Review of generating symmetric functions

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Define g_r by

$$\prod_{i=1}^{n} \frac{(tx_iz;q)_\infty}{(ux_iz;q)_\infty} = \sum_{r\in\mathbb{Z}_{\ge 0}} g_r(x_1,\ldots,x_n;u,t,q)\, z^r.$$

Since

$$(tx_iz;q)_\infty = (1-tx_iz)(1-tx_izq)(1-tx_izq^2)\cdots$$

then q_r given by

$$\prod_{i=1}^{n} \frac{1-tx_iz}{1-ux_iz} = \sum_{r\in\mathbb{Z}_{\ge 0}} q_r(x_1,\ldots,x_n;u,t)\, z^r$$

satisfy

$$q_r(x_1,\ldots,x_n;u,t) = g_r(x_1,\ldots,x_n;u,t,0)$$

Expanding

$$\prod_{i=1}^{n} \frac{1-tx_iz}{1-ux_iz} = \frac{1}{(1-ux_1z)}\cdots\frac{1}{(1-ux_nz)}(1-tx_nz)\cdots(1-tx_1z)$$

gives that if $r\in\mathbb{Z}_{>0}$ then

$$q_r(x_1,\ldots,x_n;u,t) = \sum_{k=0}^{n} \sum_{i_r\le\cdots\le i_{k+1}>i_k>\cdots>i_1} u^{r-k}(-t)^k x_{i_r}\cdots x_{i_1}$$

$$= (u-t)\sum_{k=0}^{n-1} \sum_{i_r\le\cdots\le i_{k+1}>i_k>\cdots>i_1} u^{\#\le}(-t)^{\#>} x_{i_r}\cdots x_{i_1}$$

Reviewing RSK: 'Insertion into a column of length k' (2)

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$$\binom{i}{j} \odot \left(\boxed{\begin{matrix} p_1 \\ \vdots \\ p_k \end{matrix}}, \boxed{\begin{matrix} q_1 \\ \vdots \\ q_k \end{matrix}} \right) = \left(\boxed{\begin{matrix} p_1 \\ \vdots \\ p_{\ell-1} \\ j \\ p_{\ell+1} \\ \vdots \\ p_k \end{matrix}}, \boxed{\begin{matrix} q_1 \\ \vdots \\ \vdots \\ \vdots \\ q_k \end{matrix}} \right) \odot \binom{i}{p_\ell} \quad \text{if } p_{\ell-1} < j \le p_\ell$$

$$\binom{i}{j} \odot \left(\boxed{\begin{matrix} p_1 \\ \vdots \\ p_k \end{matrix}}, \boxed{\begin{matrix} q_1 \\ \vdots \\ q_k \end{matrix}} \right) = \left(\boxed{\begin{matrix} p_1 \\ \vdots \\ p_k \\ j \end{matrix}}, \boxed{\begin{matrix} q_1 \\ \vdots \\ q_k \\ i \end{matrix}} \right) \quad \text{if } j > p_k.$$

For example:

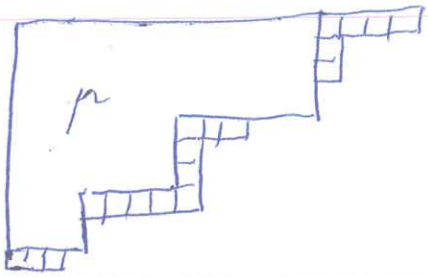
$$\begin{pmatrix} 13 & 12 & 11 & 10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ 1 & 3 & 3 & 5 & 5 & 5 & 7 & 7 & 7 & 6 & 4 & 3 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 13 & 12 & 11 & 10 & 9 & 8 & 7 & 6 \\ 1 & 3 & 3 & 5 & 5 & 5 & 7 & 7 \end{pmatrix} \odot \left(\boxed{\begin{matrix} 1 \\ 3 \\ 4 \\ 4 \\ 7 \end{matrix}}, \boxed{\begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix}} \right)$$

$$= \left(\begin{matrix} 1 & 1 & 3 & 3 & 5 & 5 & 6 & 7 & 7 \\ 3 \\ 4 \\ 5 \\ 7 \end{matrix}, \quad \begin{matrix} 1 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} \right)$$

Notice the pattern in which the boxes appear.

