For $r \in \mathbb{Z}_{>0}$ define

$$\hat{g}_r(x_1,\dots,x_n;u,t) = \sum_{k=0}^{r-1} \sum_{i_r \le \cdots \le i_{k+1} > i_k > \cdots > i_1} u^{r-1-k}(-t)^k x_{i_1}\cdots x_{i_r}$$

The Pieri rule says

$$\hat{g}_r s_\mu = \sum_{\substack{\lambda/\mu \text{ bs} \\ |\lambda/\mu| = r}} s_\lambda$$

Let $\lambda/\mu$ be a skew shape. The skew shape $\lambda/\mu$ is

- a <u>border strip</u> if $\lambda/\mu$ has at most one box on each diagonal
- a <u>vertical strip</u> if $\lambda/\mu$ has at most one box in each row
- a <u>horizontal strip</u> if $\lambda/\mu$ has at most one box in each column.

$$h_r(x_1,\dots,x_n) = \hat{g}_r(x_1,\dots,x_n;1,0) = \sum_{i_r \le \cdots \le i_1} x_{i_1}\cdots x_{i_r}$$

$$e_r(x_1,\dots,x_n) = \hat{g}_r(x_1,\dots,x_n;0,-1) = \sum_{i_r > \cdots > i_1} x_{i_1}\cdots x_{i_r}$$

$$p_r(x_1,\dots,x_n) = \hat{g}_r(x_1,\dots,x_n;u,u)\,u^{-(r-1)} = \sum_i x_i^r$$

Then

$$e_r s_\mu = \sum_{\substack{\lambda/\mu \text{ vs} \\ |\lambda/\mu| = r}} s_\lambda, \qquad h_r s_\mu = \sum_{\substack{\lambda/\mu \text{ hs} \\ |\lambda/\mu| = r}} s_\lambda$$

and

$$p_r s_\mu = \sum_{\substack{\lambda/\mu \text{ connected} \\ \text{bs}}} (-1)^{\#\text{cols}(\lambda/\mu) - 1} s_\lambda .$$

Let $k \in \mathbb{Z}_{>0}$.
For $\mu = (\mu_1, \dots, \mu_\ell)$ with $\mu_1 \geq \cdots \geq \mu_\ell > 0$ and $\mu_1 + \cdots + \mu_\ell = k$ ($\mu$ is a partition with $k$ boxes) define

$$\hat{q}_\mu = \hat{q}_{\mu_1} \hat{q}_{\mu_2} \cdots \hat{q}_{\mu_\ell} .$$

<u>Theorem</u> Let $Q$ be a standard tableau of shape $\lambda$. Let $b \in \lambda$ and suppose $Q(b) = j$.
Define

$$wt_Q^\mu(b) = \begin{cases} -t, & \text{if } j+1 \text{ is sw of } j \text{ in } Q \\ u, & \text{if } j+1 \text{ is ne of } j \text{ in } Q \\ 0, & \text{if } j+1 \text{ is ne of } j \text{ in } Q \\ & \quad j+2 \text{ is sw of } j \text{ in } Q \\ & \quad j+1 \notin J(\mu) \end{cases}$$

where $J(\mu) = \{\mu_1, \mu_1 + \mu_2, \dots, \mu_1 + \cdots + \mu_\ell\}$.
Define

$$wt_Q^\mu = \prod_{\substack{b \in \lambda \\ Q(b) \notin J(\mu)}} wt_Q^\mu(b).$$

Then let

$$K_{u,t}^{\lambda}(\mu) = \sum_{Q \in \hat{\mathcal{B}}_k^{\lambda}} \mathrm{wt}^{\mu}(Q)$$

Then

$$\hat{q}_{\mu} = \sum_{\lambda} K_{u,t}^{\lambda}(\mu)\, s_{\lambda}$$

Define a matrix $K_{u,t} = \left(K_{u,t}^{\lambda}(\mu)\right)$.

The <u>character table of $S_n$</u> is $K_{1,1}$

The <u>character table of $H_n$</u> is $K_{q,1}$

The <u>Kostka numbers</u> are $K_{\lambda\mu} = K_{1,0}^{\lambda}(\mu)$

$$\hat{q}_1 = s_{(1)}$$

$$\hat{q}_2 = u s_{(2)} + (-t) s_{(1,1)}$$

$$\hat{q}_3 = u^2 s_{(3)} + u(-t) s_{(2,1)} + (-t)^2 s_{(1,1,1)}$$

$$\hat{q}_4 = u^3 s_{(4)} + u^2(-t) s_{(3,1)} + u(-t)^2 s_{(2,1,1)} + (-t)^3 s_{(1,1,1,1)}$$

$$\begin{aligned}
\hat{q}_{(2,2)} = \hat{q}_2 \hat{q}_2 &= \hat{q}_2\left(u s_{(2)} + (-t) s_{(1,1)}\right) \\
&= u\, \hat{q}_2 s_{(2)} + (-t)\, \hat{q}_2 s_{(1,1)} \\
&= u\left(u s_{(4)} + u s_{(3,1)} + u s_{(2,2)} + (-t) s_{(2,1,1)} + (-t) s_{(3,1)}\right) \\
&\quad + (-t)\left(u s_{(3,1)} + u s_{(2,1,1)} + (-t) s_{(1,1,1,1)} + (-t) s_{(2,1,1)} + (-t) s_{(2,2)}\right) \\
&= u^2 s_{(4)} + \left(u^2 + u(-t) + u(-t)\right) s_{(3,1)} + \left(u(-t) + u(-t) + (-t)^2\right) s_{(2,1,1)} + \cdots
\end{aligned}$$

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$K_{u,t} =$

| $\lambda \backslash \mu$ | $(2)$ | $(1^2)$ |
|---|---|---|
| $(2)$ | $u$ | $1$ |
| $(1^2)$ | $-t$ | $1$ |

$K_{u,t} =$

| $\lambda \backslash \mu$ | $(3)$ | $(21)$ | $(1^3)$ |
|---|---|---|---|
| $(3)$ | $u^2$ | $u$ | $1$ |
| $(21)$ | $0+u(-t)$ | $u+(-t)$ | $1$ |
| $(1^3)$ | $(-t)^2$ | $-t$ | $1$ |

$K_{u,t} =$

| | $(4)$ | $(31)$ | $(22)$ | $(21^2)$ | $(1^4)$ |
|---|---|---|---|---|---|
| $(4)$ | $u^3$ | $u^2$ | $u^2$ | $u$ | $1$ |
| $(31)$ | | | $u^2+2u(-t)$ | | |
| $(22)$ | | | $(-t)^2+u^2$ | | |
| $(21^2)$ | | | $2u(-t)+(-t)^2$ | | |
| $(1^4)$ | | | $(-t)^2$ | | |

<u>Contents of boxes</u>

A box is $(r,c) \in \mathbb{Z}^2$.

The <u>content of a box</u> $b=(r,c)$ is

$$ct(b) = c-r.$$

then

$j+1$ is <u>sw</u> of $j$ in $Q$ if $ct(Q^{-1}(j+1)) > ct(Q^{-1}(j))$

$j+1$ is <u>ne</u> of $j$ in $Q$ if $ct(Q^{-1}(j+1)) < ct(Q^{-1}(j))$.

A <u>connected borderstrip</u> is a borderstrip $\lambda/\mu$ such that $ct(\lambda/\mu)$ is an interval in $\mathbb{Z}$.