<u>Schur functions</u> Let $n\in\mathbb{Z}_{>0}$

For $\lambda=(\lambda_1,\dots,\lambda_n)\in\mathbb{Z}^n_{\ge 0}$ with $\lambda_1\ge\cdots\ge\lambda_n$ let

$$B_n(\lambda)=\left\{\begin{array}{l}\text{SSYTs of shape }\lambda\\ \text{filled from }\{1,\dots,n\}\end{array}\right\}$$

The <u>length of λ</u> is

$$\ell(\lambda)=\#\{i\mid \lambda_i\neq 0\}.$$

Let $m,n\in\mathbb{Z}_{>0}$ and suppose λ is a partition with $\ell(\lambda)\le n$ and $\ell(\lambda)\le m$.

$$s_\lambda(x)=s_\lambda(x_1,\dots,x_n)=\sum_{P\in B_n(\lambda)} x^{wt(P)}=\sum_{P\in B_n(\lambda)} x_1^{\#1\text{'s in }P}\cdots x_n^{\#n\text{'s in }P}$$

$$s_\lambda(y)=s_\lambda(y_1,\dots,y_m)=\sum_{Q\in B_m(\lambda)} y^{wt(Q)}=\sum_{Q\in B_m(\lambda)} y_1^{\#1\text{'s in }Q}\cdots y_m^{\#m\text{'s in }Q}$$

Rewrite matrices in $M_{m\times n}(\mathbb{Z}_{\ge 0})$ as sequences:

$$M_{m\times n}(\mathbb{Z}_{\ge 0})\longleftrightarrow\left\{\begin{array}{l}\text{sequences of pairs }\binom{i}{j}\\ \text{in weakly decreasing order}\end{array}\right\}$$

$$A=(a_{ij})\longmapsto \underbrace{\binom{m}{n}\otimes\cdots\otimes\binom{m}{n}}_{a_{mn}\text{ times}}\otimes\underbrace{\binom{m}{n-1}\otimes\cdots\otimes\binom{m}{n-1}}_{a_{m,n-1}\text{ times}}\otimes\cdots\otimes\underbrace{\binom{1}{1}\otimes\cdots\otimes\binom{1}{1}}_{a_{11}\text{ times}}$$

the ordering is

$$\binom{j}{k}>\binom{i}{l}\text{ if } j>i \quad\text{and}\quad \binom{i}{l}>\binom{i}{k}\text{ if } l>k.$$

RSK insertion: "Insert" $\binom{i}{j}$ into a pair of columns".

$$\binom{i}{j} \otimes \left(\begin{array}{|c|}\hline p_1 \\ \vdots \\ p_r \\ \hline \end{array}\, , \begin{array}{|c|}\hline q_1 \\ \vdots \\ q_r \\ \hline \end{array} \right) = \left(\begin{array}{|c|}\hline p_1 \\ \vdots \\ j \\ \vdots \\ p_r \\ \hline \end{array}\, , \begin{array}{|c|}\hline q_1 \\ \vdots \\ \vdots \\ q_r \\ \hline \end{array} \right) \otimes \binom{i}{p_l} \quad \text{if } p_{l-1} < j \le p_l$$

$$\binom{i}{j} \otimes \left(\begin{array}{|c|}\hline p_1 \\ \vdots \\ p_r \\ \hline \end{array}\, , \begin{array}{|c|}\hline q_1 \\ \vdots \\ q_r \\ \hline \end{array} \right) = \left(\begin{array}{|c|}\hline p_1 \\ \vdots \\ p_r \\ j \\ \hline \end{array}\, , \begin{array}{|c|}\hline q_1 \\ \vdots \\ q_r \\ i \\ \hline \end{array} \right) \otimes \binom{\emptyset}{\emptyset} \quad \text{if } j > p_r$$

A SSYT of shape (1^r) has $\begin{array}{|c|}\hline p_1 \\ \vdots \\ p_r \\ \hline \end{array}$ with $p_1 < \cdots < p_r$.