A border strip is a skew shape λ/μ with at most one box in each diagonal



The Pieri rule

Let

$$UD_r = \left\{ \begin{array}{c} \text{sequences} \\ (i_r, \ldots, i_1) \end{array} \;\middle|\; \begin{array}{l} i_1, \ldots, i_r \in \{1, \ldots, n\},\ k \in \{0, 1, \ldots, r-1\} \text{ and} \\ i_r \le \cdots \le i_{k+1} > i_k > \cdots > i_1 \end{array} \right\}$$

and

$$B_n(\mu) = \{\text{SSYT of shape } \mu \text{ filled from } \{1, \ldots, n\}\}.$$

Then RSK gives a bijection

$$UD_r \times B_n(\mu) \longrightarrow \bigsqcup_{\substack{\lambda/\mu \text{ bs} \\ |\lambda/\mu| = r}} B_n(\lambda)$$

$$((i_r, \ldots, i_1), P) \longmapsto Q$$

which results in the identity

$$q_r(x_1, \ldots, x_n; u, t)\, s_\mu(x) = \left(\sum_{(i_r, \ldots, i_1) \in UD_r} u^{\#(\le)} (-t)^{\#(>)} x_{i_r} \cdots x_{i_1} \right) \left(\sum_{P \in B_n(\mu)} x^{wt(P)} \right)$$

$$= \sum_{\substack{\lambda/\mu \text{ bs} \\ |\lambda/\mu| = r}} (u-t)\, u^{\#\left(\substack{\text{vertical} \\ \text{steps}}\right)} (-t)^{\#\left(\substack{\text{horiz} \\ \text{steps}}\right)} \sum_{Q \in B_n(\lambda)} x^{wt(Q)}$$

$$= \sum_{\substack{\lambda/\mu \text{ bs} \\ |\lambda/\mu| = r}} (u-t)\, s_\lambda(x).$$

Specializations

(1) The elementary symmetric functions e_r are given by

$$\prod_{i=1}^{n}(1+x_i z) = \sum_{r\in\mathbb{Z}_{\geq 0}} e_r(x_1,\ldots,x_n) z^r$$

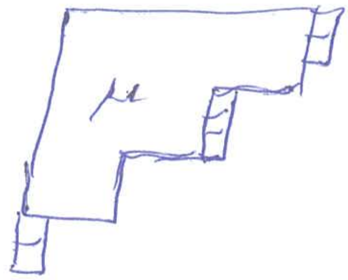
so

$$e_r(x_1,\ldots,x_n) = q_r(x_1,\ldots,x_n; 0,-1)$$

and

$$e_r(x_1,\ldots,x_n) = \sum_{i_r > i_{r-1} > \cdots > i_1} x_{i_r}\cdots x_{i_1}.$$

A vertical strip is a skew shape λ/μ with at most one box in each row.



Then

$$e_r s_\mu = \sum_{\substack{\lambda/\mu \text{ VS} \\ |\lambda/\mu| = r}} s_\lambda$$

(2) The homogeneous symmetric functions h_r are given by

$$\prod_{i=1}^{n}\frac{1}{1-x_i z} = \sum_{r\in\mathbb{Z}_{\geq 0}} h_r z^r$$

so

$$h_r(x_1,\ldots,x_n) = q_r(x_1,\ldots,x_n; 1, 0)$$

and

$$h_r(x_1,\ldots,x_n) = \sum_{i_r \leq \cdots \leq i_1} x_{i_r}\cdots x_{i_1}.$$

A horizontal strip is a skew shape λ/μ with at most one box in each row.

Then

$$h_r s_\mu = \sum_{\substack{\lambda/\mu \text{ hs} \\ |\lambda/\mu| = r}} s_\lambda$$