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Adv. Disc Math ①
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Schur functions

S_n is the group of $n\times n$ permutation matrices with matrix multiplication.

S_n acts on $\mathbb{C}[x_1,\ldots,x_n]$ by permuting $x_1,\ldots,x_n$.

$$\mathbb{C}[x_1,\ldots,x_n]^{S_n} = \{f\in\mathbb{C}[x_1,\ldots,x_n] \mid wf=f\}$$

$$\mathbb{C}[x_1,\ldots,x_n]^{\det} = \{f\in\mathbb{C}[x_1,\ldots,x_n] \mid wf=\det(w)f\}.$$

Then $\mathbb{C}[x_1,\ldots,x_n]^{S_n}$ has $\mathbb{C}$-basis

$$\{m_\lambda \mid \lambda=(\lambda_1,\ldots,\lambda_n)\in\mathbb{Z}^n_{\geq 0} \text{ and } \lambda_1\geq\cdots\geq\lambda_n\}$$

where

$$m_\lambda = \sum_{\mu\in S_n\lambda} x^\mu, \quad \text{with } x^\mu = x_1^{\mu_1}\cdots x_n^{\mu_n}$$

if $\mu=(\mu_1,\ldots,\mu_n)$. Note that

$$m_\lambda = \frac{1}{v_\lambda}\sum_{w\in S_n} wx^\lambda, \quad \text{where } v_\lambda = \mathrm{Card}(\mathrm{Stab}(\lambda))$$

So

$$m_\lambda = \frac{1}{v_\lambda} e_+ x^\lambda, \quad \text{where } e_+ = \sum_{w\in S_n} w.$$

17.09.2025
Adv. Disc. Math
A. Ram

The $\mathbb{C}$-vectorspace $\mathbb{C}[x_1,\dots,x_n]^{\det}$ has $\mathbb{C}$-basis

$$\{a_\gamma \mid \gamma=(\gamma_1,\dots,\gamma_n)\in\mathbb{Z}_{\geq 0}^n \text{ and } \gamma_1>\gamma_2>\dots>\gamma_n\}$$

where

$$a_\gamma = \sum_{w\in S_n} \det(w)\, w x^\gamma$$

Note that

$a_\gamma=0$ if there exist $i,j\in\{1,\dots,n\}$ with $i\neq j$ and $\gamma_i=\gamma_j$

Proposition There is a bijection

$$\left\{\begin{array}{l}\lambda=(\lambda_1,\dots,\lambda_n)\in\mathbb{Z}_{\geq 0}^n\\ \text{with } \lambda_1\geq\dots\geq\lambda_n\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\gamma=(\gamma_1,\dots,\gamma_n)\in\mathbb{Z}_{\geq 0}^n\\ \text{with } \gamma_1>\dots>\gamma_n\end{array}\right\}$$

$$(\lambda_1,\dots,\lambda_n) \longmapsto (\lambda_1+n-1, \lambda_2+n-2,\dots,\lambda_n+n-n)$$

$$\lambda \longmapsto \lambda+\rho, \text{ where } \rho=(n-1,n-2,\dots,2,1,0).$$

$$\gamma-\rho \longleftarrow\!\shortmid\ \gamma$$

Proposition (a) $a_\rho = \prod_{1\leq i<j\leq n}(x_i-x_j)$

(b) If $g\in\mathbb{C}[x_1,\dots,x_n]^{\det}$ then

$$\frac{1}{a_\rho}\, g\in\mathbb{C}[x_1,\dots,x_n]^{S_n}.$$

17.09.2015 (3)
Adv. Disc. Math
A. Ram

Theorem (Boson-Fermion correspondence) There is a $\mathbb{C}[x_1,\ldots,x_n]^{S_n}$-module isomorphism

$$\begin{array}{ccc} \mathbb{C}[x_1,\ldots,x_n]^{S_n} & \longrightarrow & \mathbb{C}[x_1,\ldots,x_n]^{\det} \\ f & \longmapsto & a_\rho f \\ \frac{1}{a_\rho} f & \longleftarrow\!\!\shortmid & g \end{array}$$

The <u>Schur function</u>

$$s_\lambda = \frac{a_{\lambda+\rho}}{a_\rho}$$

<u>Corollary</u> $\left\{ s_\lambda \;\middle|\; \begin{array}{l} \lambda = (\lambda_1,\ldots,\lambda_n) \in \mathbb{Z}_{\geq 0}^n \\ \lambda_1 \geq \cdots \geq \lambda_n \end{array} \right\}$

is a basis of $\mathbb{C}[x_1,\ldots,x_n]^{S_n}$.

<u>Examples</u> $S_3 = \{ \equiv, \underline{\times}, \overline{\times}, \times, \times, \times \}$

$$m_{(416)} = x^{(416)} + x^{(146)} + x^{(461)} + x^{(614)} + x^{(641)} + x^{(164)}$$

$$= x_1^4 x_2 x_3^6 + x_1 x_2^4 x_3^6 + x_1^4 x_2^6 x_3 + x_1^6 x_2 x_3^4 + x_1^6 x_2^4 x_3 + x_1 x_2^6 x_3^4$$

$$= m_{(641)}$$

$$a_{(641)} = x^{(641)} - x^{(461)} - x^{(614)} - x^{(146)} + x^{(416)} + x^{(164)}$$

$$= x_1^6 x_2^4 x_3 - x_1^4 x_2^6 x_3 - x_1^6 x_2 x_3^4 - x_1 x_2^4 x_3^6 + x_1^4 x_2 x_3^6 + x_1 x_2^6 x_3^4$$

$$= (x_1^2 - x_2^2) x_1^4 x_2^4 x_3 + (x_1^5 - x_2^5) x_1 x_2 x_3^4 + (x_1^4 - x_2^4) x_1 x_2 x_3^6.$$

$$q(x) = x_1^2x_2 - x_1x_2^2 - x_1^2x_3 - x_2x_3^2 + x_2^2x_3 + x_1x_3^2$$

$$= x_1x_2(x_1-x_2) - (x_1^2-x_2^2)x_3 + (x_1-x_2)x_3^2$$

$$= \{x_1x_2 - (x_1+x_2)x_3 + x_3^2\}(x_1-x_2)$$

$$= \{x_1(x_2-x_3) - (x_2-x_3)x_3\}(x_1-x_2)$$

$$= (x_1-x_3)(x_2-x_3)(x_1-x_2)$$