A SSYT of shape λ is a product of column shape SSYTs.

Combining RSK insertion with matrices to sequences of pairs

$$M_{m\times n}(\mathbb{Z}_{\ge 0}) \longleftrightarrow \left\{ \begin{array}{l} \text{Pairs of SSYT } (P,Q) \text{ with} \\ \text{shape}(P) = \text{shape}(Q) \\ P \text{ filled from } \{1,\ldots,n\} \\ Q \text{ filled from } \{1,\ldots,m\} \end{array} \right\}$$

If $A = (a_{ij}) \in M_{m\times n}(\mathbb{Z}_{\ge 0})$ let

$$wt(A) = \prod_{i=1}^{n} \prod_{j=1}^{m} (x_i y_j)^{a_{ij}}$$

Then

$$\sum_{A \in M_{m\times n}(\mathbb{Z}_{\ge 0})} wt(A) = \prod_{i=1}^{n} \prod_{j=1}^{n} \frac{1}{1 - x_i y_j}.$$

24.09.2025
Adv. Disc Math
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Theorem (Cauchy identity)

$$\prod_{i=1}^{m}\prod_{j=1}^{n}\frac{1}{1-x_iy_j} = \sum_{\lambda} s_\lambda(x)s_\lambda(y)$$

Proof
$$\prod_{i=1}^{m}\prod_{j=1}^{n}\frac{1}{1-x_iy_j} = \sum_{A\in M_{m\times n}(\mathbb{Z}_{\geq 0})} wt(A)$$

$$= \sum_\lambda \sum_{\substack{(P,Q)\in B_n(\lambda)\times B_m(\lambda)\\ shape(P)=shape(Q)}} x^P y^Q = \sum_{\lambda} s_\lambda(x)s_\lambda(y)$$ //.

Corollary Let $f_\lambda = \mathrm{Card}\left\{\begin{matrix}\text{standard tableaux}\\ \text{of shape } \lambda\end{matrix}\right\}$
Then
$$(x_1+\cdots+x_n)^m = \sum_{\substack{\lambda\in\mathcal{Y}_m\\ \ell(\lambda)\leq n}} s_\lambda(x) f_\lambda$$

Proof Restrict the Cauchy identity to matrices in $M_{m\times n}(\{0,1\})$ with one 1 in each row. //

Corollary $$n! = \sum_{\lambda\in\mathcal{Y}_n} f_\lambda^2$$

Proof Restrict the Cauchy identity to $n\times n$ permutation matrices.

Examples

Matrix — sequence of pairs — pair (P,Q)

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \longleftrightarrow \binom{2}{3}\otimes\binom{1}{1}\otimes\binom{1}{3} \longleftrightarrow \left(\begin{array}{ll} 1 & 3 \\ 3 & \end{array} , \begin{array}{ll} 1 & 1 \\ 2 & \end{array} \right)$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \longleftrightarrow \binom{3}{3}\otimes\binom{2}{1}\otimes\binom{1}{3} \longleftrightarrow \left(\begin{array}{ll} 1 & 3 \\ 3 & \end{array} , \begin{array}{ll} 1 & 2 \\ 3 & \end{array} \right)$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \longleftrightarrow \binom{3}{2}\otimes\binom{2}{1}\otimes\binom{1}{3} \longleftrightarrow \left(\begin{array}{ll} 1 & 3 \\ 2 & \end{array} , \begin{array}{ll} 1 & 2 \\ 3 & \end{array} \right)$$

$$\begin{pmatrix} 3 & 2 & 0 \\ 1 & 1 & 4 \\ 0 & 3 & 0 \end{pmatrix} \longleftrightarrow \binom{3}{2}\otimes\binom{3}{2}\otimes\binom{3}{2}\otimes\binom{2}{1}\otimes\binom{2}{2}\otimes\binom{2}{3}\otimes\binom{2}{3}\otimes\binom{2}{3}\otimes\binom{2}{3}$$
$$\otimes\binom{1}{1}\otimes\binom{1}{1}\otimes\binom{1}{1}\otimes\binom{1}{2}\otimes\binom{1}{2}$$

