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Representations Let $\mathbb{C}$ be a field.

A $\mathbb{C}$-algebra is a $\mathbb{C}$-vector space A with a multiplication
$$A \times A \to A$$
$$(a_1, a_2) \mapsto a_1 a_2$$
which is linear in each component, associative has an identity and satisfies distributive laws.

A representation of A is an A-module.

An A-module is a $\mathbb{C}$ vectorspace M with an A-action
$$A \times M \to M$$
$$(a, m) \mapsto m$$
which is linear in each component, associative, respects identity and satisfies distributive laws.

An irreducible, or simple, A-module is an A-module M which has no submodules except 0 and M.

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Group representations

Let G be a group.

A representation of G is a G-module.

A G-module is a $\mathbb{C}G$-module.

The group algebra of G is the algebra

$$\mathbb{C}G = \mathbb{C}\text{-span}\{g \in G\}$$

with multiplication determined by the multiplication in G.

Theorem: The group algebra of the symmetric group S_k is the algebra generated by

$$s_1, s_2, \ldots, s_{n-1} \text{ and } M_1, M_2, \ldots M_k$$

with relations

$$s_i^2 = 1, \quad s_i s_j = s_j s_i \quad \text{if } j \notin \{i-1, i+1\}$$

$$s_i s_{i+1} s_i = s_i s_{i+1} s_i \quad \text{for } i \in \{1, \ldots, k-2\}$$

$$M_1 = 0, \quad M_i M_j = M_j M_i$$

$$M_i = s_{i-1} M_{i-1} s_{i-1} + s_{i-1}.$$

Let $q \in \mathbb{C}^\times$.

The <u>Hecke algebra $H_k(q)$</u> is the $\mathbb{C}$-algebra generated by

$$T_1, \dots, T_{k-1} \quad \text{and} \quad y_1, y_2, \dots, y_k$$

with relations

$$T_i^2 = (q - q^{-1})T_i + 1, \quad T_iT_j = T_jT_i \text{ if } j \notin \{i-1, i+1\}$$

$$T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1} \quad \text{for } i \in \{1, \dots, k-2\}$$

$$y_1 = 1, \quad y_i = T_{i-1}^{-1}\, y_{i-1}\, T_{i-1}^{-1} \quad \text{for } i \in \{1, \dots, k-1\}$$

$$y_iy_j = y_jy_i \quad \text{for } i, j \in \{1, \dots, k\}.$$

Let λ be a partition with k-boxes and

$$\hat{S}_k^\lambda = \left\{ \begin{matrix} \text{standard tableaux } T \\ \text{of shape } \lambda \end{matrix} \right\}$$

<u>Theorem</u> Assume $1 \notin q^{2\mathbb{Z}_{\neq 0}}$. The map

$$\left\{ \begin{matrix} \text{partitions} \\ \text{of } k \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{isomorphism classes} \\ \text{of irreducible} \\ H_k(q)\text{-modules} \end{matrix} \right\}$$

$$\lambda \longmapsto H_k^\lambda$$

~~which~~ is a bijection, where

$$H_k^\lambda = \mathbb{C}\text{-span}\{v_T \mid T \in \hat{S}_k^\lambda\}$$

with $H_k(q)$-action determined by

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$$Y_i v_T = q^{2\,ct(T^{-1}(i))} v_T,$$

$$T_i v_T = \left(\frac{q-q^{-1}}{1-q^{2(ct(T^{-1}(i))-ct(T^{-1}(i+1)))}}\right) v_T + \left(q^{-1} + \frac{\sim}{\sim}\right) v_{s_iT}$$

where s_iT is T with i and $i+1$ switched

$v_{s_iT} = 0$ if $s_iT \notin \hat{S}^\lambda_k$.

Define

$$(s_i)_{TT} = \frac{1}{ct(T^{-1}(i)) - ct(T^{-1}(i+1))}$$

<u>Theorem</u> The map

$$\left\{\begin{matrix}\text{partitions}\\ \text{of } k\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{isomorphism classes}\\ \text{of irreducible}\\ \mathbb{C}S_k\text{-modules}\end{matrix}\right\}$$

$$\lambda \longmapsto S^\lambda_k$$

is a bijection, where

$$S^\lambda_k = \mathbb{C}\text{-span}\{v_T \mid T \in \hat{S}^\lambda_k\}$$

and $\mathbb{C}S_k$-action determined by

$$M_i v_T = ct(T^{-1}(i)) v_T,$$

$$s_i v_{T} = (s_i)_{TT}\, v_T + (1+(s_i)_{TT})\, v_{s_iT}.$$