25.04.2025 (1)
Adv. Disc. Math
A. Ram

RSK and Schur functions

The bijections

$$M_{m\times n}(\mathbb{Z}_{\geq 0}) \longleftrightarrow \left\{\begin{matrix}\text{sequences of pairs } \binom{i}{j} \\ \text{in weakly decreasing order}\end{matrix}\right\}$$

$$\longleftrightarrow \left\{\begin{matrix}\text{Pairs } (P,Q) \text{ such that} \\ P \text{ is a SSYT filled from } \{1,\ldots,n\} \\ Q \text{ is a SSYT filled from } \{1,\ldots,m\} \\ \text{shape}(P) = \text{shape}(Q)\end{matrix}\right\}$$

gives the Cauchy identity

$$\prod_{j=1}^{n}\prod_{i=1}^{m}\frac{1}{1-x_iy_j} = \prod_{j=1}^{n}\prod_{i=1}^{m}(1+x_iy_j+(x_iy_j)^2+\cdots)$$

$$= \sum_{A\in M_{m\times n}(\mathbb{Z}_{\geq 0})} \mathrm{wt}(A)$$

$$= \sum_{\lambda}\sum_{(P,Q)\in B_n(\lambda)\times B_m(\lambda)} x^{\mathrm{wt}(P)}y^{\mathrm{wt}(Q)}$$

$$= \sum_{\lambda} s_\lambda(x_1,\ldots,x_n)\,s_\lambda(y_1,\ldots,y_m).$$

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$$\left(\begin{array}{llllll} 1&1&1&2&7&9 \\ 2&3&3&7&& \\ 3&8&8&&& \\ 7&&&&& \end{array}\ ,\ \begin{array}{llllll} 1&2&2&4&6&6 \\ 3&3&4&5&& \\ 4&6&6&&& \\ 6&&&&& \end{array}\right)$$

$$= \begin{pmatrix} 6 \\ 3 \end{pmatrix} \otimes \left(\begin{array}{lllll} 1&1&1&2&9 \\ 2&3&7&7& \\ 3&8&8&& \\ 7&&&& \end{array}\ ,\ \begin{array}{lllll} 1&2&2&4&6 \\ 3&3&4&5& \\ 4&6&6&& \\ 6&&&& \end{array}\right)$$

$$= \begin{pmatrix} 6 \\ 3 \end{pmatrix} \otimes \begin{pmatrix} 6 \\ 3 \end{pmatrix} \otimes \left(\begin{array}{llll} 1&1&1&2 \\ 2&7&7&9 \\ 3&8&8& \\ 7&&& \end{array}\ ,\ \begin{array}{llll} 1&2&2&4 \\ 3&3&4&5 \\ 4&6&6& \\ 6&&& \end{array}\right)$$

$$= \begin{pmatrix} 6 \\ 3 \end{pmatrix} \otimes \begin{pmatrix} 6 \\ 3 \end{pmatrix} \otimes \begin{pmatrix} 6 \\ 7 \end{pmatrix} \otimes \left(\begin{array}{llll} 1&1&1&2 \\ 2&7&7&9 \\ 3&8&& \\ 8&&& \end{array}\ ,\ \begin{array}{llll} 1&2&2&4 \\ 3&3&4&5 \\ 4&6&& \\ 6&&& \end{array}\right)$$

$$= \begin{pmatrix} 6 \\ 3 \end{pmatrix} \otimes \begin{pmatrix} 6 \\ 3 \end{pmatrix} \otimes \begin{pmatrix} 6 \\ 7 \end{pmatrix} \otimes \begin{pmatrix} 6 \\ 8 \end{pmatrix} \otimes \left(\begin{array}{llll} 1&1&1&2 \\ 2&7&7&9 \\ 3&&& \\ 8&&& \end{array}\ ,\ \begin{array}{llll} 1&2&2&4 \\ 3&3&4&5 \\ 4&&& \\ 6&&& \end{array}\right)$$

$$= \begin{pmatrix} 6 \\ 3 \end{pmatrix} \otimes \begin{pmatrix} 6 \\ 3 \end{pmatrix} \otimes \begin{pmatrix} 6 \\ 7 \end{pmatrix} \otimes \begin{pmatrix} 6 \\ 8 \end{pmatrix} \otimes \begin{pmatrix} 6 \\ 8 \end{pmatrix} \otimes \left(\begin{array}{llll} 1&1&1&2 \\ 2&7&7&9 \\ 3&&& \end{array}\ ,\ \begin{array}{llll} 1&2&2&4 \\ 3&3&4&5 \\ 4&&& \end{array}\right)$$

$$= \begin{pmatrix} 6 \\ 3 \end{pmatrix} \otimes \begin{pmatrix} 6 \\ 3 \end{pmatrix} \otimes \begin{pmatrix} 6 \\ 7 \end{pmatrix} \otimes \begin{pmatrix} 6 \\ 8 \end{pmatrix} \otimes \begin{pmatrix} 6 \\ 8 \end{pmatrix} \otimes \begin{pmatrix} 5 \\ 3 \end{pmatrix} \otimes \left(\begin{array}{llll} 1&1&1&2 \\ 2&7&9& \\ 7&&& \end{array}\ ,\ \begin{array}{llll} 1&2&2&4 \\ 3&3&4& \\ 4&&& \end{array}\right)$$

$$= \begin{pmatrix} 6&6&6&6&6 \\ 3&3&7&8&8 \end{pmatrix} \otimes \begin{pmatrix} 5 \\ 3 \end{pmatrix} \otimes \begin{pmatrix} 4 \\ 1 \end{pmatrix} \otimes \left(\begin{array}{lll} 1&1&2 \\ 2&7&9 \\ 7&& \end{array}\ ,\ \begin{array}{lll} 1&2&2 \\ 3&3&4 \\ 4&& \end{array}\right)$$