$$\longleftrightarrow (1\,1\,1\,2\,2,\ 1\,1\,1\,1)$$

then $\left(\begin{array}{lllll} 1 & 1 & 1 & 2 & 2 \\ 2 & 3 & 3 & 3 & 3 \end{array} , \begin{array}{lllll} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 \end{array} \right)$

then $\left(\begin{array}{llllll} 1 & 1 & 1 & 1 & 2 & 2 \\ 2 & 3 & 3 & 3 & 3 & \end{array} , \begin{array}{llllll} 1 & 1 & 1 & 1 & 1 & 2 \\ 2 & 2 & 2 & 2 & 2 & \end{array} \right)$

then $\left(\begin{array}{lllllll} 1 & 1 & 1 & 1 & 2 & 2 & 3 \\ 2 & 2 & 2 & 3 & 3 & 3 & \end{array} , \begin{array}{lllllll} 1 & 1 & 1 & 1 & 1 & 2 & 3 \\ 2 & 2 & 2 & 2 & 2 & 3 & \end{array} \right)$

then $\left(\begin{array}{llllllll} 1 & 1 & 1 & 1 & 2 & 2 & 3 & 3 \\ 2 & 2 & 2 & 2 & 3 & 3 & & \end{array} , \begin{array}{llllllll} 1 & 1 & 1 & 1 & 1 & 2 & 3 & 3 \\ 2 & 2 & 2 & 2 & 2 & 3 & & \end{array} \right)$

$$\binom{3}{2}\otimes\binom{3}{2}\otimes\binom{3}{2}\otimes\binom{2}{1}\otimes\binom{2}{2}\otimes\binom{2}{1}\otimes\binom{2}{3}\otimes\binom{2}{3}\otimes\binom{2}{3}\otimes\binom{1}{1}\otimes\binom{1}{1}\otimes\binom{1}{1}\otimes\binom{1}{2}\otimes\binom{1}{2}$$

$$\simeq\binom{3}{2}\otimes\binom{3}{2}\otimes\binom{3}{2}\otimes\binom{2}{1}\otimes\binom{2}{2}\otimes\binom{2}{3}\otimes\binom{2}{3}\otimes\binom{2}{1}\otimes\binom{2}{3}\otimes$$

$$(1\,1\,1\,2\,2,\ 1\,1\,1\,1\,1)$$

$$\simeq\binom{3}{2}\otimes\binom{3}{2}\otimes\binom{3}{2}\otimes\binom{2}{1}\otimes\left(\begin{array}{ccccc}1&1&1&2&2\\2&3&3&3&3\end{array},\ \begin{array}{ccccc}1&1&1&1&1\\2&2&2&2&2\end{array}\right)$$

$$\simeq\binom{3}{2}\otimes\binom{3}{2}\otimes\binom{3}{2}\otimes\left(\begin{array}{cccccc}1&1&1&1&2&2\\2&3&3&3&3&\end{array},\ \begin{array}{cccccc}1&1&1&1&1&2\\2&2&2&2&2&\end{array}\right)$$

$$\simeq\binom{3}{2}\otimes\left(\begin{array}{ccccccc}1&1&1&1&2&2&3\\2&2&2&3&3&3&\end{array},\ \begin{array}{ccccccc}1&1&1&1&1&2&3\\2&2&2&2&2&3&\end{array}\right)$$

$$\simeq\left(\begin{array}{cccccccc}1&1&1&1&2&2&3&3\\2&2&2&2&3&3&&\end{array},\ \begin{array}{cccccccc}1&1&1&1&1&2&3&3\\2&2&2&2&3&&&\end{array}\right)$$