07.08.2025
AdvDiscMath
Lect 6 A. Ram

(1)

Posets and Lattices

Two examples:

(1) The subset lattice.

Let $n \in \mathbb{Z}_{>0}$. The <u>subset lattice</u> is

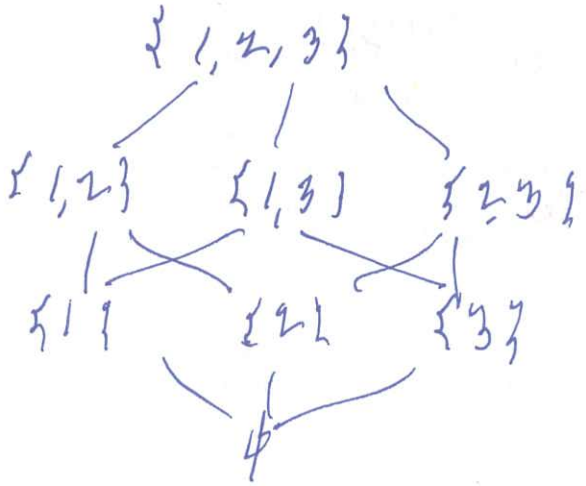
$$S(n) = \{\text{subsets of } \{1, \ldots, n\}\}$$

with partial order given by inclusion.

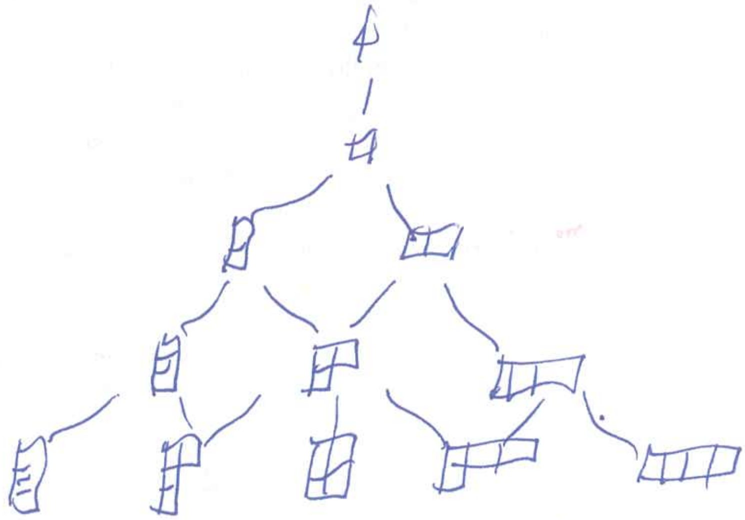
(2) The <u>Young lattice</u> is

$$\mathbb{Y} = \{\text{partitions}\}$$

with partial order given by inclusion.



The subset lattice $S(3)$



Young lattice $\mathbb{Y}$

Then

$$S(n) = \bigsqcup_{k=0}^{n} S(n)_k \quad \text{where}$$

$$S(n)_k = \left\{ \begin{matrix} \text{subsets of } \{1, \ldots, n\} \\ \text{with } k \text{ element} \end{matrix} \right\}, \quad \mathrm{Card}(S(n)_k) = \binom{n}{k}.$$

Posets and Lattices

07.08.2015
Adv. Disc. Math
Lect 6. A. Ram

Let S be a set.

A relation on S is a subset of $S \times S$.

Write $x \leq y$ if $(x,y) \in \leq$

A partially ordered set is a set P with a relation $\leq$ on P such that

(a) If $x \in P$ then $x \leq x$

(b) If $x,y,z \in P$ and $x \leq y$ and $y \leq z$ then $x \leq z$.

(c) If $x,y \in P$ and $x \leq y$ and $y \leq x$ then $x = y$.

The Hasse diagram of P is the graph with

Vertices: P

Edges: $x \to y$ if $x < y$ and there does not exist $z \in P$ with $x < z < y$.

A maximal chain in P is a sequence $(x_1, x_2, \ldots, x_n)$ in P such that

(a) If $i \in \{1, \ldots, n-1\}$ then $x_i < x_{i+1}$ and there does not exist $z \in P$ with $x_i < z < x_{i+1}$

(b) There does not exist $z \in P$ such that $x_n < z$.

(c) There does not exist $z \in P$ such that $z < x_1$.

Lattices

Let P be a poset and $E \subseteq P$.

The infimum, or greatest lower bound, of E in P is $l \in P$ such that

(a) If $e \in E$ then $l \le e$,

(b) If $m \in P$ and m satisfies
if $e \in E$ then $m \le e$
then $m \le l$.

The supremum, or least upper bound, of E in P is $s \in P$ such that

(a) If $e \in E$ then $e \le s$

(b) If $t \in P$ and t satisfies
if $e \in E$ ~~then~~ then $e \le t$
then $s \le t$.

Notation If $k \in \mathbb{Z}_{>0}$ and $x_1, \ldots, x_k \in P$ use the notation

$$\inf(x_1, \ldots x_k) = \inf(\{x_1, \ldots x_k\}) \quad \text{and}$$
$$\sup(x_1, \ldots, x_k) = \sup(\{x_1, \ldots, x_k\}).$$

A lattice is a poset P such that

if $x, y \in P$ then $\inf(x,y)$ and $\sup(x,y)$ exist in P.

Modular Lattices

Let P be a lattice. Use the notation

$$x \wedge y = \inf(x,y) \quad \text{and} \quad x \vee y = \sup(x,y)$$

The language is $x \wedge y$ is "x meet y"

and $x \vee y$ is "x join y".

A modular lattice is a lattice P such that

if $m, n, p \in P$ and $p \leq m$ then

$$m \vee (n \wedge p) = (m \vee p) \wedge n.$$

Theorem Let A be a $\mathbb{Z}$-algebra and let V be an A-module.

Let

$$\mathcal{G}(V) = \{A\text{-submodules of } V\}$$

partially ordered by inclusion.

Then

$$\mathcal{G}(V) \text{ is a modular lattice.}$$

Proposition Let A be a $\mathbb{Z}$-algebra and let V be an A-module.

Let $M, N, P \in \mathcal{G}(V)$.

(a) (infimums exist)

$$\inf(M,N) = M \cap N = \{v \in V \mid v \in M \text{ and } v \in N\}$$

07.08.2025
Adv. Disc. Math. (5)
Lect 6 A. Ram

(b) (supremums exist)

$$\sup(M,N) = M+N = \{m+n \mid m\in M \text{ and } n\in N\}$$

(c) (modular law) If $P\subseteq M$ then

$$M + (N\cap P) = (M+N)\cap P.$$

(d) (modular property)

$$\frac{M+N}{M} \cong \frac{N}{M\cap N}$$

<u>Proposition</u> Let $A = \mathbb{F}_q$ and let V be an A-module so that $V = \mathbb{F}_q^n$, where $n = \dim(V)$ as an $\mathbb{F}_q$-vector space.

Then

$$\mathbb{G}(\mathbb{F}_q^n) = \bigsqcup_{k=0}^{n} \mathbb{G}(\mathbb{F}_q^n)_k$$

where

$$\mathbb{G}(\mathbb{F}_q^n)_k = \left\{ \begin{array}{c} \mathbb{F}_q\text{-subspaces } W \text{ of } \mathbb{F}_q^n \\ \text{with } \dim(W) = k \end{array} \right\}$$

and

$$\mathrm{Card}(\mathbb{G}(\mathbb{F}_q^n)_k) = \begin{bmatrix} n \\ k \end{bmatrix}$$