$$= \begin{pmatrix} 6&6&6&6&6&5 \\ 3&3&7&8&8&3 \end{pmatrix} \otimes \begin{pmatrix} 4 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 4 \\ 7 \end{pmatrix} \otimes \left(\begin{array}{lll} 1&1&2 \\ 2&9& \\ 7&& \end{array}\ ,\ \begin{array}{lll} 1&2&2 \\ 3&3& \\ 4&& \end{array}\right)$$

$$= \begin{pmatrix} 6 & 6 & 6 & 6 & 6 & 5 \\ 3 & 3 & 7 & 8 & 8 & 3 \end{pmatrix} \infty \begin{pmatrix} 4 \\ 1 \end{pmatrix} \infty \begin{pmatrix} 4 \\ 7 \end{pmatrix} \infty \begin{pmatrix} 4 \\ 7 \end{pmatrix} \infty \begin{pmatrix} 1 & 1 & 2, & 1 & 2 & 2 \\ 2 & 9 & , & 1 & 3 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 6 & 6 & 6 & 6 & 5 & 4 & 4 & 4 \\ 3 & 3 & 7 & 8 & 8 & 3 & 1 & 7 & 7 \end{pmatrix} \infty \begin{pmatrix} 3 \\ 2 \end{pmatrix} \infty \begin{pmatrix} 1 & 1 & 2, & 1 & 2 & 2 \\ 9 & & , & 3 & & \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 6 & 6 & 6 & 6 & 5 & 4 & 4 & 4 \\ 3 & 3 & 7 & 8 & 8 & 3 & 1 & 7 & 7 \end{pmatrix} \infty \begin{pmatrix} 3 \\ 2 \end{pmatrix} \infty \begin{pmatrix} 3 \\ 9 \end{pmatrix} \infty (1\,1\,2, 1\,2\,2)$$

$$= \begin{pmatrix} 6 & 6 & 6 & 6 & 6 & \cancel{6} & 5 & 4 & 4 & 4 & 3 & 3 \\ 3 & 3 & 7 & 8 & 8 & \cancel{3} & 3 & 1 & 7 & 7 & 2 & 9 \end{pmatrix} \infty \begin{pmatrix} 2 \\ 1 \end{pmatrix} \infty (1\,2, 1\,2)$$

$$= \begin{pmatrix} 6 & 6 & 6 & 6 & 6 & 5 & 4 & 4 & 4 & 3 & 3 \\ 3 & 3 & 7 & 8 & 8 & 3 & 1 & 7 & 7 & 2 & 9 \end{pmatrix} \infty \begin{pmatrix} 2 \\ 1 \end{pmatrix} \infty \begin{pmatrix} 2 \\ 1 \end{pmatrix} \infty (2, 1)$$

$$= \begin{pmatrix} 6 & 6 & 6 & 6 & 6 & 5 & 4 & 4 & 3 & 3 & 3 & 2 & 2 & 1 \\ 3 & 3 & 7 & 8 & 8 & 3 & 1 & 7 & 7 & 2 & 9 & 1 & 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 2 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

<u>Restrict to matrices in $M_{m\times n}(\{0,1\})$ with one 1 in each row</u>

<u>Example</u>

$$\begin{pmatrix} 0&0&1\\ 1&0&0\\ 0&0&1\end{pmatrix} = \begin{pmatrix}3\\3\end{pmatrix}\otimes\begin{pmatrix}2\\1\end{pmatrix}\otimes\begin{pmatrix}1\\3\end{pmatrix}$$

$$= \begin{pmatrix}3\\3\end{pmatrix}\otimes\begin{pmatrix}2\\1\end{pmatrix}\otimes(3,1)$$

$$= \begin{pmatrix}3\\3\end{pmatrix}\otimes(13, 12) = \left(\begin{array}{ll}1&3\\3\end{array},\ \begin{array}{ll}1&2\\3\end{array}\right)$$

$$\left\{\begin{array}{c} A\in M_{m\times n}(\{0,1\}) \\ \text{with one 1 in each} \\ \text{row}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{Pairs }(P,Q) \\ P \text{ SSYT filled from } \{1,\ldots,n\} \\ Q \text{ SYT, shape}(P) = \text{shape}(Q)\end{array}\right\}$$

The identity is

$$(x_1+x_2+\cdots+x_n)^m = \sum_\lambda s_\lambda(x) f_\lambda$$

where f_λ is the number of SYT of shape λ.

<u>Restrict to permutation matrices</u> $\left(n! = \sum_\lambda f_\lambda^2\right)$.

<u>Example</u>

$$\begin{pmatrix} 0&0&1\\ 1&0&0\\ 0&1&0\end{pmatrix} = \begin{pmatrix}3\\2\end{pmatrix}\otimes\begin{pmatrix}2\\1\end{pmatrix}\otimes\begin{pmatrix}1\\3\end{pmatrix}$$

$$= \begin{pmatrix}3\\2\end{pmatrix}\otimes\begin{pmatrix}2\\1\end{pmatrix}\otimes(3,1)$$

$$= \begin{pmatrix}3\\2\end{pmatrix}\otimes(13, 12) = \left(\begin{array}{ll}1&3\\2\end{array},\ \begin{array}{ll}1&2\\3\end{array}\right)$$

$$\left\{\begin{array}{c}\text{permutations} \\ \text{in } S_n\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{Pairs }(P,Q)\text{ with} \\ P \text{ SYT, } Q \text{ SYT} \\ \text{shape}(P) = \text{shape}(Q)\end{array}\right\}$$