then $G(\mathbb{F}_2^3)$ is



with representatives of G/P_1 on level 1 and representatives of G/P_2 on level 2.

Another way to encode this poset is via the Fano plane



so that the inclusion of points in lines matches the poset $G(\mathbb{F}_2^3)$.