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Adv. Disc. Math

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Skew shapes and the Littlewood-Richardson rule

A *lattice permutation* is a word

$$p = \boxed{i_1}\otimes\cdots\otimes\boxed{i_k} \quad \text{in } B(\square)^{\otimes k}$$

such that

if $r\in\{1,\dots,k\}$ and $j\in\{1,\dots,n-1\}$

then

$$\#\left(\boxed{j} \text{ in } \boxed{i_1}\otimes\cdots\otimes\boxed{i_r}\right) \geq \#\left(\boxed{j+1} \text{ in } \boxed{i_1}\otimes\cdots\otimes\boxed{i_r}\right).$$

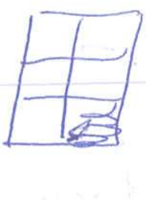
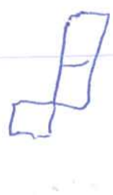
Theorem A word p is a lattice permutation if and only if p is a highest weight element of $B(\square)^{\otimes k}$.

Let $\lambda=(\lambda_1,\dots,\lambda_n)$ and $\mu=(\mu_1,\dots,\mu_n)$ be partitions such that $\mu\subseteq\lambda$.

Let

$$\lambda/\mu = \left\{\begin{matrix}\text{boxes in } \lambda \\ \text{that are not in } \mu\end{matrix}\right\}$$

Example $\lambda=(2,2,2)$ and $\mu=(1,1)$

$\lambda =$  $\supseteq \mu =$  and $\lambda/\mu =$ 

A <u>SSYT of shape λ/μ filled from $\{1, \ldots, n\}$</u> is a function $T: \lambda/\mu \to \{1, \ldots, n\}$ such that

(a) If $(r,c), (r,c+1) \in \lambda/\mu$ then $T(r,c) \le T(r,c+1)$

(b) If $(r,c), (r+1,c) \in \lambda/\mu$ then $T(r,c) < T(r+1,c)$.

Let

$$B(\lambda/\mu) = \left\{ \begin{array}{l} \text{SSYT of shape } \lambda/\mu \\ \text{filled from } \{1, \ldots, n\} \end{array} \right\}$$

<u>Examples</u> $n=3$.

$$B\left(\begin{array}{|c|c|}\hline & \\ \hline \end{array}\right) = \left\{ \begin{array}{cccc} \begin{array}{cc} & \boxed{1} \\ \boxed{1} & \boxed{2} \end{array}, & \begin{array}{cc} & \boxed{1} \\ \boxed{2} & \boxed{2} \end{array}, & \begin{array}{cc} & \boxed{1} \\ \boxed{2} & \boxed{3} \end{array}, & \begin{array}{cc} & \boxed{1} \\ \boxed{3} & \boxed{3} \end{array}, \\ \begin{array}{cc} & \boxed{1} \\ \boxed{1} & \boxed{3} \end{array}, & \begin{array}{cc} & \boxed{2} \\ \boxed{1} & \boxed{3} \end{array}, & \begin{array}{cc} & \boxed{2} \\ \boxed{2} & \boxed{3} \end{array}, & \begin{array}{cc} & \boxed{2} \\ \boxed{3} & \boxed{3} \end{array} \end{array} \right\}$$

$$B\left(\begin{array}{cc} & \square \\ & \square \\ \square & \end{array}\right) = \left\{ \begin{array}{ccc} \begin{array}{cc} & \boxed{1} \\ & \boxed{2} \\ \boxed{1} & \end{array} & \begin{array}{cc} & \boxed{1} \\ & \boxed{2} \\ \boxed{2} & \end{array} & \begin{array}{cc} & \boxed{1} \\ & \boxed{2} \\ \boxed{3} & \end{array} \\ \begin{array}{cc} & \boxed{1} \\ & \boxed{3} \\ \boxed{1} & \end{array} & \begin{array}{cc} & \boxed{1} \\ & \boxed{3} \\ \boxed{2} & \end{array} & \begin{array}{cc} & \boxed{1} \\ & \boxed{3} \\ \boxed{3} & \end{array} \\ \begin{array}{cc} & \boxed{2} \\ & \boxed{3} \\ \boxed{1} & \end{array} & \begin{array}{cc} & \boxed{2} \\ & \boxed{3} \\ \boxed{2} & \end{array} & \begin{array}{cc} & \boxed{2} \\ & \boxed{3} \\ \boxed{3} & \end{array} \end{array} \right\}$$

Let $k = $ (# of boxes in λ/μ) 14.09.2025 3

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<u>Theorem</u> There is a unique crystal structure on $B(\lambda/\mu)$ such that

$$\begin{aligned} B(\lambda/\mu) &\xleftrightarrow{\Phi} B(\square)^{\otimes k} \\ T &\longmapsto (\text{arabic reading of } T) \end{aligned}$$

The <u>skew Schur function</u> $s_{\lambda/\mu}$ is

$$s_{\lambda/\mu} = \mathrm{char}(B(\lambda/\mu))$$

<u>Corollary</u> (of the above and yesterday)

$$s_{\lambda/\mu} = \sum_{p \in B(\lambda/\mu)^+} s_{wt(p)}$$

where

$$B(\lambda/\mu)^+ = \left\{ \begin{array}{l} \text{highest weight elements} \\ \text{of } B(\lambda/\mu) \end{array} \right\}$$

Let

$$B(\lambda/\mu)^+_\nu = \left\{ \begin{array}{l} \text{highest weight elements} \\ p \in B(\lambda/\mu) \text{ with } wt(p) = \nu \end{array} \right\}$$

Then:

$$s_{\lambda/\mu} = \sum_{\nu \in \mathbb{Z}^n} \sum_{p \in B(\lambda/\mu)^+_\nu} s_\nu = \sum_\nu c^\lambda_{\mu\nu} s_\nu$$

where

$$c^\lambda_{\mu\nu} = \mathrm{Card}(B(\lambda/\mu)^+_\nu).$$

Corollary (of lattice permutation characterization of highest weight elements)

$$c_{\mu\nu}^{\lambda} = \#\left\{ \begin{array}{l} \text{SSYT of shape } \lambda/\mu \text{ of weight } \nu \\ \text{such that the arabic reading} \\ \text{of } T \text{ is a lattice permutation} \end{array} \right\}$$

A Littlewood-Richardson filling of λ/μ is a SSYT of shape λ/μ such that the arabic reading word of T is a lattice permutation.

Examples If $\lambda = (2,2,0)$ and $\mu = (1,0,0)$ then

$$c_{\mu\nu}^{\lambda} = \begin{cases} 1, & \text{if } \nu = (2,1,0) \\ 0, & \text{otherwise} \end{cases}$$

If $\lambda = (2,2,1)$ and $\mu = (1,1,0)$ then

$$c_{\mu\nu}^{\lambda} = \begin{cases} 1, & \text{if } \nu = (2,1,0) \\ 1, & \text{if } \nu = (1,1,1) \\ 0, & \text{otherwise.} \end{cases}$$

The $c_{\mu\nu}^{\lambda}$ are Littlewood-Richardson coefficients