18.09.2015 Adv. Disc Math A. Ram ①

## Crystals and Schur functions

Every crystal is a direct sum of components of $B(\square)^{\otimes k}$.

Let $B$ be a crystal. The data of $B$ includes

$$wt: B \longrightarrow \mathbb{Z}^n$$

and $\tilde{f}_1, \ldots, \tilde{f}_{n-1}$ which are matrices with entries in $\{0,1\}$.

The <u>character of $B$</u> is

$$\mathrm{char}(B) = \sum_{p \in B} x^{wt(p)}$$

Let

$$\tilde{e}_1 = \tilde{f}_1^t, \quad \tilde{e}_2 = \tilde{f}_2^t, \quad \ldots, \quad \tilde{e}_{n-1} = \tilde{f}_n^t.$$

A <u>highest weight element of $B$</u> is $p \in B$ such that if $i \in \{1, \ldots, n-1\}$ then $\tilde{e}_i p = 0$.

Let $B^+ = \{\text{highest weight elements of } B\}$.

<u>Theorem</u> Let $B$ be a crystal.

(a) $\mathrm{char}(B) \in \mathbb{C}[x_1, \ldots, x_n]^{S_n}$

(b) $$\mathrm{char}(B) = \sum_{p \in B^+} s_{wt(p)}.$$

<u>Corollary</u> (Weyl character formula)

$$\mathrm{char}(B) = s_\lambda$$

if $B$ has a unique highest weight element $p^+$ and $wt(p^+) = \lambda$.

Recall that

$$\mathbb{C}[x_1, \ldots, x_n]^{S_n} = \left\{ f \in \mathbb{C}[x_1, \ldots, x_n] \;\middle|\; \begin{matrix} \text{if } w \in S_n \text{ then} \\ wf = f \end{matrix} \right\}$$

$$= \left\{ f \in \mathbb{C}[x_1, \ldots, x_n] \;\middle|\; \begin{matrix} \text{if } i \in \{1, \ldots, n-1\} \\ \text{then } s_i f = f \end{matrix} \right\}$$

where $s_i$ is the transposition that switches $i$ and $i+1$.

<u>Proof of (a)</u>

Let $j \in \{1, \ldots, n-1\}$ and $p \in B$ and let

$$S_j(p) = \{ \tilde{f}_j^{\varphi_i(p)} p, \ldots, \tilde{f}_j p, p, \tilde{e}_i p, \ldots, \tilde{e}_i^{\varepsilon_i(p)} p \}$$

be the $j$-string of $p$. Define

$$s_j p \in S_j(p) \text{ such that } wt(s_j p) = s_j\, wt(p).$$

(One must show that this definition makes sense)

Then

$$s_j(s_j p)$$

and

$$s_j \cdot \mathrm{char}(B) = \sum_{p \in B} x^{s_j wt(b)} = \sum_{q \in B} x^{wt(s_j p)}$$

$$= \mathrm{char}(B).$$

So $\mathrm{char}(B)$ is a symmetric function.

<u>Proof idea for (b)</u> Let $e_- = \sum_{w \in S_n} \det(w) w$

Then, since $\mathrm{char}(B)$ is a symmetric function,

$$\frac{\mathrm{char}(B)\, a_\rho}{a_\rho} = \frac{1}{a_\rho} \mathrm{char}(B) e_- x^\rho$$

$$= \frac{1}{a_\rho} e_- \mathrm{char}(B) x^\rho$$

$$= \frac{1}{a_\rho} \sum_{p \in B} e_- x^{wt(p)} x^\rho$$

$$= \sum_{p \in B} \frac{a_{wt(p)+\rho}}{a_\rho} = \sum_{p \in B} s_{wt(p)}$$

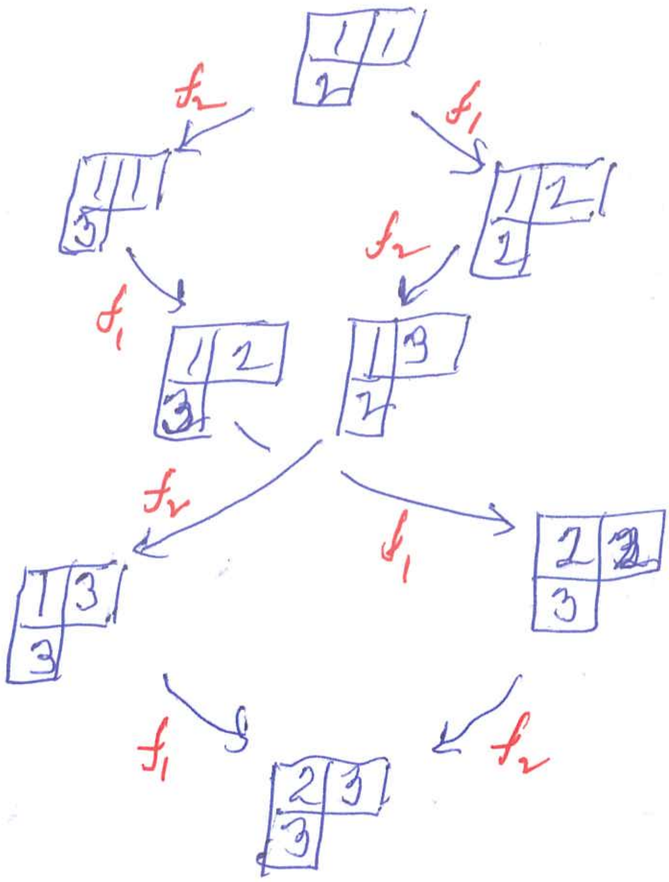
where

$$a_\gamma = e_- x^\gamma = \sum_{w \in S_n} \det(w) x^{w\gamma}$$

and $\rho = (n-1, n-2, \ldots, 2, 1, 0)$.

and $s_\gamma = \dfrac{a_{\gamma+\rho}}{a_\rho}$

Example: $n = 3$ and $p = (2,1,0)$ and $B$ is



$$\operatorname{char}(B) = x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + x_1 x_2 x_3 + x_1 x_2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2$$

$$= s_{(210)} + s_{(201)} + s_{(120)} + s_{(111)} + s_{(111)} + s_{(102)} + s_{(021)} + s_{(012)}$$

$$= \frac{a_{(420)}}{a_p} + \frac{a_{411}}{a_p} + \frac{a_{(330)}}{a_p} + \frac{a_{321}}{a_p} + \frac{a_{321}}{a_p} + \frac{a_{312}}{a_p} + \frac{a_{231}}{a_p} + \frac{a_{222}}{a_p}$$

$$= \frac{a_{(420)}}{a_p} + 0 + 0 + \frac{a_{321}}{a_p} + \frac{a_{321}}{a_p} - \frac{a_{321}}{a_p} - \frac{a_{321}}{a_p} + 0 = s_{(420)}.$$